**Lancaster University Management School**
**Working Paper**
**2008/002**

# Voronoi game on disjoint open curves

M Dziubinski

The Department of Economics
Lancaster University Management School
Lancaster LA1 4YX
UK

# Voronoi Game On Disjoint Open Curves

Marcin Dziubiński [a]

[a] *Department of Economics, Lancaster University, Lancaster LA1 4YX, UK*

**Abstract**

Two players are endowed with resources for setting up $N$ locations on $K$ open curves of identical lengths, with $N > K \geq 1$. The players alternately choose these locations (possibly in batches of more than one in each round) in order to secure the area closer to their locations than that of their rival's. The player with the highest secured area wins the game and otherwise the game ends in a tie. Earlier research has shown that, if an analogical game is played on disjoint closed curves, the second mover advantage is in place only if $K = 1$, while for $K > 1$ both players have a tying strategy. It was also shown that this results hold for open curves of identical lengths when rules of the game additionally require players to take exactly one location in the first round. In this paper we show that the second mover advantage is still in place for $K \geq 1$ and $2K - 1 \leq N$, even if the additional restriction is dropped, while $K \leq N < 2K - 1$ results in the first mover advantage. We also study a natural variant of the game, where the resource mobility constraint is more stringent so that in each round each player chooses a single location and we show that the second mover advantage re-appears for $K \leq N < 2K - 1$ if $K$ is an even number.

*Key words:* Competitive locations, Disjoint spaces, Winning/Tying strategies, Equilibrium configurations.
*JEL:* C72, D21, D72.

## 1 Introduction

Games involving choice of locations has long been an important area of study in economics. The corresponding literature centers around the seminal work by Hotelling (1929) which considers a profit maximizing firm's decision about optimal location when the consumers are located uniformly on a line segment. Subsequently, this was extended to the celebrated circular city model in Chamberlin (1953) and later by Salop (1979). While in Hotelling (1929),

*Email address:* `m.dziubinski@lancaster.ac.uk` (Marcin Dziubiński).

Chamberlin (1953) and Salop (1979) simultaneous-move games are considered, Prescott and Visscher (1977) and Economides (1986) study the problem when firms are allowed to enter sequentially on a line segment and circular city respectively to show that the outcomes of a sequential location game can differ significantly from those obtained in a simultaneous-move scenario. In Cheong et al. (2002) the existence of a winning strategy for the second mover is shown, even for a single round location game played on a two dimensional closed plane. In Chawla et al. (2003) an upper bound for the size of the first mover disadvantage is provided in a game where firms compete to maximize market shares and consumers are distributed over a $d$-dimensional Euclidean space. For an overview of competitive facility location models see Eiselt and Laporte (1989) and Tobin et al. (1989).

A recent work in this respect is a game of influence studied by Ahn et al. (2004), where there are two players who are each endowed with the same number $N$ of facilities to locate (possibly in batches of more than one facilities) on a closed or an opened curve in a sequential manner. In order to win the game, a player must secure as much area as possible that is closer to its locations than those of its competitor. Each player faces a resource mobility constraint such that not all facilities can be located in the first round. They show that in such a game (to be described precisely in section 2) where play must involve at least two rounds, the second mover always has a winning strategy and the game would always result in a tie if players were forced to end the game in a single round. The result for the case of the game played on open curves was established with additional restriction, that players are forced to take exactly one location in the first round.

A variant of the above mentioned games of influence is where players compete over a collection of disjoint areas in which locations can be placed. For example, retail chains set up stores in different cities or countries. Such variant of the game played on multiple disjoint closed curves was studied by Datta et al. (2007) and it was shown that additional spaces of this type lead to existence of tying strategies for both players. This paper addresses analogical location game played on a family of disjoint open curves.

We show that the second mover advantage, as in Ahn et al. (2004), holds as long as $2K - 1 \leq N$, while in the case of $K \leq N < 2K - 1$ we have the first mover advantage. This improves the result from Ahn et al. (2004), in particular, as we do not require additional restrictions in rules of the game for the case of $K = 1$ for our result to hold.

We also study a natural variant of this game, where each player takes exactly one location at each round. In this extended game we show that the second mover advantage reappears for $K \leq N < 2K - 1$, if and only if $K$ is an even number.

The rest of the paper is structured as follows. In section 2 we define the game. Section 3 states and proves our results and the paper concludes in section 4.

## 2 The multiple line-segments game

A *Voronoi game*, as introduced in Ahn et al. (2004), involves two players called White ($W$) and Black ($B$) each having $N$ points to place in a metric space $S$, so that no two points can occupy the same location. After the points are placed, a division of space into areas of points lying closer to a particular point than the others (i.e. a Voronoi diagram) is created. The player whose points attract bigger area in total wins the game.

We extend this game to multiple disjoint metric spaces (c.f. Datta et al. (2007)) and focus our attention on Voronoi games played on multiple disjoint open spaces of identical lengths. Since we want to study only a natural class of distance function, where distance between any two points is simply the length of interval between them, so we assume from now on, that all these open spaces are line-segments of equal lengths.

Players place their points in rounds and that player $W$ places his points first at each round. Moreover, (i) each player must place at least one point in each round, (ii) in the first round when play begins, $W$ cannot place all $N$ points (perhaps because not all resources are available at the beginning of the game),[1] (iii) the game ends only after all players have placed all $2N$ points and (iv) at any round, the total points placed so far by $B$ cannot exceed that of $W$.[2]

This results in a sequential game where roles (that is first and second mover identities) cannot be reversed and the number of rounds is endogenous and can be controlled by $W$ subject to the restriction that there must be at least 2 rounds. The objective of each player is to maximize the total length of fragments of the curves that are closer to that player's locations than to the

[1] Condition (ii) enforces at least two rounds of the game. In the case of one round game, either there is a simple winning strategy for the first mover or the second mover can win using strategies for games with at least two rounds. We will comment on the one round variant of the game throughout the paper.

[2] This is basically a condition required to preserve the first and second mover identities over any play. These identities could be preserved even with the assumption that players place equal number of points in each period. In this sense, the condition given in Ahn et al. (2004) and used here is general and hence weaker. We shall also study a natural variant of this game where each player must place exactly one point in each round.

other one's, so that a player wins if and only if the area it secures is strictly the largest one. Otherwise there is a tie.

It is shown in Ahn et al. (2004) that when this game is played on a closed curve, then $B$ always has a winning strategy, though $W$ can bring its size of influence arbitrarily close to that of $B$'s. An analogical result for this game played on an open curve is also shown, however under additional restriction requiring players to place exactly one point in the first round.

In Datta et al. (2007) a generalized version of the game, with players placing their points on multiple disjoint close curves of equal lengths, is studied to show the existence of tying strategies for both players. This result shows that it is possible to enforce fair spreading of influence with rules of the game in question. It is also shown in Datta et al. (2007) that the second mover advantage is in place again, in a natural extension where players place exactly one point in each round.

Our objective is to check if extending the settings of the game from one to multiple open-curves of identical lengths leads to similar results as in case of moving from one to multiple closed curves. We now give a formal presentation of a Voronoi game on disjoint line-segments of equal lengths.

Let $\{W, B\}$ be the set of players, where $W$ stands for White and $B$ stands for Black. *The game on the family of disjoint line-segments* is defined by a pair $\left\langle N, \{L_j\}_{j=1}^{K} \right\rangle$, such that $N > K \geq 1$ and $\{L_j\}_{j=1}^{K}$ is a family of $K$ disjoint line-segments having equal lengths.

The game ends when each player $p \in \{W, B\}$ selects a *total* of $N$ points on $K$ line-segments. The set of points selected by $W$ is $\Gamma \subseteq \bigcup_{j=1}^{K} L_j$ and the set of points selected by $B$ is $\Omega \subseteq \bigcup_{j=1}^{K} L_j$. Players re-arrive in alternating sequence with $W$ moving first, and are in principle allowed to place points in batches.

Let $\Gamma_r$ be the set of points that $W$ places in round $r \geq 1$ while $\Omega_r$ be the same for $B$. The game ends when all $2N$ points are placed on the line-segments. [3] We will use $w \in \Gamma$ ($b \in \Omega$) to denote a point placed by $W$ ($B$) during the game. We will call points placed by the player $W$ *white points* and those placed by the player $B$ *black points*.

As discussed above, the game has the following conditions:

(1) $|\Gamma_r|, |\Omega_r| \geq 1$, for every $r \geq 1$.
(2) $|\Gamma_1| < N$.

[3] Note that we put no restriction on how players distribute these points across the line-segments (some line-segments are allowed to remain empty, in which case it is ignored while computing payoffs).

(3) $\sum_{i=1}^{r} |\Gamma_i| \geq \sum_{i=1}^{r} |\Omega_i|$, for every $r \geq 1$.
(4) $\sum_{i\geq 1} |\Gamma_i| = \sum_{i\geq 1} |\Omega_i| = N$.

The endogenously determined number of rounds in a given play of the game will be denoted by $R$. Obviously $\Gamma = \bigcup_{r=1}^{R} \Gamma_r$ and $\Omega = \bigcup_{r=1}^{R} \Omega_r$. Notice that the restrictions of the game imply that $R \geq 2$.

Let $L$ be a line-segment and let $(x, y)$ be an ordered pair of elements of $L$. Then $d(x, y) = |x - y|$ is the distance between $x$ and $y$. Notice that $d(x, y) = d(y, x) \in [0, 1]$. Given an interval $(x, y)$, the volume (or length) of $(x, y)$ is $d(x, y)$. Let

$$A_W(L) = \left\{ x \in L : \min_{w \in L\cap\Gamma} d(x, w) < \min_{b \in L\cap\Omega} d(x, b) \right\}$$

be a set of points on $L$ that are closer to points placed by $W$ on $L$ than to points placed there by $B$. Similarly we can define $A_B(L)$. Notice that $A_W(L)$ and $A_B(L)$ are finite sets of pairwise disjoint line-segments covering $L$.

Let $A$ be a finite set of intervals and let $V(A)$ denote the volume (sum of lengths) of the intervals in $A$. When the game ends, each player $p$ receives a score $S_p$ equal to the volume of the set of intervals constituting the set of points closest to positions chosen by that player over all line-segments, that is

$$S_p = \sum_{k=1}^{K} V(A_p(L_k))$$

for $p \in \{W, B\}$. Given these scores, the payoff of the players is $u_p(S_p, S_q) = S_p - S_q$, where $\{p, q\} = \{W, B\}$. We say that the game is *a tie* if $S_p = S_q$, while player $p$ *wins* if $S_p > S_q$. A strategy is a contingent plan for every possible history of the game. We do not need to define this general notion formally although we lay out complete specifications of the strategies we report. We will use uppercase letters $X$, $Y$ to denote pure strategies. Strategy $X$ is called *a winning strategy* (*a tying strategy*) for player $p$ if no matter what player $q$ does, by using $X$ player $p$ guarantees that $S_p > S_q$ ($S_p \geq S_q$).

### *2.1 Definitions and preliminary observations*

We first develop some concepts and notations. Let $P_W$ and $P_B$, such that $P_W \cup P_B = P$, be sets of all white and all black points in $P$, respectively. Then an interval $(x, y) \subseteq L$ such that $\{x, y\} \subseteq P_W$ ($\{x, y\} \subseteq P_B$) is called a *white (black) interval.* An interval that is neither white nor black is called a *bichromatic interval* and an interval which is not bichromatic shall be at times referred to as a *monochromatic interval.* Additionally the segment between the left end of a line-segment and the leftmost point placed on it is called a *left* *segment*. The notion of a *right segment* is defined symmetrically. A *border segment* is either a left or a right segment. The union of the left and the right segment is called a *border interval*. A border interval is called *balanced* if both border segments have equal sizes. Similar to non-border intervals, a border interval can be white, black, monochromatic or bichromatic.

We will use $w^L$ ($b^L$) to denote the number of white (black) points placed on line-segment $L$. We will also use $I^W(L)$ ($I^B(L)$) to denote the number of white (black) intervals on $L$.

Let $m$ be a positive natural number and $L$ a line-segment. Then the set of *key positions*[4] on $L$ determined by $m$ is the set

$$\kappa(L,m) = \{p \in L : p = 1/(2m) + l/m, \text{ where } l \in \{0,\ldots,m-1\}\}.$$

A point placed in a key position will be called a *key point* and an interval formed by two key points will be called a *key interval*. Key positions $1/(2m)$ and $1 - 1/(2m)$ are called *extreme key positions* and a point placed in any of them is called an *extreme key point*. Notice that border intervals, like any other intervals, may be key intervals as well. A border segment formed by a key point is called a *key border segment*.

The following lemmas and a corollary, which are generalizations of the lemmas presented in Ahn et al. (2004) for more than one line-segment, will be useful.

**Lemma 1** *Let $\{L_k\}_{k=1}^K$ be a family of line-segments. Then*

$$\sum_{k=1}^K I^W(L_k) - \sum_{k=1}^K I^B(L_k) = \sum_{k=1}^K w^{L_k} - \sum_{k=1}^K b^{L_k},$$

*i.e. the difference between the number if white points and black points placed on the family of line-segments is equal to the difference between the number of white and black intervals on that family of line-segments.*

**Proof.** In Ahn et al. (2004) it is shown that for any line-segment $L$ it holds that $I^W(L) - I^B(L) = w^L - b^L$. Then $\sum_{k=1}^K \left(I^W(L_k) - I^B(L_k)\right) = \sum_{k=1}^K \left(w^{L_k} - b^{L_k}\right)$, and so $\sum_{k=1}^K I^W(L_k) - \sum_{k=1}^K I^B(L_k) = \sum_{k=1}^K w^{L_k} - \sum_{k=1}^K b^{L_k}$. ■

The following corollary is immediate from the above lemma.

**Corollary 2** *Let $\{L_j\}_{j=1}^K$ be a family of line-segments where each of the players $W$ and $B$ placed the same number of points. Then the number of white and black intervals is the same.*

[4] We use the term key position here for what was called a key point in Ahn et al. (2004). We prefer to use a term key point to refer to a point placed at a key position.

**Lemma 3** *Let $\{L_k\}_{k=1}^{K}$ be a family of line-segments with key positions with respect to some $M$ for each line-segment. Assume that:*

*(i)* *all $KM$ key positions are covered,*
*(ii)* *there exists a line-segment $L$ with $b^L < M$ black points and*
*(iii)* *each line-segment contains at most one white interval and none of them contains a white key interval.*

*Then there must exist a bichromatic key interval. Moreover, if all white intervals are border intervals, then there must exist a non-border bichromatic key interval.*

**Proof.** Let $L$ be a line-segment with $b^L < M$ black points. Since there can be at most one white interval on $L$, so, by Lemma 1, it must be that $w^L \leq b^L + 1 \leq M$. Notice first that it cannot be that all key points are white. This is because if it was the case, then there would be $M$ white key intervals on $L$, and after distributing $b^L < M$ black points on them, there would have to be at least one empty white key interval, which violates point (iii). Hence there must be at least one white and at least one black key point on $L$, as all key positions are taken and $b^L < M$. If there is at least one white and at least one black key point on $L$, then there are at least two bichromatic intervals there. Suppose that there are $Q$ black key points and $M - Q$ white key points on $L$. Then two cases are possible: (a) $Q \geq M - Q$ and (b) $M - Q > Q$.

Assume first that case (a) holds and that there are $P$ white key intervals on $L$. Then, by Lemma 1, there are $2Q - M + P$ black key intervals and $M - (2Q - M + 2P) = 2(M - Q - P) \geq 2$ bichromatic key intervals on $L$. Now all remaining $\leq M - Q - 1$ black points are distributed over the key intervals on $L$. Since there is no empty white key interval and there is at most one white interval on $L$, so there must be at least one black point in each white key interval. This leaves $\leq M - Q - P - 1$ black points to be distributed over remaining black and bichromatic key intervals. Suppose that all of them are distributed over bichromatic key intervals. Then there are $2(M-Q-P) - (M-Q-P-1) = M-Q-P+1$ bichromatic key intervals without a black point inside. Since $M - Q - P \geq 1$, so there are at least 2 bichromatic key intervals without a black point inside. Also, all remaining $\leq Q$ white points must be distributed over $M$ key intervals. If there is no white interval on $L$, then all of these points must be placed inside black intervals, and there will be at least two empty bichromatic key intervals, one of them not being a border interval. If there is a white interval on $L$ then at most one of remaining white intervals can be placed inside a bichromatic key interval. Thus there will be at least one empty bichromatic key interval and, if the white interval is a border one, the empty bichromatic key interval must be a non-border interval.

Now assume that case (b) holds and that there are $P$ black key intervals on $L$. Then, by Lemma 1, there are $M - 2Q + P$ white and $M - (M - 2Q + 2P) = 2(Q - P) \geq 2$ bichromatic key intervals on $L$. Now all remaining $\leq M - Q - 1$ black points are distributed over the key intervals on $L$. Since there are no empty white key intervals and there is at most one white interval on $L$, so there must be at least one black point in each white key interval. This leaves $\leq Q - P - 1$ black points to be distributed over remaining black and bichromatic key intervals. Suppose all of them are distributed over bichromatic key intervals. Then there are $2(Q - P) - (Q - P - 1) = Q - P + 1 \geq 2$ bichromatic key intervals without a black point inside. Also, all remaining $\leq Q$ white points must be distributed over $M$ key intervals. By arguments analogical to those used for case (a), there must remain at least one empty bichromatic key interval and this interval cannot be a border one, if there is a border white interval on $L$. This completes the proof. ■ We now move to our main results.

## 3 Results

We will consider two cases separately: the one with $K \mid N$ and the other with $K \nmid N$.

### *3.1 Case of $K \mid N$*

We show that for any game $\left\langle N, \{L_j\}_{j=1}^{K} \right\rangle$ with $K \mid N$ and $N > K$, the second mover has a winning strategy for any $K \geq 1$. Notice that this case includes the case studied in Ahn et al. (2004) as a special case.

Consider Strategy $Y_L^*$ presented below. Player $B$ places one or two points at each round, apart from the last one, when he has to place all remaining points (proceeding according to the strategy). Key positions are defined on each of line-segments with respect to $N/K$. Player $B$ has to monitor the number of white border key intervals, which can change between rounds and which is denoted by $M(r)$, for a given round $r$.

**Theorem 4** *Let $\left\langle N, \{L_j\}_{j=1}^{K} \right\rangle$ define a game on a family of disjoint line-segments, with $N > K$. If $K \mid N$ then $Y_L^*$ is a winning strategy for $B$.*

**Proof.** The game played when $B$ uses $Y_L^*$ can be divided into three subsequent stages: (a) taking key positions ($B$ plays option (a)), (b) breaking white intervals ($B$ plays option (b)) and (c) the last move ($B$ plays option (c)).

**Strategy $Y_L^*$**

**if** *there is an empty key position left* **then**

**(a)** place a point on a key position, preferring extreme key positions to the other and preferring line-segments with one extreme white key point to the other

**else if** *$B$ has $> M(r) + 1$ points left* **then**

**(b)** **if** *there exits a non-border white interval* **then**

**(1)** place a point inside a maximal white non-border interval

**else if** *there exists an unbalanced white border interval with one key border segment and length of the other border segment is $l$* **then**

**(2)** place a point in the key border segment of this interval at distance $< K/(2N) - l$ from the key point

**else if** $M(r) = 0$ **then**

**(3)** place a point in a non-border bichromatic key interval at distance of $< K/N - l$ from its white end point, where $l$ is length of a maximal white border interval

**else**

**(4)** place two points in a white border key interval creating a balanced black border interval of length $> l$, where $l$ is length of a maximal white non-key interval

**else**

**(c)** **if** $M(r) > 0$ **then**

**(1)** place all points in different white key border segments at distance $< \delta/(M(r) + 2)$ from white key points, where $\delta$ is the difference between a key interval and maximal white non-key interval (if such interval exists), or the difference between a key border segment and maximal white non-key border segment (otherwise)

**else if** *there is exactly one white interval and its length is $l$* **then**

**(2)** place a point in a bichromatic key interval at distance $< K/N - l$ from its white endpoint

**else if** *there is a non-border white interval* **then**

**(3)** place a point in a non-border white interval

**else**

**(4)** place a point in a non-border bichromatic key interval at distance $< K/N - l$ from the white endpoint, where $l$ is length of a maximal white border interval

---

Suppose that the game ended in the first stage. This means that $B$ took all $N$ key positions. By Lemma 1, the number of white and black intervals is the same after the game is over and every black interval must be larger than white interval (as all black intervals are key intervals). Moreover any white border segment must be smaller than black border segment and if there is no white interval, there must be a white border segment, as $W$ must have placed a point within a black border key interval. Thus if the game ended in the first stage, $B$ must be a winner.

Now suppose that the game ended after the first stage, which means that $W$ took at least one border key position. Observe that option (b) is always applicable. Clearly, if one of cases (1) or (2) holds, then the corresponding moves by $B$ are possible. If neither of above cases hold, then all white intervals must be border ones with neither border segment being a key segment. Moreover at least one such interval exists, as $W$ has at least one point more on the line-segments.

Suppose that case (3) holds, meaning that there is no white border key interval and each line-segment contains at most one white interval, and this interval is a border one. Then, by Lemma 3, there exists a non-border bichromatic key interval, and so the corresponding $B$'s move is applicable (notice that since there are $< N$ black points placed on the line-segments, so there must be a line-segment with $< N/K$ black points).

Obviously if case (4) holds, then the corresponding move is applicable, as there is at least one white border key interval.

Before showing that all moves in stage (c) are applicable and make $B$ win the game, we will show three claims, crucial for this part of the proof.

**Claim 5** *Assume that $W$ took $M$ border key intervals in stage (a). Then $W$ must be $\geq M+1$ points ahead before $B$'s first move after stage (a).*

**Proof.** Notice that every time $W$ takes a white border key interval, he either places two points to take both border key positions or he places one point (per border key interval) to take one remaining empty key position. In the second case, he must be at least two points ahead (per border key interval) before $B$'s move in the previous round, as otherwise $B$ would have the remaining free border key interval in his move in the previous round. Thus $W$ is $\geq M+1$ points ahead before any $B$'s move at any round $r$ during stage (a) and before $B$'s first move after this stage. Above argument is valid only for $N > K$, so that two points are needed to take a border key interval. ■

**Claim 6** *If $M(r) > 0$ at the beginning of stage (c), then $B$ must have exactly $M(r)+1$ points left before his move at this stage.*

**Proof.** By Claim 5, $W$ is at least $M+1$ points ahead after stage (a), which means that $B$ has at least $M+1$ points after stage (a). Hence, if no point is placed in stage (b), then the claim is true. Suppose that at least one point was placed in stage (b). Player $B$ places one or two points in each round $r$ of this stage and before he does this, he has $> M(r)+1$ points left. Obviously, if he places exactly one point, then he will have $\geq M(r)+1$ points left after his move. Assume that he places two points. This means that case (4) is applied and the two points are placed within a white border key interval. Thus the number of white border key intervals falls by 1 and the number of points $B$ has left falls by 2, and so after his move $B$ has $\geq M(r)+1$ points left at the beginning of the next round. ■

**Claim 7** *Suppose that there exists a white non-border key interval at the beginning of stage (c). Then at the beginning of stage (c), there must exist a white border key interval, all black intervals are key intervals and either*

*(i) there exists a white non-key interval or*
*(ii) there exists a white non-key border segment and a black key border segment.*

**Proof.** Assume that $B$ took $Q$ key positions in stage (a). Rules of the game and strategy $Y_L^*$ guarantee that $Q > 0$.

Let $M$ denote the number of white border key intervals created in stage (a). Assume first that $M = 0$. If no white key interval was created in stage (a), then the claim is trivially satisfied. Hence assume that there is at least one white key interval created in stage (a). Then there must exist a bichromatic key interval (as otherwise there would have to exist at least one white border key interval). This means that there are $\leq N-Q-1$ white non-border key intervals. Since $B$ has $N-Q$ points left after stage (a) and $N-Q > 1$ (as there is at least one white key interval and taking it requires two white points, if $N > K$), so $B$ will place at least one point in stage (b), starting with breaking white key intervals and he has enough points to break them all, as he will place $N-Q-1$ points in this stage. This shows that if there are no white border key intervals after stage (a), then there cannot be a white non-border key interval at the beginning of stage (c).

Suppose now that there exists a non-border white key interval before $B$'s move at the beginning of stage (c). This means that the only option $B$ could apply throughout stage (b) was (b)(1) (as he would try to break white non-border key intervals first). It implies, in particular, that the number of white border key intervals will not change during stage (b) and will be $M$ at the beginning of stage (c). By what we have shown above, this implies that $M > 0$, so there must exist a white border key interval then.

Moreover, since $B$ never created a black interval in stage (b), as he was only breaking white key intervals in this stage, so all black intervals must be key intervals at the beginning of stage (c).

Since there are $\leq N-Q$ white key intervals created in stage (a) and $M$ of them are border key intervals, so there are $\leq N-Q-M$ white non-border key intervals created in stage (c). If $B$ did not place any point in stage (b), then he must have $\geq M+1$ points left after stage (a), that is $N-Q \geq M+1$. If $B$ placed at least one point in stage (b), then he broke $N-Q-(M+1)$ white non-border key intervals during this stage, which means that there are $\leq N-Q-M-(N-Q-(M+1)) = 1$ white non-border key intervals at the beginning of stage (c).

For the last part of the proof, notice first that if there is a bichromatic key interval after stage (a), then the number of white non-border key intervals is $\leq N-Q-1-M$. Hence $N-Q > M+1$, so $B$ will place at least one point in stage (b), and he will break all white non-border key intervals in this stage, as he can break $N-Q-(M+1)$ of them.

Now assume that there is a white non-border key interval at the beginning of stage (c). As we have shown above, this implies that $M(r) = M > 0$ at the beginning of this stage and that $B$ could only be breaking non-border white key intervals throughout stage (b). Also, there cannot exist a bichromatic key interval, and so there are $Q$ black and $N-Q$ white key intervals created in stage (a).

If there is a white non-key interval at the beginning of stage (c), then our claim holds. If there is no such interval at the beginning of stage (c), then all $Q > 0$ points $W$ placed in stage (b) must have been placed within $Q$ black key intervals, each point in a different one (notice that the only black intervals are key intervals as $B$ could not create any other black interval in stage (b)).

Since there were no bichromatic key intervals created in stage (a) and $B$ was taking border key positions first, so there is at least one black border key interval. Hence one of white points must be placed there, and so there must exist a white non-key border segment and a black key border segment after all white points are placed. ■

Now we will show that option (c) of Strategy $Y_L^*$ is always applicable during stage (c) and that $B$ will win the game after this stage. Suppose that case (1) holds. This means that there are $M(r) > 0$ white border key intervals. By Claim 6, $B$ has exactly $M(r)+1$ points left in this case. Suppose that there are $Q$ white intervals before $B$'s move. Since $B$ has $M(r)+1$ points left, so, by Lemma 1, there exist $Q \geq M(r)+1$ white and $Q-M(r)-1 \geq 0$ black intervals.

Two cases are possible at this stage: (i) there exist a white non-border key interval or (ii) all white key intervals are border intervals. If case (i) holds, then, by Claim 7, all black intervals are key intervals and either there exist a non-key white interval or a non-key white border segment and a key black border segment. Hence the move corresponding to case (1) of option (c) is applicable and $B$ places $M(r)+1$ points in different white key border segments, breaking all white border key intervals and creating one black border interval. Now the area of all old $Q - M(r) - 1$ black intervals is not smaller than the area of a maximal of $Q - M(r) - 1$ of remaining white intervals and $B$ is losing on $M(r)$ bichromatic border intervals with white key border segment by $< M(r)\delta/(M(r)+2)$, where $\delta$ is as defined in Strategy $Y_L^*$.

If there exists a white non-key interval, then the newly created black interval is larger than it by $> \delta - 2\delta/(M(r)+2)$, and it is easy to check that the difference between the area controlled by $B$ and the area controlled by $W$ is $> 0$. Hence $B$ wins in this case.

If there is no white non-key interval then the remaining one white interval is a key interval and is larger than the newly created black interval by $< 2\delta/(M(r)+2)$, while $B$ has black key border segment bigger than some white non-key border segment by $\delta$. Again, it is easy to check that difference between area controlled by $B$ and area controlled by $W$ is $> 0$, and so $B$ wins in this case as well.

Now suppose that case (ii) holds. Then all of $Q - M(r)$ white intervals which are not border key intervals are non-key intervals. Strategy $Y_L^*$ guarantees that for any black interval there exists a unique smaller white interval and so only one of these $Q - M(r)$ white intervals can be bigger than all black intervals. Moreover, by Lemma 1, $Q - M(r) \geq 1$, as $B$ has $M(r)+1$ points left and so the move corresponding to case (1) of option (c) is applicable. As in case (i), $B$ places $M(r)+1$ points in different white key border segments, breaking all white border key intervals and creating one black border interval larger than a maximal of white intervals by $> \delta - 2\delta/(M(r)+2)$. On the other hand, $B$ is losing on $M(r)$ bichromatic border intervals with a white key border segment by $< M(r)\delta/(M(r)+2)$. It is easy to check that the difference between the area controlled by $B$ and area controlled by $W$ is $> 0$. Hence $B$ wins the game.

Observe that, by Claim 7, all white intervals must be non-key intervals, if there are no white border key intervals. Now suppose that case (2) holds. Then, by Lemma 3, there exists a bichromatic key interval and so $B$'s move is applicable and $B$ wins. This is because, by similar arguments to those used for case (ii) above, after $B$ places his last point, for each black interval there exists a unique smaller white interval, all black border segments being a part of a bichromatic border interval are larger than their opposite white border segments and there is at least one black interval at the end of the game.

If case (3) holds, then there are at least two white intervals and no white key interval. Thus the corresponding move is applicable and, by the same arguments as those used above, $B$ wins the game.

Lastly, suppose that case (4) holds. Then, by Lemma 3, the corresponding move of $B$ is applicable and, by similar arguments to those used for case (2),

$B$ wins. ■

**Remark 8 (One-by-one variant of the game)** *Notice that Strategy $Y_L^*$ is a valid winning strategy for $B$ even for one-by-one variant of the game, where each player places exactly one point at each round. This is because it never asks $B$ to place more points than his opponent did in the same round, throughout stages (a) and (b). Moreover if $W$ places exactly one point at each round, then $B$ will have to place exactly one point in the last stage (c).*

In the case of $N = K$ the situation changes and the first mover advantage appears for $K > 2$. As we show below, the following Strategy $Y_L'$ is a winning strategy for $W$ in this case. Key positions are points in the middle of line-segments. We will refer to them as *middle positions* in this case (and we will call points placed at this positions *middle points*), for reasons that will become clear when the case $N = 2K - 1$ will be considered.

Player $W$ places one or two points at each round. Two points may be placed only once in the whole game and only if $2\,|\,K$.

---

**Strategy $Y_L'$**

---

**if** *there is an empty middle position left* **then**

(a) **if** $2\,|\,K$, *there are exactly* 2 *middle positions left and all opponent's points are middle points* **then**
  place two points taking both the remaining middle positions
 **else**
  place a point on a middle position

**else**

(b) **if** *there exists a non-border interval of the opponent* **then**

(1)  place a point inside a maximal non-border interval of the opponent
 **else**

(2)  place a point in a border segment of the opponent of size $1/2$, at a distance $< 1/2 - l$ from opponent's point, where $l$ is length of maximal border segment of the opponent smaller than $1/2$

---

**Theorem 9** *Let $\left\langle N, \{L_j\}_{j=1}^{K} \right\rangle$ define a game on a family of disjoint line-segments, with $N = K > 2$. Then $Y_L'$ is a winning strategy for $W$.*

**Proof.** The game played by a player using Strategy $Y_L'$ can be divided into two subsequent stages: (a) taking middle positions – the player plays option (a) and (b) breaking intervals of the opponent – the player plays option (b). Notice that moves prescribed for $W$ in stage (a) are valid only for $K > 2$.

Suppose first that $W$ took all middle positions, and so the game ended in stage (a). Then obviously he has to be a winner, since all intervals he created are larger than intervals of the opponent (as length of these intervals is 1), all border segments he created are larger than border segments of the opponent (as length of all these border segments is 1/2) and there must be at least one empty white border segment of length 1/2 and at least one black border segment of length $< 1/2$ at the end of the game, as number of border segments of length 1/2 is $2N > N$ for $N > 0$.

Now suppose that there is at least one black middle point after stage (a). Let $Q$ be the number of white middle points. Then $N - Q > 0$ is the number of black middle points. Notice that Strategy $Y'_L$ guarantees that $Q > N - Q$. Notice also that there will always be an empty black segment of length 1/2 throughout stage (b). This is because number of rounds in stage (b) is $N - Q$ and there are $2(N - Q)$ black border segments of length 1/2 after last move of $W$ in stage (a) (recall that $N - Q > 0$). To show that all $W$'s moves in stage (b) are applicable, we need to show that in case (2) of option (b) there must always be a black border segment of length $< 1/2$.

If $B$ places at least one point not in a middle position in stage (a), then obviously after this stage there is at least one black border segment of length $< 1/2$. If $B$ places his points in middle positions only, then the strategy guarantees that $W$ is the last one to take a middle position in stage (a) (recall that $K > 2$). Hence $B$ will have to create a black border segment of length $< 1/2$ in his subsequent move. This shows that there will be a black border segment of length $< 1/2$ at $W$'s first move in stage (b). Since throughout this stage $W$ places his points in non-border black intervals or in black border segments of length 1/2, so there will always exist at least one black non-key border segment throughout this stage.

Observe also that whenever $B$ does not create a new border segment of length $< 1/2$ in stage (b), he must create a black non-border interval, so $W$ will play according to case (1) of option (b) in his next move. This shows that whenever $W$ plays according to case (2) of option (b), there is a unique black border segment of length $< 1/2$ smaller than a white border segment created by $W$'s move. Since the strategy guarantees that there are more white border segments of length 1/2 than black border segments of length 1/2 after the game, and for any white border segment of length $< 1/2$ there exists a unique smaller black border segment, so $W$ must win the game. ■

Strategy $Y'_L$ is not valid for one-by-one variant of the game, as it may require $W$ to place two points in cases where $2 \,|\, K$. Indeed, as we show below, in this variant of the game Strategy $Y'_L$ is a winning strategy for $B$, if $2 \,|\, K$. If $2 \nmid K$, then Strategy $Y'_L$ is a valid strategy for $W$ even in one-by-one variant of the game and, as shown above, it is a winning strategy.

**Theorem 10** *Let* $\left\langle N, \{L_j\}_{j=1}^{K} \right\rangle$ *define a game on the family of disjoint line-*

*segments with $N = K$ and assume that players place exactly one point at a time. If $2 | K$, then $Y'_L$ is a winning strategy for $B$.*

**Proof.** The proof is very similar to proof of Theorem 9 and most of the arguments are analogical to those used there. The only difference, which is crucial to the result, is that now with $2 \mid K$, player $B$ will be the last one to take a middle position in stage (a) (if $W$ does not play outside middle positions in this stage). Thus there will be a white border segment of length $< 1/2$ at the beginning of stage (b). ■

**Corollary 11** *Let $\left\langle N, \{L_j\}_{j=1}^{K} \right\rangle$ define a game on a family of disjoint line-segments, with $N = K = 2$. Then $Y'_L$ is a winning strategy for $B$.*

**Proof.** For $N = 2$ the restrictions of the game lead effectively to a one-by-one game, so Theorem 10 applies. ■

**Remark 12 (One round game)** *Notice that if $W$ could place all $N$ points in the first round, then $W$ could win by taking all key positions in his first move. As it is shown in proof of Theorem 4, a player who manages to control all key positions must be a winner.*

### *3.2 Case of $K \nmid N$*

The case of $K \nmid N$ is arguably more difficult than the case of $K \mid N$, as it is impossible to define key positions uniformly for all line-segments, so that there are $N$ key positions in total. It turns out that an extension of Strategy $Y^*_L$ with new cases, that can appear only when $K \nmid N$, is a winning strategy for $B$, if $N > 2K$. Before introducing the aforementioned extension, we will define a notion of advantage a player can have, which will be important in the definition of the extended strategy.

**Definition 13** *Let $\{L_j\}_{j=1}^{K}$ be a family of line-segments with white and black points placed on them. We say that player $p$ has an advantage of size $\alpha_\sigma$ if for any interval $I$ of $p$, there exists a unique interval $\sigma(I)$ of the opponent such that*

$$\sum_{I \in \mathcal{I}^p} (I - \sigma(I)) + \beta = \alpha_\sigma > 0,$$

*where $\mathcal{I}^p$ denotes the set of all intervals of $p$ and $\beta$ is the difference between $p$'s and opponent's border segments of bichromatic border intervals. The assignment $\sigma$ is called* an advantageous assignment.

Consider the following extension of Strategy $Y_L^*$, which covers two new cases, one for option (b) and one for option (c), that can appear if $K \nmid N$. Key positions are determined with respect to $\lfloor N/K \rfloor$.

**Extended strategy $Y_L^*$**

**if** *there is an empty key position left* **then**

**(a)** $\vdots$

**else if** $B$ *has* $> M(r) + 1$ *points left* **then**

**(b)** $\vdots$

**else if** $M(r) = 0$ *and there exists a bichromatic key interval* **then**

**(3)** place a point in a non-border bichromatic key interval within a distance of $< K/N - l$ from its white end point, where $l$ is length of a maximal white border interval

**else if** $M(r) > 0$ **then**

**(4)** place two points in a white border key interval creating a balanced black border interval of length $> l$, where $l$ is length of a maximal white non-key interval

**else**

**(5)** place a point in a white border interval to keep advantage

**else**

**(c)** $\vdots$

**else if** *there is a bichromatic key interval* **then**

**(4)** place a point in a non-border bichromatic key interval at a distance $< K/N - l$ from the white endpoint, where $l$ is length of a maximal white border interval

**else**

**(5)** place a point in a white border interval to keep advantage

---

Notice that, as we proved above, if $K \mid N$, then none of these new cases would be played by $B$. They may be used only when $K \nmid N$. We will refer to extended version of Strategy $Y_L^*$ as $Y_L^*$.

**Theorem 14** *Let $\left\langle N, \{L_j\}_{j=1}^{K} \right\rangle$ define a game on a family of disjoint line-segments, with $N \geq 2K$. Then $Y_L^*$ is a winning strategy for $B$.*

**Proof.** The result holds for the case $K \mid N$, as stated in Theorem 4. Now assume that that $K \nmid N$. Like before, the game played when $B$ uses Strategy $Y_L^*$ can be divided into three subsequent stages (a), (b) and (c). Unlike in the case of $K \mid N$, the game cannot end in the first stage and $W$ can take all key positions.

First we would like to note that this new setting affects neither proof of Claim 5 nor Claim 6 and these claims hold for extended Strategy $Y_L^*$ as well.

Proof of Claim 7 used the fact that there is at least one black key point, which does not have to be true in case of $K \nmid N$. Nevertheless even a stronger version of the claim holds in this case, as stated below.

**Claim 15** *There is no empty white non-border key interval at the beginning of stage (c) and hence there exists a white non-key interval then.*

**Proof.** Assume that $B$ took $Q$ key positions in stage (a) ($Q \geq 0$ this time). Let $P$ denote the number of all key positions ($2K \leq P < N$). Then there are $\leq P - Q$ white key intervals at the beginning of stage (b).

Let $M$ denote the number of white border key intervals created in stage (a). Then there are $\leq P - Q - M$ white non-border key intervals after this stage. Suppose that $P - Q - M > 0$, then $P - Q \geq M + 1$ and, since $N > P$, $N - Q > M + 1$. This means that $B$ will place at least one point in stage (b) and he can break all white non-border intervals during this stage, as the number of points he can place in stage (b) is $N - Q - (M + 1) \geq P - Q - M$. Thus there can be no white non-border key intervals at the beginning of stage (c).

Since $W$ is $M(r)+1$ points ahead at the beginning of stage (c), so, by Lemma 1, there are $\geq M(r) + 1$ white intervals. Because $M(r)$ is the number of white border key intervals and these are the only white key intervals, so there must be at least one white interval which is not a key interval. ■

Observe that if $B$ ends the game by playing according to one of cases (c)(1)–(4), then he must win the game. This is because there did exist either a white border key interval or a bichromatic key interval throughout stage (b) and at the beginning of stage (c), and so case (b)(5) was never played. Hence the argumentation used in proof of Theorem 4 is valid here (as we shown above, Claim 6 still holds and Claim 7 is implied by stronger Claim 15).

Suppose then, that $B$ played according to case (c)(5) in the last round. This means that it was also possible that he played according to case (c)(5) throughout stage (b). If all these moves were applicable, then he will obviously win, as he will have an advantage and there is the same number of white and black intervals after the game. Hence it is enough to show that $B$'s moves in cases (b)(5) and (c)(5) are always applicable. The following two claims are crucial for this to hold.

**Claim 16** *Let $r$ be a round at which B plays according to case (b)(5) or (c)(5). Then either there exists a black interval or an unbalanced bichromatic border interval with black key border segment, before $B$'s move at round $r$.*

**Proof.** Suppose either (b)(5) or (c)(5) is to be played at some round $r$. This means that there are no white key intervals, no bichromatic key intervals and each line-segment contains at most one white interval, which is a border interval.

By Lemma 3, each line-segment must contain $\geq \lfloor N/K \rfloor$ black points. Since $N < K\lceil N/K \rceil$ (as $K \nmid N$), so there must be a line-segment $L$ with $w^L \leq \lfloor N/K \rfloor$ white points. Thus it must be, that $w^L \leq b^L$ on this segment. If $b^L > w^L$, then by Lemma 1, there must exist a black interval on $L$.

Assume that $b^L = w^L = \lfloor N/K \rfloor$. Notice that in this case it is impossible that all key points on $L$ are white. This is because according to Strategy $Y_L^*$, $B$ never places one point within a white border key interval, so this would mean that there is a white key interval on $L$, which is impossible. Assume then that all key points are black on $L$. Then either there is a white and a black interval on $L$, or all intervals are bichromatic. The second case means that there is an unbalanced bichromatic border interval with a black key border segment.

Lastly, assume that there exist black and white key points. If there is a white interval on $L$ then, by Lemma 1, there must be a black interval on $L$ as well. Otherwise, all intervals are bichromatic and each key interval contains exactly one point. This means that the border key interval must be black and must contain a white point. This is because according to $Y_L^*$, $B$ never places one point within white key interval, and the only case when he can place one point within a bichromatic border key interval is (c)(2). If there is one white point within a black border key interval, then there is an unbalanced bichromatic interval with a black key border segment. ∎

**Claim 17** *Let $r_1$ and $r_2$ be subsequent rounds at which $B$ is to play according to case (b)(5) or (c)(5). Then*

$$\left(\mathcal{I}_{r_1}^B \cup \mathcal{KI}_{r_1}^{BW}\right) \cap \left(\mathcal{I}_{r_2}^B \cup \mathcal{KI}_{r_2}^{BW}\right) \neq \varnothing,$$

*where $\mathcal{I}_r^B$ denotes the set of black intervals before $B$'s move at round $r$ and $\mathcal{KI}_r^{BW}$ denotes the set of unbalanced bichromatic border intervals with black key border segment, before $B$'s move at round $r$.*

**Proof.** Notice that after playing according to case (b)(5) at round $r_1$, player $B$ can play according to (b)(1) only, at any round between $r_1$ and $r_2$ (because cases (b)(2)–(4) cannot reappear at this stage of the game).

Thus he is never creating any black intervals or any black key border segments between these rounds. By Claim 16 we know that one of these must remain between rounds $r_1$ and $r_2$, which completes the prove. ∎

Now we will show how $B$ can maintain an advantage when playing according to case (b)(5) or (c)(5) at round $r$. By Claim 16, there is either a black interval or an unbalanced bichromatic border interval with a black key border segment before $B$'s move at round $r$. Strategy $Y_L^*$ guarantees that whenever a black interval is created, there exists a unique white interval not larger than it. Moreover, equal sizes of these intervals are possible in case of key intervals only, and since there are no white key intervals at this point of the game, it must be that for any black interval there exists a strictly smaller white interval. Hence there must exist an advantageous assignment for $B$.

Let $\sigma^*$ be an advantageous assignment that maximizes the advantage and let

$$a(r) = \min\left(\left\{I - \sigma^*(I) : I \in \mathcal{I}_r^B\right\} \cup \left\{\delta^B(I) : I \in \mathcal{KI}_r^{BW}\right\}\right),$$

where $\delta^B(I)$ is the difference between black and white border segments of bichromatic interval $I$. Observe also that since $W$ is $\geq 1$ points ahead before $B$'s move, so, by Lemma 1, there is at least one more white interval than black intervals.

If $B$ places his point in a white border interval which is not assigned to any black interval by $\sigma^*$ and creates a black border segment smaller than the opposite white border segment by $< a(r)/K$, then $B$ will maintain the advantage. Moreover, if at some round $r' > r$ he is to play according to case (b)(5) or (c)(5) again, then he will still be able to maintain some advantage, as Claim 17 guarantees that $a(r') \geq a(r)$ and there cannot be more than $K$ white border segments with black point placed by playing according to case (b)(5) or (c)(5). ∎

**Remark 18 (One round game)** *Notice that if $K \nmid N$, then the proof above works even if $W$ places all his points in the first round. Hence, in this case, Strategy $Y_L^*$ is a valid winning strategy for $B$ even in a one round game.*

**Remark 19 (One-by-one variant of the game)** *Extended Strategy $Y_L^*$ is a valid winning strategy for $B$ even for the one-by-one variant of the game, just like its version for $K \mid N$ case.*

In the case of $K < N < 2K$ the result depends on $K$. We start by analysing case of $K < N \leq 2K - 2$, for which the following Strategy $Y_L'$, which is an extension of strategy $Y_L'$ used for $N = K$, is a winning strategy for $W$. Like in case of $N = K$, key positions are middle positions of line segments and, like before, we will refer to them as middle positions. Player $W$ places either one or two points at each round.

**Theorem 20** *Let $\left\langle N, \{L_j\}_{j=1}^K \right\rangle$ define a game on a family of disjoint line-segments, with $K < N \leq 2K - 2$. Then $Y_L'$ is a winning strategy for $W$.*

**Proof.** Like in case of $N = K$, the game played by a player using extended Strategy $Y_L'$ can be divided into two subsequent stages: (a) taking middle positions (player $W$ plays option (a)) and (b) breaking intervals of the opponent (player $W$ plays option (b)).

**Extended strategy $Y'_L$**

**if** *there is an empty middle position left* **then**

(a) ⋮

**else**

(b) ⋮

**else if** *there are two border segments of the opponent with different lengths* **then**

(2) place a point in a maximal border segment of the opponent at distance $< (l - l')/4$ from opponent's point, where $l$ is length of maximal border segment of the opponent and $l'$ is length of maximal border segments of the opponent smaller than $l$

**else**

(3) place a point in a border segment of the opponent at distance $(1 - 2l)/(4(K - 1))$ from opponent's endpoint, where $l$ is length of border segment of the opponent

If $W$ never plays according to case (b)(3), then clearly he will win the game. Arguments here are similar to those used in proof of Theorem 9 – for any white border segment there will be a unique smaller black border segment and there are no white non-border intervals. Like in proof of Theorem 9 it is crucial here that at the beginning of stage (b) there exists at least one black border segment of length $< 1/2$.

Throughout the rest of proof we will use $m$ to denote the size of a minimal black border segment, when $W$ was placing his first point not in the middle of a line-segment. As argued in proof of Theorem 9, $m < 1/2$.

Since $N \leq 2K - 2$, so at the end of the game there must exist either one line-segment without black points (and hence with only one white point in the middle) or two line-segments with only one black point in each (and hence with at most two white points). Notice that if case (b)(3) is applied at least once during the game, then it must be that each line segment with only one black point either contains two white points (if the black point is in the middle) or a white border segment of size 1/2 (if the black point is not in the middle). This is because, if case (b)(3) holds, than all black border segments must be of length $\leq m$.

This means that when the game is over and case (b)(3) was applied at least once, there will be either $\geq 2$ white border segments of length 1/2, or one white border segment of length 1/2 and $\geq 2$ white border segments of length $> 1/2 - (1/2 - m)/4 = 3/8 + m/4$, or $\geq 4$ white border segments of length $> 3/8 + m/4$.

Suppose that $2 \nmid K$. In this case both players place exactly one point at each round. Assume that at some round $r$, $W$ plays according to case (b)(3), creating a white border segment of length $l - \alpha$, where $l$ is length of black border segment. If in his move at round $r$ player $B$ creates a black border segment of size $> l$ and $r$ is not the last round, then in the next round player $W$ will place his point in this black border segment. If player $B$ creates a black border segment if size $l$ and $r$ is not the last round, then in next round player $W$ will have to play according to case (b)(3) again. If $r$ is the last round than $B$ can create a black border segment of length $\leq 1/2 - \varepsilon$, where $\varepsilon > 0$ and may be arbitrarily small (recall that maximal white border segment may have length $\leq 1/2$).

Notice that if player $B$ maximizes his outcome, than at the end of the game there will be no black non-border intervals. This is because this black border interval would have to be created in the last round and would have to be created by placing a point in a black border segment. It is more profitable for $B$ to place a point in a white border segment in the last round, than in a black one.

Hence there must be $K$ black and $K$ white border segments after the game is over (by Lemma 1, the numbers of white and black intervals must be equal). Now three cases are possible: (i) there are two white border segments of size $1/2$, or (ii) there is one white border segment of size $1/2$ and $\geq 2$ white border segments of size $> 3/8 + m/4$, or (iii) there are $\geq 4$ white border segments of size $> 3/8 + m/4$. In all cases (i) – (iii), the remaining white border segments could have been created by applying case (b)(3), and so their length is $\geq l - \alpha$, where $\alpha$ is the distance from the black end of a border segment, at which white point is placed to create a white border segment and $l$ is length of maximal black border segment before $W$'s move in the last round. The length of each of the black border segments (apart from the one created in the last round) must be $\leq l$, and it is easy to check that in all cases (i) – (iii), $\alpha = (1-2m)/(4(K-1))$ makes the difference between area controlled by $W$ and the area controlled by $B$ positive, so that $W$ wins.

Now Suppose that $2 \mid K$. It is possible in this case, that player $W$ places two points at some round, and so $B$ have to place two points at some round as well. If $W$ placed exactly one point each round, then the same argumentation as this used above shows that he wins. Assume then, that $W$ placed two points at the last round of stage (a) to take two remaining middle positions. It is possible then, that $B$ places two points in the last round creating two black border segments of length $\leq 1/2 - \varepsilon$, where $\varepsilon > 0$ and may be arbitrarily small. If the game ends with at most one such black border segment, then the same arguments as those used for case $2 \nmid K$ show that $W$ wins.

Hence assume that $B$ created two black border segments of size $\le 1/2 - \varepsilon$. Then after stage (a) there are $K/2+1$ white middle points and $K/2-1$ black middle points. Now remaining $\le 2K-2-(K/2-1) = 3K/2-1$ black points are distributed over line-segments in such way that two of them are placed in white border segments of size $1/2$. Suppose that at the end of the game there are no white border segments of length $1/2$. This means that each line-segment with white a middle point contains $\ge 2$ black points and there are $\le 3K/2-1-2(K/2+1) = K/2-3$ black points that are placed in remaining $K/2-1$ line-segments (notice that this situation is possible only if $K/2-3 \ge 0$, that is when $K \ge 6$). This means that there are $\ge 4$ white border segments of length $> 3/8 + m/4$ created in black border segments of size $1/2$ at the end of the game. The remaining $K-4$ white border segments are of size $\ge l-\alpha$, where $\alpha$ is the distance from the black end of a border segment in which white border segment is created and $l$ is the length of a maximal black border segment before $W$'s move in the last round. Length of each of the black border segments (apart from the two created in the last round) must be $\le l$ and it is easy to check that $\alpha = (1-2m)/(4(K-1))$ makes the difference between area controlled by $W$ and area controlled by $B$ positive, and $W$ must be the winner.

The above case was possible only for $K \ge 6$. The remaining case to consider is $K=4$ (the case of $K=2$ is considered already, as $2K-2=K$ for $K=2$ and so it is the situation where $N=K$). If $K=4$ then, as shown above, it is impossible that there is no white border segment of length $1/2$ at the end of the game. Suppose then, that there is at least one such white border segment. Then there are $\le 3K/2-1-(2(K/2+1)-1) = K/2-2 = 0$ black points to be placed elsewhere. Hence, in the worst case, $W$ has one border segment of length $1/2$, two border segments of length $> 3/8+m/4$ and remaining $K-3$ white border segments have length $\ge l-\alpha$. Again, it is easy to check that $\alpha = (1-2m)/(4(K-1))$ makes the difference between area controlled by $W$ and area controlled by $B$ positive, and $W$ must be the winner. ■

Like in case of Strategy $Y'_L$ for $N=K$, the extended Strategy $Y'_L$ is not valid for one-by-one variant of the game, as it may require $W$ to place two points in cases where $2\,|\,K$. Similar to the case $N=K$, in this variant of the game, Strategy $Y'_L$ is a winning strategy for $B$, if $2\,|\,K$. If $2 \nmid K$, then the extended Strategy $Y'_L$ is a valid winning strategy for $W$ even in the one-by-one variant of the game.

**Theorem 21** *Let $\left\langle N, \{L_j\}_{j=1}^{K} \right\rangle$ define a game on the family of disjoint line-segments with $K < N \le 2(K-1)$ and assume that players place exactly one point at a time. If $2\,|\,K$, then $Y'_L$ is a winning strategy for $B$.*

**Proof.** The proof is very similar to proof of Theorem 20 and most of the arguments are analogical to those used there. The crucial point now is that in case of the one-by-one game with $2 \mid K$, player $B$ will be the last one to take a middle position in stage (a) (if $W$ does not play outside middle positions in this stage). Thus there will be a white border segment of length $< 1/2$ at the beginning of stage (b).

Let $m$ denote the size of a minimal white border segment, when $B$ is placing his first point not in the middle of a line-segment. As pointed out above, $m < 1/2$.

If case (b)(3) is never applied during the game then $B$ wins, as argued in proof of Theorem 20. Otherwise one of the following three cases must hold at the end of the game (cf. proof of Theorem 20): (i) there are two black border segments of size 1/2, or (ii) there is one black border segment of size 1/2 and $\geq 2$ black border segments of length $> 3/8 + m/4$, or (iii) there are $\geq 4$ black border segments of length $> 3/8 + m/4$.

Strategy $Y'_L$ guarantees that at the end of the game there will be no white non-border intervals and there must be $K$ black and $K$ white border segments after the game is over (by Lemma 1). The size of white border segments must be $\leq m$ and, unlike in the case of $W$ using Strategy $Y'_L$ in the unrestricted variant of the game, if $W$ creates a white border segment larger than other white border segments, then $B$ will place a point in this segment in his move, so the opponent cannot create a larger border segment in the last round. Hence the size of all white border segments must be $\leq m$.

In all cases (i) – (iii), the remaining black border segments could have been created by applying case (b)(3), so their length is $\leq l - \alpha$, where $\alpha$ is a distance between the white end of a border segment and a black end of the black border segment created in it, while $l$ is length of maximal white border segment before $B$'s move in the last round. Length of each of the white border segments must be $\leq l$ and it is easy to check that $\alpha = (1 - 2m)/(4(K - 1))$ makes difference between area controlled by $B$ and area controlled by $W$ positive, so that $B$ must be the winner. ■

For the last case of $N = 2K - 1$, we will show that there is a winning strategy for $B$. The strategy we present below combines Strategies $Y'_L$ and $Y^*_L$ with some additional moves, covering new cases that may arise in this setting. Again, notion of key position and key point is used and key positions are determined with respect to $\lceil N/K \rceil = 2$. Notions of middle positions and middle points are also important as $B$ may switch to using Strategy $Y'_L$ during the course of the game.

**Theorem 22** *Let $\left\langle N, \{L_j\}_{j=1}^{K} \right\rangle$ define a game on a family of disjoint line-segments, with $N = 2K - 1$, $K \geq 2$. Then $Y'^*_L$ is a winning strategy for $B$.*

**Strategy $Y_L'^*$**

**if** *W placed exactly one point taking a key position in the first round* **then**

(1) **if** $r = 1$ **then**

  place a point in the middle of the line-segment with white point

 **else**

  play according to Strategy $Y_L'$

**else if** *there exists a white key point and two black key points in different line-segments* **then**

(2) play according to Strategy $Y_L^*$

**else if** *$W$ took all $2(K-1)$ key positions on $K-1$ line-segments* **then**

(3) **if** $r = 1$ **then**

  place a point in the middle of a non-border white interval

 **else**

  place a point in each empty white non-border interval, fix some positive $\delta < 1/4$

  **if** *there exists an empty line-segment* **then**

   place a point in an empty line-segment

  **else if** *there exists a line-segment with only one, white, point* **then**

   place two points in a line-segment with one white point, on its both sides at a distance $\delta$ from it

  place each of remaining points in a different white border key interval at a distance $< (1-4\delta)/(2(K-2))$ from white point

**else if** *there exists a white key point* **then**

 **if** $r = 1$ **then**

(4.1)  take two key positions in different line-segments, taking line-segment with one white key point first

 **else**

(4.2)  take a key position in a line-segment without black key point, taking line-segment with one white key point first

**else if** *$B$ has $> 1$ points left* **then**

(5) place a point in a key position, taking empty line-segments first

**else**

(6) place a point in a maximal white non-border interval, maximal bichromatic interval or maximal white border segment, depending on which one of these moves gives largest area; if the point is placed within bichromatic interval or white border segment, then place it close enough to white point to win the game

---

**Proof.** Strategy $Y_L'^*$ consists of two main options that may be taken by player $B$ at the beginning of the game. These options are recognized in the first round, after player $W$ placed his points. The options are as follows: (i) player $W$ placed exactly one point, taking a key position (case (1)) and (ii) player

$W$ either placed $\geq 2$ points or did not take a key position in his first move.

If option (i) is taken, then player $B$ continues his play using strategy $Y'_L$ and $B$'s strategy concentrates around notion of middle positions. As we observed in previous proofs, the game played by a player using Strategy $Y'_L$ can be divided into two subsequent stages: (a) taking middle positions – a player plays option (a) and (b) breaking intervals of the opponent – a player plays option (b).

If $B$ never plays according to case (b)(3), then clearly he will win the game. Arguments here are similar to those used in proof of Theorem 9: for any black border segment there will be a unique smaller white border segment and there are no non-border black intervals. It is crucial here, that at the beginning of stage (b) there exists at least one white border segment of length $< 1/2$, the one created in the first round (recall that $W$ took a key position determined by 2, which is not a middle position).

If $B$ plays according to case (b)(3) at least once during the game, then there must exist a black border segment of length $> 1/2 - (1/2 - 1/4)/4 = 7/16$, at the end of the game. For assume that $B$ took $Q \leq K$ middle points in stage (a) (which means that $W$ took remaining $K - Q \geq 0$ of them). If there were no black border segment of length $> 7/16$ at the end of the game, then $W$ would have to place at least one white point in each of black border segments in line-segments with black point in the middle and at least one white point in each half of each line-segment with one white point in the middle. Such play by $W$ requires $2K$ points and hence is impossible, so there must exist at least one black border segment of length $> 7/16$ at the end of the game. On the other hand all white border segments must have length $\leq 1/4$ (as case (b)(3) was applied at least once during the game, and there exist a white border segment of length $1/4$ after the first round).

Since Strategy $Y'_L$ guarantees that there are no white non-border intervals at the end of the game, so numbers of white and black border segments must be the same then (recall that by Lemma 1 there must be the same number of white and black intervals at the end of the game). Thus there are $K$ black border segments and at most $K - 1$ of them can be smaller than any white border segment. The difference between black border segment of length $> 7/16$ and a minimal white border segment is $> 7/16 - m$, where $m$ is length of minimal white border segment. On the other hand, the difference between remaining white border segments and remaining black border segments is $< (K-1)(1-2m)/(4(K-1)) = 1/4 - m/2$. Thus the difference between area controlled by $B$ and area controlled by $W$ is $> 7/16 - m - (1/4 - m/2) = 3/16 - m/2 \geq 1/16$, as $m \leq 1/4$. Hence $B$ wins the game.

If option (ii) is taken, then $B$'s strategy concentrates around notion of key positions and he starts by placing his points in key positions. Since number of key positions is greater than number of points each player has, so $B$ cannot simply apply Strategy $Y_L^*$ here. As we will show below, this strategy can be safely applied only when at least one key position is taken by player $W$ and $B$ can secure two key positions for himself. Apart from that, two other situations are possible: one where player $W$ takes $2(K-1)$ key positions on $K-1$ line-segments and another, where player $W$ does not take any key positions throughout the game.

Assume first that at some point of the game there was one white key point and two black key points in two different line-segments. The game in this case can be divided in three stages, like it was in previous cases of $B$ using Strategy $Y_L^*$: (a) taking key positions, (b) breaking white intervals – $B$ plays option (b) and (c) the last move – $B$ plays option (c). The game can never end in the first stage, as $B$ has at least two key positions $W$ has at least one.

For validity of Strategy $Y_L^*$, observe that Lemma 3 can be used to show that moves for cases (b)(3), (c)(2) and (c)(4) are applicable, like in proof of Theorem 4 (as there will always exist a line-segment with $< 2$ black points). Applicability of moves for cases (b)(1), (b)(2), (b)(4) and (c)(3) is also easy to see (cf. proof of Theorem 4).

For case (c)(1) we need to show first the following, weaker, replacements for Claims 5 and 6.

**Claim 23** *Assume that $W$ took $M$ border key intervals in stage (a). Then $W$ must be $\geq M$ points ahead before $B$'s first move after stage (a).*

**Proof.** If player $B$ places only one point at each round throughout stage (a), then, by the same arguments as those used in proof of Claim 5, player $W$ must be $\geq M+1$ points ahead before $B$'s first move after stage (a).

Notice that the only round when player $B$ can place more than one point in stage (a) is the first round when case (4.1) of Strategy $Y_L'^*$ is applied. This case is applied after player $W$ places at least two points in the first round, leaving at least two line-segments with empty key positions (as otherwise option (1) would be taken by $B$ or case (3) would applied). Since player $B$ places two points in this case and then places exactly one point throughout the rest of stage (a), so $W$ must be $\geq M$ points ahead after stage (a).

Above argumentation is valid only for $N > K$, so that two points are needed to take a border key interval. ■

**Claim 24** *If $M(r) > 0$ at the beginning of stage (c), then either*

*(i) B has exactly $M(r)+1$ points left or*
*(ii) B has exactly $M(r)$ points left and all black points are key points.*

**Proof.** If player $W$ was $\geq M+1$ points ahead before $B$'s first move after stage (a), then, by the same arguments as those used in proof of Claim 6, $B$ must have exactly $M(r)+1$ points left before his move at the beginning of stage (c). Similarly, if player $W$ was $M$ points ahead before $B$'s first move after stage (a) and he placed at least one point in stage (b).

If $W$ was $M$ points ahead before $B$'s first move after stage (a) and he placed no point in stage (b), then all black points must be key points at the beginning of stage (c). ■

Claim 7 still holds, however it requires a different proof, which we give below.

**Proof of Claim 7.** Assume that $B$ took $Q$ key positions in stage (a). Since player $B$ switched to Strategy $Y_L^*$, so $2 \leq Q < 2K$. Observe also that if $B$ placed any point in stage (b), then this point must have been placed within white non-border key interval (as there exists a white non-border key interval at the beginning of stage (c)). This means, in particular, that no black interval can be created in stage (b) and so all black intervals at the beginning of stage (c) must be key intervals.

If there is a white key non-border interval at the beginning of stage (c), then there must exist a white border key interval (in the same line-segment), as there are 2 key positions on each line-segment and $B$ does not place any point within white border intervals. Moreover, since $Q \geq 2$, so the number of white key points is $2K - Q \leq 2K - 2 < N$ and there must be at least one white point which is not a key point, at the beginning of stage (c).

If at least one of points $W$ placed out of key position was placed within white or bichromatic interval, then there must exist a white non-key interval at the beginning of stage (c), as $B$ does not place any points within white non-key intervals. Similarly, if at least two of these points were placed within a black key interval, then there must exist a white non-key interval at the beginning of stage (c), as well.

Assume that each of these points was placed in a different black key interval. If any of them was placed in black border segment, then there exists a white non-key border segment and a black key border segment (the one in the same line-segment). If all of them were placed in black non-border key intervals, then in each of line-segments with these intervals there is a black border key interval (as there are 2 key positions on each line-segment). ■

Now we will show that $B$'s move for case (c)(1) is applicable. If this case holds, then there are $M(r) > 0$ white border key intervals. By Claim 24, $B$

has either $M(r)$ or $M(r)+1$ points to place. Suppose first that he has $M(r)$ points to place. Then, by Claim 24, all black points are key points. Moreover, since there are 2 key positions on each line-segment, so each line-segment with white border key interval contains also at least one white non-border interval. Hence there are $Q \geq 2M(r)$ white intervals and, by Lemma 1, there are $Q - M(r)$ black intervals. All of these intervals must be key intervals, as all black points are key points.

Also, since every line-segment with white border key interval contains a non-border key interval, so there must exist a white non-key interval or a white non-key border segment and a black key border segment. This is because either one of these line-segments contains a non-key non-border white interval or all of them contain a white non-border key interval. In the second case Claim 7 applies. Hence the move corresponding to case (1) of option (c) is applicable and $B$ places one point in each white border key interval. After all points are placed, player $B$ loses by $< M(r)\delta/(M(r)+2)$ on bichromatic border segments with white key border segment and wins by $\geq \delta$ on remaining intervals. It is easy to check that the difference between area controlled by $B$ and area controlled by $W$ is positive and so $B$ wins in this case.

If $B$ has $M(r)+1$ points to place, then it can be shown that $B$ wins by using the same argumentation as in proof of Theorem 4. Similarly with $B$'s moves for cases (2)–(4) of option (c). Thus we have shown that if $B$ can switch to Strategy $Y_L^*$, he can win the game.

Assume now that $B$ could not switch to Strategy $Y_L^*$. Then one of two situations must have occurred during the course of the game: either $W$ took $2(K-1)$ key positions on $K-1$ line-segments, so that $B$ was unable to secure key positions in two different line-segments or $W$ did not take a key position till the end of the game.

Suppose that the first situation is the case. Then the game must have two rounds and it must be that player $B$ placed only one point in the first round (if he placed two points, than he would have taken 2 key positions, so this situation would not be possible). Two situations are possible in the first round: (i) player $W$ placed $2(K-1)$ points taking all these key positions or (ii) player $W$ placed exactly one point, not taking a key position and $B$ placed exactly one point taking a key position on a different line-segment.

In situation (i), in the second and last round player $W$ places his last point in a new line-segment or in a line segment with key white interval creating a non-key white interval. In situation (ii) player $W$ places $2(K-1)$ points on $K-1$ line-segments without black point, taking all $2(K-1)$ key positions (notice that it means that there will be a line-segment with three white points then).

Now three cases are possible before $B$'s move in the last round: (i) there is an empty line-segment, (ii) there is a line-segment with one white point and without black points and (iii) there is a line-segment with one black key point and without white points.

If case (i) holds, then it could have been created by situation (i) only, so there is already a white non-border interval with black point inside. In his last move, player $B$ starts by placing $K-1$ points in white non-border intervals and one point in empty line-segment. If case (iii) holds, then it could have been created by situation (ii) only, so there is a line-segment with one black point only. In his last move, player $B$ starts by placing $K$ points in white border intervals.

Observe that at this point situation in two of these cases is the same for $B$, before his remaining $K-2$ are placed. Player $B$ places these $K-2$ points in white border key intervals (notice that if $K=2$, then no point is placed in white border key intervals). After this move there are no white non-border intervals, $B$ loses by $(K-2)(1-4\delta)/(2(K-2)) = 1/2-2\delta$ on border intervals with white key border segment (if $K>2$) and wins by $\geq 1/2$ over remaining white interval. Thus $B$ wins the game by $\geq 2\delta$.

If case (ii) holds, then it could have been created by situation (i) only, so there is already a white non-border interval with black point inside. Player $B$ places $K-2$ points in white non-border intervals, breaking all of them. Then he places two points in the line segment with one white point and remaining $K-2$ points in white border key intervals (again, if $K=2$, then no point is placed in white border key intervals). After this move $B$ loses by $(K-2)(1-4\delta)/(2(K-2)) = 1/2-2\delta$ on border intervals with white key border segment (if $K>2$) and wins by $> 1-2\delta-1/2 = 1/2-2\delta$ on the two remaining white intervals. Thus $B$ wins the game.

For the last situation that can occur when option (ii) is taken by player $B$, assume that $W$ did not take any key position throughout the game. In his last move, player $B$ plays according to case (6) of Strategy $Y_L'^*$. Before $B$'s move there are $K-2$ line-segments with two black key points and 2 line-segments with one black key point. Notice also, that either there exists a white border segment or a non-border white interval, as $W$ placed more points than $B$ and, by Lemma 1, there must exist a white interval. Hence $B$'s move corresponding to case (6) is applicable, as long as it is possible to gain enough area to win the game.

Assume first that one of line-segments with one black key point contains no white point. If $B$ took a key position on another line-segment with one black key point, then at the end of the game there would be $K-1$ line-segments with $2(K-1)$ black key points and $2K-1$ white points and one line-segment with one black point only. Notice that $W$ can be winning by $< 1/2$ on the $K-1$ line-segments, as he has only one point more there than $B$ (so by Lemma 1 there is one white interval more than black intervals), none of his border segments can be larger than black border segment and all white intervals must have length $< 1/2$, while all black ones have length $1/2$. On the other hand, $B$ is winning by 1 on the line-segment with one black key point only and hence $B$ would win the game. Observe that Strategy $Y_L'^*$ may ask $B$ to place his last point in some place other then the key position in a line-segment with one black point only, but it would only increase his advantage.

Now assume that one of line-segments with one black key point contains $\geq 3$ white points. If $B$ placed his last point on a key position in another line-segment with one black key point, then at the end of the game there would be $K-1$ line-segments with $2(K-1)$ black key points and $2(K-1)-2$ white points. Notice that $B$ must be winning by $\geq 1$ on the $K-1$ line-segments, as by Lemma 1 there are two black intervals more than white intervals, all black intervals have length $1/2$ and are larger than white intervals, and none of black border segments can be smaller than white border segment. On the other hand, $B$ is losing on the line-segment with one black key point only by $< 1$ and so he would win the game. Again, if Strategy $Y_L'^*$ asked $B$ to place his point in a different place than a key position in a line-segment with one black point, then he would only increase his advantage.

The remaining cases to analyse are the following: (i) each of line-segments with one black key point contains exactly two white points, (ii) one of line-segments with one black key point contains exactly one white point and another exactly two white points and (iii) each of line-segments with one black point contains exactly one white point.

Consider case (i) first. The $K-2$ line-segments with $2(K-2)$ black key points contain $2(K-2)-1$ white points, so, by analogical arguments to those used above, $B$ is winning there by $\geq 1/2$. If any of the two line-segments with one black key point contains a black border segment of length $\geq 1/4$, then $B$ is losing by $< 1/2$ there, as area controlled by $W$ on this line-segment is $< 3/4$. Hence $B$ is winning on $K-1$ line-segments and $B$ can win the game by taking a key position on the remaining line-segment, as he will be winning there after such move. If Strategy $Y_L'^*$ asked $B$ to play anywhere else, then he would only increase his advantage.

If none of black border segments of length $1/4$ is empty on the two line-segments with one black key point, then there are white border intervals in each of them. If any of these white intervals has length $< 1/2$, then $B$ can win the game by taking key position in another line-segment. Suppose then that each of these white intervals has length $\geq 1/2$. Suppose that $B$ places his point in maximal white border segment of maximal of these white border intervals, at a distance $\delta$ from the white point. Let $a_1$, $a_2$, $b_1$ and $b_2$ denote lengths of

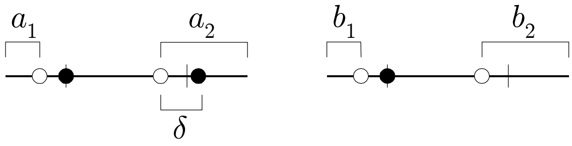


Fig. 1.

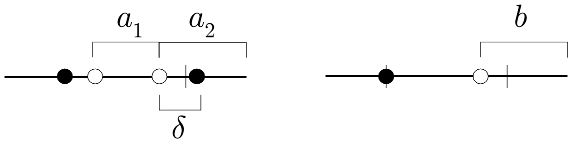


Fig. 2.

white border segments (as presented in Fig. 1), so that $a_2 \geq b_2$. Then after $B$'s move difference between area controlled by $W$ and area controlled by $B$ on the two line-segments is $a_1+b_1+b_2-a_2+\delta \leq a_1+b_1+\delta < \frac{1}{2}$, for $\delta < 1/2-(a_1+b_1)$, so $B$ wins the game. If Strategy $Y_L'^*$ asked $B$ to play anywhere else, then he would only increase his advantage.

Consider case (ii) now. The $K-2$ line-segments with $2(K-2)$ black key points contain $2(K-2)$ white points in this case, so $B$ is winning there by some $\varepsilon > 0$. If the line-segment with one white point and one black key point contains a white border segment of length $\leq 1/4$, then $B$ is not losing on this line-segment and $B$ can win the game by placing his last point in a key position on another line-segment with only one black key point. If the strategy asked $B$ to play anywhere else, then he would only increase his advantage.

If the line-segment with one white point and one black key point contains a white border segment of length $> 1/4$, then two cases are possible: (a) there is a non-border white interval on line-segment with one black key point and two white points or (b) there is a white border interval there.

Suppose that case (a) holds and assume that player $B$ placed his last point in maximal white border segment, at a distance $\delta$ from white point. Let $a_1$, $a_2$ and $b$ denote lengths of white border segments (as presented in Fig. 2) and assume (without loss of generality) that $a \geq b$. Notice that $a_1 < 1/2$, as $a_2 > 1/4$. After $B$'s move, the difference between area controlled by $B$ and area controlled by $W$ on the two line-segments is $1/2 - a_1 - b + a_2 - \delta \geq 1/2 - a_1 - \delta > 0$, for $\delta < 1/2 - a_1$. Hence $B$ wins the game and if Strategy $Y_L'^*$ asked him to play anywhere else, then he would only increase his advantage.

Now suppose that case (b) holds and assume that player $B$ placed his last point in maximal white border segment, at a distance $\delta$ from white point. Let $a_1$, $a_2$ and $b$ denote lengths of white border segments (as presented in Fig. 3)

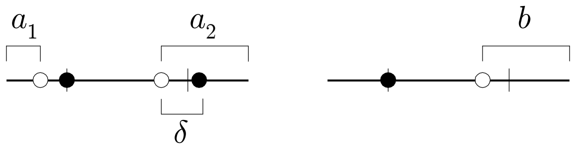


Fig. 3.

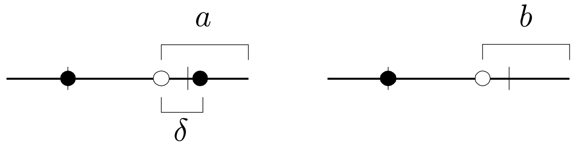


Fig. 4.

and assume (without loss of generality) that $a \geq b$. Then, after $B$'s move, the difference between area controlled by $W$ and area controlled by $B$ on the two line-segments is $a_1 + b - a_2 + \delta - 1/4 \leq a_1 - 1/4 + \delta < 0$, for $\delta < 1/4 - a_1$. Thus $B$ wins the game and if Strategy $Y_L'^*$ asked him to play anywhere else, then he would only increase his advantage.

Lastly, consider case (iii). The $K-2$ line-segments with $2(K-2)$ black key points contain $2K-3$ white points in this case, so $B$ is losing there by $1/2-\varepsilon$ (where $\varepsilon > 0$). If any of the line-segments with one white point and one black key point contains a white border segment of length $\leq 1/4$, then $B$ is not losing on this line-segment and $B$ can win the game by placing his last point in a key position on the other line-segment with only one black key point, as he will be winning by $\geq 1/2$ on this line-segment after his move. If the strategy asked $B$ to play anywhere else, then he would only increase his advantage.

If both the line-segments contain white border segments of length $> 1/4$, then let $a$ and $b$ denote lengths of white border segments (as presented in Fig. 4) and assume that $a \geq b$. After $B$'s move, the difference between area controlled by $B$ and area controlled by $W$ on the two line-segments is $a - b - \delta + 1/2 \geq 1/2 - \delta > 1/2 - \varepsilon$, for $\delta < \varepsilon$. Hence $B$ wins the game and if Strategy $Y_L'^*$ asked him to play anywhere else, then he would only increase his advantage. This completes the proof ■

**Remark 25 (One round game)** *Notice that we never used assumption that $W$ takes more than one round to place all his points in proof above. Even in the case of him taking $2(K-1)$ key positions on $K-1$ different line-segments, where we studied two round game for identifying all possible cases only. Hence Strategy $Y_L'^*$ is a valid winning strategy for $B$ even for the one round game.*

**Remark 26 (One-by-one variant of the game)** *Notice that Strategy $Y_L'^*$ is a valid winning strategy for $B$ even for one-by-one variant of the game, as* *it never asks $B$ to place more points than his opponent did in the same round.*

### *3.3 Opponent's defence*

Although in all possible configuration of the game studied above there is a winning strategy for one of the players, it is always possible for a player to make the size of his defeat as small as he wants, as showed below.

**Theorem 27** *Let $\left\langle N, \{L_j\}_{j=1}^{K} \right\rangle$ define a game on a family of disjoint line-segments. Then for any player $P \in \{B, W\}$ and any $\varepsilon > 0$, $P$ can make the difference between his and his opponent's area $< \varepsilon$ at the end of the game.*

**Proof.** Assume a player keeps placing his points at key positions or middle positions, where relevant (if possible), or in his opponents non-border intervals (if possible), or in his opponents border intervals at a distance $< \varepsilon/N$ from end point of maximal of the two border segments, or in bichromatic intervals with opponents point being a key point, at a distance $< \varepsilon/N$ from this point. Then at the end of the game a difference between any of his and any of opponent's intervals will be $< \varepsilon/N$ (similarly with his and his opponents border segments). Since there can be at most $N$ intervals of a player at the end of the game, so he cannot lose the game by $\geq \varepsilon$. ■

## 4 Conclusions

We have studied a two-player Voronoi game of Ahn et al. (2004) played on multiple disjoint open curves of equal lengths. We have shown that there is either the first or the second mover advantage, depending on number of points players have to place and number of open curves they will play on. Strategies we proposed make as little assumptions as possible about number of points players place in each round, in particular they guarantee that the second mover never has two place more points than his opponent in his moves. This makes these strategies valid for more restricted variants of the game, the one were players place exactly one point at each round, in particular.

Our results show that, arguably a desirable property of existence of a tying strategy (i.e. "fairness"), is not in place for disjoint open curves, as it was in the case of analogical game played on more than one closed curves Datta et al. (2007). The interesting question in this respect is, whether it is possible to enforce fairness by designing rules of the game so that only number of points placed by players at each round are restricted and players identity is preserved, while positions at which points are placed are not affected by the rules. Our hypothesis is that it is impossible without affecting positions where points are placed.

Another interesting direction of further research would be to study Voronoi games in question when played on possibly intersecting multiple disjoint curves. This would lead two Voronoi games on graphs (see for example Dürr and Thang (2007)).

## References


Ahn, H.-K., Cheng, S.-W., Cheong, O., Golin, M., van Oostrum, R., 2004. Competitive facility location: the voronoi game. Theoretical Computer Science 310 (1–3), 457–467.

Chamberlin, E. H., February 1953. The product as an economic variable. Quarterly Journal of Economics 67 (1), 1–29.

Chawla, S., Rajan, U., Ravi, R., Sinha, A., 2003. Min-max payoffs of a location game. Tech. rep., CMU CS.

Cheong, O., Har-Peled, S., Linial, N., Matoušek, J., 2002. The one-round voronoi game. In: SCG '02: Proceedings of the eighteenth annual symposium on Computational geometry. ACM Press, New York, NY, USA, pp. 97–101.

Datta, D., Dziubiński, M., Roy, J., 2007. A Location Game on Disjoint Circles. SSRN eLibrary.

Dürr, C., Thang, N. K., 2007. Nash equilibria in voronoi games on graphs. URL http://www.citebase.org/abstract?id=oai:arXiv.org:cs/0702054

Economides, N., 1986. Minimal and maximal product differentiation in Hotellings duopoly. Economic Letters 21, 67–71.

Eiselt, H. A., Laporte, G., 1989. Competitive spatial models. European Journal of Operational Research 39 (3), 231–242.

Hotelling, H., 1929. Stability in competition. Economic Journal 39 (1), 41–57.

Prescott, E. C., Visscher, M., Autumn 1977. Sequential location among firms with foresight. Bell Journal of Economics 8 (2), 378–393.

Salop, S. C., Spring 1979. Monopolistic competition with outside goods. Bell Journal of Economics 10 (1), 141–156.

Tobin, R. L., Friesz, T. L., Miller, T., 1989. Existence theory for spatially competitive network facility location models. Annals of Operations Research 18 (1), 267–276.