Effective description of the gap fluctuation for chaotic Andreev billiards

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We present a numerical study of the universal gap fluctuations and the ensemble averaged density of states (DOS) of chaotic two-dimensional Andreev billiards for finite Ehrenfest time \( \tau_E \). We show that the distribution function of the gap fluctuation for small enough Ehrenfest time can be related to that derived earlier for zero Ehrenfest time. An effective description based on the random matrix theory is proposed giving a good agreement with the numerical results. A systematic linear decrease of the mean field gap with increasing Ehrenfest time \( \tau_E \) is observed but its derivative with respect to \( \tau_E \) is in between two competing theoretical predictions and close to that of the recent numerical calculations for Andreev map. The exponential tail of the density of states is interpreted semiclassically.

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Recently, mesoscopic ballistic two dimensional normal (N) dots in contact with a superconductor(S) have been extensively studied. Such hybrid systems are commonly called Andreev billiards.1–5 In the most recent works, interest has shifted to mesoscopic fluctuations of the excitation spectrum of these systems.6–9 Since the subgap spectrum determines the tunneling conductance of an N-S contact this is an essential question both experimentally and theoretically.

Based on the semiclassical treatments and random matrix theory (RMT), it was shown by Melsen et al.2 that integrable Andreev billiards are gapless, whereas systems with classically chaotic dots possess an energy gap on the scale of the Thouless energy \( E_T = \hbar / (2 \tau_D) \), where \( \tau_D = \pi A / (W v_F) \) is the mean dwell time in the normal dot (here \( A \) is the area of the normal dot, \( W \) is the width of the superconducting region, and \( v_F \) is the Fermi velocity). For such systems, it is assumed that \( \delta_N \ll E_T \ll \Delta \), where \( \delta_N = 2 \pi \hbar^2 / (m A) \) is the mean level spacing of the isolated normal dot with effective mass \( m \) of the electrons and \( \Delta \) is the bulk order parameter of the superconductor.4 In further studies6,7 it was concluded that in chaotic cases, the lowest energy level \( E_1 \) of the system varies from sample to sample with a universal probability distribution \( P(x) \) given in Ref. 6 if the energy levels \( E_1 \) are rescaled as \( x = (E_1 - E_{g \text{RMT}}) / \Delta_g \), where

\[
E_{g \text{RMT}} = 2 \gamma^{1/2} E_T,
\]

\[
\Delta_g \text{RMT} = c' M^{1/3} \delta_N.
\]

Here \( \gamma = \frac{1}{2} (\sqrt{5} - 1) \) is the golden ratio, \( c' = [(15 - 6 \sqrt{5}) / 20]^{1/3} / 2 \pi \), \( M = \text{Int}[k_F W / \pi] \) is the number of open channels in the \( S \) region and \( k_F \) is the Fermi wave number (\( \text{Int[...] stands for the integer part} \)). The resulting distribution \( P(x) \) yields the RMT values6 \( \langle x \rangle = 1.21 \) and \( \sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = 1.27 \).

Equations (1) are strictly valid only in the RMT limit, i.e., when the Ehrenfest time \( \tau_E = (1 / \lambda) \ln (L / \lambda_F) \) tends to zero (\( \tau_E \) is the time needed for a wave packet of minimal size \( \lambda \approx 2 \pi / k_F \) to spread to the characteristic size \( L \) of the classically chaotic dot with Lyapunov exponent \( \lambda \)). For finite but small enough Ehrenfest time, Silvestrov et al.,10 and Vavivlov and Larkin11 predicted that to lowest order in \( \tau_E / \tau_D \) the mean-field gap \( E_g \) decreases linearly by increasing the ratio \( \tau_E / \tau_D \). The first numerical evidence for the distribution \( P(x) \) in the RMT limit and the dependence of \( E_g \) on the ratio \( \tau_E / \tau_D \) was presented by Jacquod et al.8 modeling the hybrid system with the one-dimensional Andreev map.

From an experimental point of view, more realistic candidates for studying quantum chaos in hybrid systems would be two-dimensional Andreev billiards with classically chaotic normal region. However, to date no numerical study of these systems confirms the predictions of the RMT, except for the mean field density of states.2 The purpose of this paper is to present (for the first time) a numerical study of the gap fluctuation and the density of states in two-dimensional chaotic Andreev billiards. Our aim is to investigate, how the distribution of \( E_1 \) changes from the RMT prediction when the Ehrenfest time is finite. We find that our results are well parametrized by assuming that \( P(x) \) retains its RMT form (and therefore \( \langle x \rangle \) and \( \sigma_x \) are unchanged), consequently the scaling parameters \( E_{g \text{RMT}} \) and \( \Delta_{g \text{RMT}} \) in Eq. (1) are replaced by \( \tau_E \)-dependent quantities \( E_g \) and \( \Delta_g \), given by

\[
E_g = \langle E_1 \rangle - \langle x \rangle \Delta_g,
\]

\[
\Delta_g = \delta E_1 / \delta x,
\]

where \( \langle E_1 \rangle \) is the mean value, \( \delta E_1 = \sqrt{(\langle E_1^2 \rangle - \langle E_1 \rangle^2)} \) is the standard deviation of \( E_1 \). Although the theoretical distribution of \( E_1 \) is unknown, its mean and standard deviation can be numerically estimated from the data of an ensemble of the SA billiard. Given the above RMT values of \( \langle x \rangle \) and \( \delta_x \), \( E_g \) and \( \Delta_g \) are determined. Our work is motivated by the fact that different predictions exist for the slope of the linear dependence of the mean field gap \( E_g \) on \( \tau_E / \tau_D \), and that the question whether an universal function for the gap distribution still exists has not been addressed. As a further support of our model it is also shown that this effective description, based on the RMT form for \( P(x) \), predicts quite accurately the edge of the ensemble averaged DOS.
In our numerical work we calculated the exact energy levels of the so-called Sinai-Andreev (SA) billiard in which the normal dot is a Sinai billiard (see Fig. 1). The energy levels of the Andreev billiards are the positive eigenvalues \( E \) (measured from the Fermi energy) of the Bogoliubov-de Gennes equation.\(^\text{12}\) To obtain the exact energy levels of an Andreev billiard we used the recently derived general and quantum mechanically exact secular equation expressed in terms of the scattering matrix \( S_0(E) \) of the normal region.\(^\text{13}\) The scattering matrix \( S_0(E) \) was calculated by expanding the wave function in the N region in terms of Bessel functions.

To ensure that the long classical trajectories (compared to the characteristic length of the system) starting and ending at the N-S interface are truly chaotic the following geometrical constrains should be applied: \( h = a - W \) (the superconductor is placed at the top of the vertical border of the Sinai billiard), \( R + W \approx a \) and \( R \approx a / \sqrt{2} \). Otherwise, there may exist arbitrary long trajectories without bouncing on the circular part of the Sinai billiard. For such intermittent trajectories the return probability decays as \( P_r(s) \sim 1/s^3 \) (\( s \) is the length of the trajectory) and this results in a gapless energy spectrum of the Andreev billiard.\(^\text{2,4}\)

Figure 2 shows our numerical results for the integrated distribution \( F(x) = \int_0^x P(x') dx' \) together with the theoretical prediction [the distribution \( P(x) \) is shown in the inset]. In our numerics we used 5000 slightly different realizations of the SA billiard by varying the geometrical parameters \( R, \) \( W \) and the Fermi wave numbers \( k_F \) (for the parameters of the SA billiard, see Ref. 14). From Eq. (2) we found that \( E_g \approx 0.5 \) \( E_T \) and \( \Delta_g \approx 0.118 \) \( E_T \) and they are different from those given in the RMT limit, \( E^R_{g,\text{RMT}} = 0.6 \) \( E_T \) and \( \Delta^R_{g,\text{RMT}} = 0.097 \) \( E_T \). It is clear from the figure that the numerical result for \( F(x) \) [using Eq. (1)] is also different from that of the theoretical prediction in the RMT limit. However, the agreement between our numerical results obtained using Eq. (2) and the universal distribution function \( F(x) \) is good. Similar numerical results were found for other ensembles of the SA billiard.

We now argue that the above deviations in values of \( E_g \) and \( \Delta_g \) from the RMT predictions are consequences of their systematic dependence on the ratio of the Ehrenfest time and the dwell time as predicted in Refs. 10 and 11. To show this, we calculated these parameters for several ensembles with different values of \( \tau_E / \tau_D \). Since the characteristic length of the system is uncertain, two definitions of the Ehrenfest time, proposed in Ref. 11 and used in numerical simulations of Ref. 8, were here adapted for numerical calculations:

\[
\tau_E^{(1)} = \frac{1}{2\lambda} \ln \frac{W^2}{\lambda_F L_c}, \quad (3a)
\]

\[
\tau_E^{(2)} = \frac{1}{2\lambda} \ln \frac{L_{av}}{\lambda_F}. \quad (3b)
\]

where \( L_c \) is the average length of the part of the trajectory lying between two consecutive bounces at the curved boundary segment of the Sinai billiard and \( L_{av} = \pi A/K \) with perimeter \( K \) of the billiard is the mean chord length in the normal region. They are parametrically different, but their numerical values are of the same magnitude. For the numerical results shown in Fig. 2 the ratio of the Ehrenfest time and the dwell time is about \( \tau_E^{(1)} / \tau_D = 0.1 \) and \( \tau_E^{(2)} / \tau_D = 0.26 \) (for calculation of the Lyapunov exponent, see Ref. 14). From our numerical study a systematic decrease of \( E_g \) as a function of ratio \( \tau_E^{(1)} / \tau_D \) is found as shown in Fig. 3 (for details of the system parameters, see Ref. 15). A similar result has been obtained.
we found that $\alpha=0.7 \pm 0.2$ and $\beta=0.95 \pm 0.02$. Similar results were obtained by using (3b): $\alpha=0.9 \pm 0.3$ and $\beta=1.10 \pm 0.07$. According to the two competing theories the universal values of $\alpha=0.23$ and $\beta=1$ (Ref. 11) and $\alpha=2$ and $\beta=1$ (Ref. 10) were predicted, while from the numerics for Andreev map only $\alpha=0.59 \pm 0.08$ is universal value. From our numerics the value $\alpha$ is in between the two theoretical predictions and within the numerical errors it agrees with the result for the Andreev map. We found that the values of $\Delta_s$ obtained from (2b) are slightly greater than those predicted from (1b). However, no obvious functional form can be deduced from our data for the dependence of $\Delta_s$ on $t_E/\tau_D$.

Our results summarized in Figs. 2 and 3 suggest that for systems with nonzero Ehrenfest time an effective description of the gap distribution can be formulated by means of the universal function $P(x)$ and the Ehrenfest time dependent parameters $E_g$ and $\Delta_s$ obtained from Eq. (2).

To further support the effective description of the universal gap fluctuation we compare the numerically obtained ensemble averaged DOS $\langle g(E) \rangle$ with that found from our effective description. The ensemble averaged DOS $\langle Q_{\text{eff}}(E) \rangle$ in the effective description can be calculated from $\langle Q_{\text{RMT}}(x) \rangle$ given in the RMT limit using the Ehrenfest time dependent parameters $E_g$ and $\Delta_s$ when “scaling back” the variable $x = (E - E_g)/\Delta_s$ into the energy variable $E$. In the RMT limit $\langle Q_{\text{RMT}}(x) \rangle = -\alpha x R^2(x) + \{A'(x)^2 + \frac{1}{2} A(x)[1 - f_{x}^{-1} A(x)] \}$ is again a universal function of $x$ (see note 20 in Ref. 6). It can be seen from Fig. 4 that the agreement between our numerically obtained ensemble averaged DOS $\langle g(E) \rangle$ and $\langle Q_{\text{eff}}(E) \rangle$ is excellent at the edge of the spectrum (where the theory is valid) for ratio $\tau_E^{(1)}/\tau_D \approx 0.063$ (the same holds for other ratios not shown). For larger energies the DOS is around $2/\Delta_N$ as it is expected. For convenience the mean field DOS (Ref. 2) in the RMT limit is also plotted in the figure. In the case of the two-dimensional Andreev billiards the ensemble averaged DOS $\langle Q(E) \rangle$ shown in Fig. 4 is the first numerical evidence for the prediction $\langle Q_{\text{eff}}(E) \rangle$ obtained from the theoretical result $\langle Q_{\text{RMT}}(x) \rangle$ (see Fig. 2 of Ref. 6).

From Fig. 4 one can also see that the agreement between the Bohr-Sommerfeld approximation (BS) (Ref. 2) and our numerics is quite good at the bottom of the spectrum. This suggests a semi-classical explanation. In a work by Schomerus and Beenakker close correspondence has been found between the morphology of the phase space and the density of low energy excitations. Reference 4 finds that the semi-classical prediction of the DOS for systems with fully chaotic phase space has no definite gap but it becomes exponentially small below the energy $= 0.5 E_T$. Furthermore in case of systems with mixed phase space and strong coupling of the regular islands to the superconductor, the above defined “gap” is substantially reduced, namely by a factor $\tau_D/\tau^{*}$, where $\tau^{*}$ is the mean dwell time of trajectories in the chaotic part of the phase space.

We applied the concept of the above defined gap reduction to our SA billiard but in somewhat different way as in Ref. 4. We took account of the effect of the superconducting lead (i.e., an opening), consequently, the finiteness of the trajectories. Therefore, a trajectory is considered to be chaotic enough and used in the calculation of the $\tau^{*}$ if the width of the bunch of trajectories starting in its N-S neighborhood at the N-S interface is increased to the characteristic length scale of the billiard before returning to the superconducting lead. For those trajectories, which bounce only on the strait walls or bounce on the circular part only once before returning to the lead this stretching is significantly smaller for the parameters of the SA billiard used in our calculations. (Let us note that the ratio of these trajectories goes to zero for $W \rightarrow 0$). In this respect, the phase space of our system can be considered to be mixed. From our numerics we found that for the ensemble corresponding to Fig. 4 the effective mean dwell time $\tau^{*} = 1.66 \tau_D$ hence the semiclassically obtained gap is reduced to the value $= 0.3 E_T$. This result is just about the energy value where the numerically found DOS becomes exponentially small and thus it confirms the semiclassical picture developed in Ref. 4. Note also that below the value $= 0.3 E_T$ (i.e., for $x \leq -2$) the RMT DOS $\langle Q_{\text{RMT}}(x) \rangle$ is also exponentially small.

While the BS approximation is quite successful in predicting the density of low energy excitations, it is clear from Fig. 4 that the edge of the spectrum can be better predicted using our effective description. This may also reveal the limits of the BS approximation. (We observed similar deviations for other ensembles not shown.)

In summary we have numerically shown that for finite but small enough Ehrenfest time the distribution of the rescaled first energy level $E_1$ of an ensemble of chaotic Andreev bil-
liards can be treated by an effective description. In this model the scaling parameters $E_g$ and $\Delta_g$ extracted from the data of the ensemble rescale the distribution of $E_1$ such that it agrees with $P(x)$ given in the RMT limit. Our numerical results also show that to lowest order in $\tau_E/\tau_D$ the mean field gap $E_g$ decreases linearly with the Ehrenfest time but the slope is between the two competing theories and close to that of the recent numerical calculations for the Andreev map. Calculation of the ensemble averaged DOS gives further confirmation of our effective description. Our numerics suggest that the exponential tail of the DOS can be well interpreted semiclassically.

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8. In our numerical calculations the bulk superconductor gap $\Delta$ was fixed: $\Delta=0.02E_F$. The width $W$ of the S region varies in $[0.19a,0.191a]$, $R$ in the range $[0.868a,0.874a]$ (this variation corresponds to $0.4\lambda_F$). Values of $E_F$ were chosen between $E_F^0\pm0.004E_F^0$ with such an $E_F^0$ that the number of open channels in the S region is always $M=25$. The Lyapunov exponent $\lambda$ was obtained by following trajectories from $4\cdot10^5$ uniformly distributed initial conditions. For the above given range in $R$ the Lyapunov exponent slightly changes in the range $[1.65,1.68]$.
9. In Fig. 3 the number of open channels varies between $M=18$ and 30, whereas the width $W$ between 0.16 $a$ and 0.22$a$, and the ranges in $R$ and $E_F$ are the same as in Ref. 14. The number of realizations is between 2500 and 5000.
10. As can be seen from Eq. (2a) one needs to evaluate both the error (i.e., standard deviation) of the numerical approximation of $(E_i)$ and that of $\Delta_j$. It can be shown that both standard deviations are enhanced by the correlations of $E_i$, thereby the error of $E_i$ is also affected. The standard deviation of $(E_i)$ can be formulated as $\sigma(E_i) = \sqrt{(N'/N)\sigma(E_i)}$, where $N$ is the number of different realizations, $\sigma(E_i)$ is the standard deviation of $E_i$. $N'$ is a characteristic of the correlations. In our studies $N'$ was in the range $15-55$. The error of $\Delta_j$ is estimated by calculating $\Delta_j$ separately for parts of our ensemble and analyzing its fluctuation.

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