Quantitative Equity Portfolio Management: An Industry Perspective

Presentation 1: Issues Relating to Constructing Multifactor Models for Equity

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Motivation of Study

- Following Grinold (1989, 1994), an active portfolio manager’s objective can be characterized as maximizing the Information Ratio (IR) or value added of her portfolio in the presence of various portfolio constraints.

- Faced with multiple information sources, a portfolio manager can employ linear factor models to generate return forecasts for the set of securities in his/her investment universe.

- Given a set of signals and portfolio constraints, optimal portfolio construction can therefore be reduced to the problem of selecting an optimal weighting scheme for these factors that will provide a forecast for any given security.

- It is well known however, that choosing factor weights to maximize a factor model’s forecasting ability is not the same as selecting weights so that a portfolio’s Information Ratio (IR) is maximized. In fact it has been demonstrated that least squares regression of linear factor models will not generally lead to estimators that maximize unconditional Sharpe Ratios (Sentana, 2005).

- In this presentation we attempt to formally derive a framework that produces optimal factor weights which exploits the differential characteristics of the factors themselves to arrive at a factor weighting scheme that generates optimal valued added portfolios under various portfolio constraints, namely portfolio turnover.
In Active Quantitative portfolio management, multi-factor models are developed and applied to

- Generate risk estimates (covariance matrix of returns) to control portfolio risk in the optimisation process

  Examples include Barra and Northfield Risk Models

- Linearly combine multiple sources excess return forecasts (Alpha Modelling)

In this presentation we will focus on the latter and consider how to form optimal combinations of return forecasts for generating value added within an equity portfolio
Introduction: What is a Multifactor Model?

- Multifactor models are used to explain the expected returns on a cross-section of assets.
- The factors represent common components of the variance of security returns which contribute to the expected return.
- Typically, we assume a linear relationship between a security’s return and the common factors:

\[ r^i_t = b^i_1 F^1_t + b^i_2 F^2_t + \ldots + b^i_K F^K_t + e^i_t \quad E(e^i_t) = 0 \]

- Represents the excess return to security \( i \) at end of period \( t \).
- Represents the return to Factor \( j \) at end of period \( t \).
- Represents the specific (idiosyncratic) return to security \( i \) at end of period \( t \).
- Represents the exposure of security \( i \) to common Factor \( j \) (e.g. Book to Price), at the start of the period.

In Active Quantitative portfolio management, multi-factor models are used in:

- Generating Risk Estimates
- Generating Excess Return Forecasts (Alpha)
Introduction: How do we optimally combine alpha factors?

- **Alpha models almost always employ multiple factors instead of a single one**
  
  Such factors include:
  
  - Sentiment Based (e.g., Price Momentum, Earnings Revisions)
  - Value Based (e.g., Book to Price, Dividend Yield)
  - Quality (Change in Working Capital, Growth in Capital Expenditures)

- **We define a Multifactor model as one that linearly combines scores of individual alpha factors to create a composite forecast of excess returns**

- **As an active manager, the most important elements at a portfolio level (in addition to portfolio constraints) can be summarised by**
  
  - Returns
  - Risk
  - Transaction Costs and Turnover

- **If the primary objective of an active manager is to identify value added opportunities, then at an alpha level these portfolio level considerations are summarised by**
  
  - Alpha Factor Performance (Factor Returns)
  - Alpha Factor Risk (Volatility in the Factor Returns)
  - Stability of Alpha Factors (Serial Correlation of Alpha factors)

- **This study looks at how to optimally form a linear combination of alpha forecasts to meet a portfolio managers investment objectives**
Overview of existing methods to construct alpha

- **The aim in alpha construction is to determine the optimal linear combination of an existing set of alpha factors**
  
i.e. we solve for the optimal weighting of a linear multifactor model
  \[ \alpha_t^i = \omega_1 f_{1t}^i + \omega_2 f_{2t}^i + \ldots + \omega_k f_{kt}^i \]

  The alpha factors are typically re-scaled and represent excess return forecasts themselves

  The factor weights are typically scaled to sum to unity

  For portfolio construction, a scaled version of this alpha forecast is then used as the expected excess return input

- **Traditionally we can consider 3 broad approaches to determine the weighting scheme** \( \omega \)
  
  Equal / Static Weighting

  Weighting schemes based on factor performance

  Weighting Schemes based maximising investor utility functions

- **We will consider each in turn and highlight how the proposed scheme rests within one of these approaches**
  
  In so doing we highlight a continuum of alpha weighting scheme approaches that range in complexity and richness
Overview of existing methods to construct alpha

- **The Equally Weighted Approach**  \( \omega_k = \frac{1}{K} \)

  **Advantages:** straightforward to implement, no errors in estimation, no additional turnover caused by time varying weights

  **Disadvantages:** fails to take into account relative performance of factors, fails to consider potential correlation between alpha factors

  In the absence of any additional information, a scheme that equally weights each factor is the most sensible and intuitive and allows the greatest potential for alpha source diversification

  This approach is often used a benchmark or base case with which to assess alternative weighting schemes
Overview of existing methods to construct alpha

- **Weighting schemes based on factor performance**

  This approach attempts to utilise the linear decomposition of returns into common factors and their factor returns

  Factor return forecasts are first generated, which are then mapped into the alpha factor weights

  Commonly, the forecasts of each factor’s performance can be generated via a number of techniques including:
  - Macroeconomic based models
  - ARIMA Models
  - Markov Switching Models
  - Historical averages obtained from regression analysis [e.g., Fama-Macbeth regression methodology]

  **Advantages:**
  - straightforward to implement
  - intuitive

  **Disadvantages:**
  - In the absence of forward looking factor return forecasting models, the factor return forecasts based on historical averages are backward looking in nature and susceptible to market turning points
  - Such approaches only consider factor performance (returns) as important when determining factor weights; they do not include the impact of the correlation or risk of the factors themselves
Weighting schemes based on investor utility functions

- **Weighting Schemes Based investor utility functions**

  Issue with previous schemes is that they fail to consider the active portfolio manager’s true objective function in deriving optimal alpha weighting scheme

  Following Grinold (1989, 1994), an active portfolio manager’s objective can be characterized as maximizing the Information Ratio (IR) or value added of her portfolio in the presence of various portfolio constraints

  There have been several approaches that attempt to align the use of factors with principle of maximizing the investor’s objective function

- **Brandt, Santa-Clara and Valkanov (2009)**

  Propose an approach whereby the weight in each stock is modeled as a linear function of the firm’s characteristics (factors), such as its market capitalization, book-to-market ratio, and lagged return.

  The coefficients of this function are found by optimizing the investor’s average utility (their objective function) of the portfolio’s return over a given sample period

  \[
  \max_{\{w_{i,t}\}_{i \in \mathbb{N}}} \mathbb{E}_t \left[ u(r_{p,t+1}) \right] = \mathbb{E}_t \left[ u \left( \sum_{i=1}^{N_t} w_{i,t} r_{i,t+1} \right) \right]
  \]

  \[
  w_{i,t} = \bar{w}_{i,t} + \frac{1}{N_t} \theta^T \tilde{x}_{i,t}
  \]

  Though intuitively appealing, the problem is not analytically tractable, and is reliant on numerical procedures that are not guaranteed to be solvable. Solution can also potentially result in unbounded portfolio weights
Weighting schemes based on investor utility functions

- Sorenson, Qian, Hua, and Schoen (2004)

  Qian Hua and Sorenson (2004, 2005): Extend the framework of Qian and Hua (2004) and show how one can derive the factor weighting scheme that maximises a portfolio’s Information Ratio (IR)

  By specifying a straightforward mean-variance objective function for the portfolio manager and as well as simplifying assumptions about factor score correlations, their approach yields an analytically tractable solution for the factor weights

  In particular they highlight that the IR of an long-short mean variance optimised strategy is directly related average performance of the alpha model (via the Information Coefficient) of a strategy as well the consistency of the forecasts over time

  \[
  \begin{align*}
  \text{avg}(r_{pt}) &= \text{avg}(IC)\sqrt{N} \sigma_a \overline{\text{dis}(r_i)} \\
  \text{std}(r_{pt}) &= \text{std}(IC)\sqrt{N} \sigma_a \overline{\text{dis}(r_i)} \\
  \end{align*}
  \]

  \[
  IR = \frac{\text{avg}(r^a_t)}{\text{std}(r^a_t)} = \frac{\text{avg}(IC)}{\text{std}(IC)}
  \]

  They further demonstrate that the active risk of a strategy is not simply the target tracking error \(\sigma_a\) but it is also a function of the strategy risk of the investment strategy (captured by \(\text{std}(IC)\))

  Using these insights they form optimal alpha weights so that the ratio of IC to its std deviation is maximised
Weighting schemes based on investor utility functions

- **Sneddon (2008): Incorporating Transaction Costs in Optimal Alpha Construction**

  Sneddon considers the problem of optimal factor weighing schemes by considering an even more realistic objective function that takes into account portfolio turnover and transaction costs.

  This approach derives a multiperiod IR in a semi-analytical framework under transaction costs – overweighting “tortoise” factors that have slow but stable IC decay and underweighting “hare factors” – high but fast decaying.

  While posing a more realistic metric to maximise (net IR), the main drawback with this approach is that it fails to incorporate strategy risk in constructing optimal factor weights.

- **Qian Sorenson and Hua, 2007 (QSH) Extension for transaction costs**

  Attempt to introduce realistic portfolio construction objectives by modelling the cost of implementation and by constructing optimal alphas n the presence of transaction costs.

  The main drawback is that, unlike Sneddon (2008), transaction costs are introduced by constraining the portfolio to achieve a target alpha autocorrelation which in principle determines drives portfolio turnover.

  This implementation is ad-hoc at best and not a realistic description of the investment manager's objective function.
Weighting schemes based on investor utility functions

- In this presentation, we propose a new approach that represents a synthesis of previous methods to deal with optimal alpha construction under more realistic portfolio construction settings; namely transaction costs.

- **To redress the shortcomings of the last two approaches**
  
  We use the analytically elegant and tractable framework developed by QSH that link strategy performance and strategy risk of the alpha model to optimal alpha weights.

  We then utilize the insights developed by Sneddon (2008) by introducing transaction costs into the objective function of the portfolio manager. The key is to link transaction costs and portfolio turnover to alpha factor serial correlation.

- **We develop the framework in steps; first considering the original IR maximization problem developed by QSH and then extending the problem for the case of transaction costs**
Optimal Factor Weighting:
The basic problem in the absence of transaction costs

- Active Manager – every period seeks to generate a portfolio \((h)\) that maximise value added subject to constraints:

\[
\max_U = h_t' \alpha_t - \frac{1}{2} \lambda h_t' \Sigma_t h_t \\
\text{s.t. } h_t' 1 = 0, \quad h_t' X_t = 0 \quad \text{and} \quad \sqrt{h_t' \Sigma_t h_t} = \sigma_{at}
\]

- The portfolio is dollar neutral and neutral to all risk factors \((X)\) through linear equality constraints and targets an active risk of \(\sigma_a\) via the tracking error constraint.

- \(\Sigma\) represents the diagonal matrix of idiosyncratic variances with typical element, \(\sigma_i\)

- Given a set of \(K\) alpha factor forecasts, \(f^k (k = 1, K)\), the manager’s problem is to construct a composite alpha forecast for returns by selecting the factor weights \((\omega)\)

\[
E(r_{it}) = \alpha_{it} = \sum_{k=1}^{K} \omega^k f_{it}^k
\]

- The objective: select \(\omega\) so that the ex-post IR (over the last \(T\) periods) of the value added portfolio \(h\) is maximised:

\[
\max_{\omega} IR = \frac{\text{avg}(r_t^a)}{\text{std}(r_t^a)} \\
\text{subject to } \sum_k \omega^k = 1 \quad \text{and} \quad 0 \leq \omega^k \leq 1
\]
Optimal Factor Weighting: The basic problem in the absence of transaction costs

Generating a solution to the basic problem: decomposing the portfolio return

Given the objective function, the optimal portfolio weights can be solved analytically (dropping time subscripts for notational ease)

\[ h = \lambda^{-1} \sum_{t} \tilde{\alpha} \quad \text{where} \quad \tilde{\alpha} = [\alpha - X(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}\alpha] \]

\[ h_i = \lambda^{-1} \cdot \frac{\tilde{\alpha}_i}{\sigma_i^2} \]

Using this basic solution, we can express the expected excess return on the portfolio \( h \) of \( N \) assets as

\[ r_i^a = \sum_{i=1}^{N} h_i r_i = \lambda^{-1} \sum_{i=1}^{N} \left( \frac{\tilde{\alpha}_i}{\sigma_i} , \frac{r_i}{\sigma_i} \right) \]

\[ r_i = \text{mean adjusted residual return to security } i \text{ such that in the cross section } \text{avg}(r/\sigma) = 0 \]

Employing the definition of covariance between the terms \( \frac{\tilde{\alpha}_i}{\sigma_i} \text{ and } \frac{r_i}{\sigma_i} \) yields the following (see appendix 1)

\[ r_i^a = (N-1)\lambda^{-1} \text{corr} \left( \frac{\tilde{\alpha}_i}{\sigma_i} , \frac{r_i}{\sigma_i} \right) \text{disp} \left( \frac{\tilde{\alpha}_i}{\sigma_i} \right) \text{disp} \left( \frac{r_i}{\sigma_i} \right) \]

We denote this correlation between the risk adjusted alpha and risk adjusted return as the Risk Adjusted Information Coefficient (IC):

\[ IC = \text{corr} \left( \frac{\tilde{\alpha}_i}{\sigma_i} , \frac{r_i}{\sigma_i} \right) \]
Optimal Factor Weighting:
The basic problem in the absence of transaction costs

- **Generating a solution to the basic problem: decomposing the portfolio’s IR**

  We can simplify the expression for the active portfolio return by substituting out the risk aversion parameter due to the fact the portfolio has a tracking error constraint set to $\sigma_a$

  $$\sigma_a^2 = \sum_{i=1}^{N} \left( h_i^2 \sigma_i^2 \right) \Rightarrow \lambda = \frac{1}{\sigma_a} \cdot \sqrt{\sum_{i=1}^{N} \left( \frac{\tilde{\alpha}_i}{\sigma_i} \right)^2}$$

  Assuming the cross sectional average of the risk adjusted alpha is approximately zero,

  $$\lambda = \frac{1}{\sigma_a} \cdot \sqrt{\sum_{i=1}^{N} \left( \frac{\tilde{\alpha}_i}{\sigma_i} \right)^2} = \frac{1}{\sigma_a} \sqrt{N-1} \text{disp} \left( \frac{\tilde{\alpha}_i}{\sigma_i} \right)$$

  We therefore have the following expression for the realised single period excess return on the portfolio

  $$r^a = \sqrt{N-1} \sigma_a \text{IC disp} \left( \frac{r_i}{\sigma_i} \right)$$

  From this we can obtain the expected excess return to the portfolio as

  $$\text{avg}(r^a) = \text{IC} \sqrt{N-1} \sigma_a \text{disp} \left( \frac{r_i}{\sigma_i} \right)$$

  and the expected active risk as

  $$\text{std}(r^a) = \text{std} \left( \text{IC} \right) \sqrt{N-1} \sigma_a \text{disp} \left( \frac{r_i}{\sigma_i} \right)$$

  Note the active risk of a portfolio is a function of both the (exante) tracking error ($\sigma_a$) **AND** the volatility of the alpha model’s IC (strategy risk)
Optimal Factor Weighting:
The basic problem in the absence of transaction costs

Generating a solution to the basic problem

We can use the previous insight that IR of an optimal risk constrained portfolio is directly linked to the ratio of the model IC and its variability over time.

Given a set of $K$ forecasting signals, $f^k$ ($k = 1, K$), the manager’s problem is to construct a composite alpha forecast for returns by selecting the factor weights, $\omega$

$$\alpha_{it} = \sum_{k=1}^{K} \omega_k \tilde{f}_{it}^k \quad \text{where} \quad \tilde{f}_{it}^k \sim iid(0,1)$$

The objective: select $\omega$ so that the ex-post IR is maximised

$$\max_{\omega} \ IR = \frac{\text{avg}(r^a_t)}{\text{std}(r^a_t)} = \frac{\text{avg}(IC)}{\text{std}(IC)} \quad \text{subject to} \quad \sum_{k} \omega^k = 1 \quad \text{and} \quad 0 \leq \omega^k \leq 1$$

Based on the composite alpha definition, it can be easily shown that under the assumption that the factors are standardised,

$$IC = \text{corr} \left( \frac{1}{\sigma_i} \sum_{j=1}^{k} \omega_j \tilde{f}^j_i, \tilde{r}_i \right) = \tau^{-1} \left( \sum_{k=1}^{K} \omega_k' IC_k \right) \quad \text{where} \quad IC_k = \text{corr} \left( \frac{\tilde{f}^k_i}{\sigma_k}, \tilde{r}_i \right)$$

$$\tau = \sqrt{\omega' \Phi \omega} \quad \text{and} \quad \Phi = \text{correlation matrix of factors}$$
Optimal Factor Weighting:
The basic problem in the absence of transaction costs

- Generating a solution to the basic problem

Based on the composite alpha definition, we can therefore represent the maximisation problem as

\[ \text{avg}(IC) = \tau^{-1} \left( \sum_{k=1}^{K} \omega_k \cdot IC_k \right) \]

\[ \text{std}(IC) = \tau^{-1} \sqrt{\omega^T \Sigma_{IC} \omega} \quad \text{where} \quad \Sigma_{IC} = \text{covariance matrix of factor ICs} \]

\[ \Rightarrow \max_{\omega} IR = \frac{\text{avg}(r_t^a)}{\text{std}(r_t^a)} = \frac{\text{avg}(IC)}{\text{std}(IC)} = \frac{\sum_{k=1}^{K} \omega_k \cdot IC_k}{\sqrt{\omega^T \Sigma_{IC} \omega}} \]

subject to \[ \sum_{k} \omega_k^2 = 1 \quad \text{and} \quad 0 \leq \omega_k^2 \leq 1 \]

The solution to this maximisation is straightforward and tractable:

\[ \omega = \frac{IC^T \Sigma_{IC}^{-1}}{IC^T \Sigma_{IC}^{-1} IC} \]

The alpha weight of a factor depends not only on its risk reward trade-off (based in IC) but also on its performance correlation to other factors in the model.
Optimal Factor Weighting: Extending the Basic Problem to account for transaction costs

- Active Manager – every period seeks to generate a long/short portfolio \((h)\) that maximises value added subject to constraints and turnover penalties:

\[
\max_{h_t} \quad h_t' \alpha_t - \frac{1}{2} \lambda h_t' \Sigma_t h_t - \frac{1}{2} \eta (h_t - h_{t-1})' I(h_t - h_{t-1})
\]

We assume that turnover, often expressed as \(|\Delta h_t|\) can be approximated with the quadratic term \(\Delta h_t, \Delta h_t\)
\(\eta\) effectively maps turnover to a transaction cost estimate.

To make the analysis tractable we assume for simplicity that risk aversion \((\lambda)\) is stable over time and that the residual risk is constant and identical across all firms, then we have the solution as

\[
h_t = \frac{\alpha_t + \eta h_{t-1}}{\lambda \sigma^2 + \eta} = \frac{\alpha_t + \eta h_{t-1}}{\beta}, \quad \text{where} \quad \beta = \lambda \sigma^2 + \eta \quad \text{and} \quad \Sigma_t = \sigma^2 I
\]

Using this, we obtain an expression for the realised alpha (return) of the portfolio as

\[
r_{pt} \equiv \sum_i h_{it} \times r_{it} = \sum_i \left( \frac{\alpha_{it} + \eta h_{it-1}}{\beta} \times r_{it} \right)
\]

Recursively substituting \(h\), we obtain

\[
r_{pt} = \sum_{p=0}^{\infty} \left( \frac{\eta^p}{\beta^{p+1}} \sum_i (\alpha_{it-p} \times r_{it}) \right) \quad \text{which can be truncated at some appropriate lag,} \quad p^*
\]
Decomposing the alpha

- We assume that the alpha is a weighted composite of k factor scores

\[ \alpha_{it} = \kappa \sum_{k=1}^{K} \omega_k \tilde{f}_{it}^k \quad \text{where} \quad \tilde{f}_{it}^k \sim iid(0,1) \quad \kappa = [(N-1)\omega'\omega]^{-0.5} \]

The factors have been standardized and are uncorrelated with another. \( \kappa \) is a scalar that ensures the composite has unit variance. We assume it is relatively stable over time.

Therefore at time \( t \) the expression in \( r_{pt} \) can be written as

\[ \sum_i (\alpha_{it} \times r_{it}) = \kappa \sum_k \left( \omega_k \sum_i \tilde{f}_{it}^k \times r_{it} \right) = \kappa \sigma_r \sum_k \left( \omega_k \times (N-1) \times corr(\tilde{f}_{it}^k, r_i) \right) \]

The portfolio return can therefore be expressed in terms of lagged factor ICs:

\[ r_{pt} = \kappa (N-1) \sigma_r \sum_{p=0}^{\infty} \frac{\eta_p}{\beta_{p+1}} \sum_k \left( \omega_k \times IC_{pt}^k \right) \quad \text{where} \quad IC_{pt}^k \equiv corr(\tilde{f}_{t-p}^k, r_i) \]

\[ = \kappa (N-1) \sigma_r \sum_{p=0}^{\infty} \left( \frac{\eta_p}{\beta_{p+1}} \omega'IC_{pt} \right) \quad \text{where} \quad IC_{pt} \equiv \begin{bmatrix} \text{corr}(\tilde{f}_{t-0}^1, r_i) \\ \vdots \\ \text{corr}(\tilde{f}_{t-K}^K, r_i) \end{bmatrix} \]
Decomposing the ex-post IR

- Using this expression we can now consider the objective of the value added fund manager when selecting optimal factor weights

- The objective: select $\omega$ so that the ex-post IR (over the last $T$ periods) of the value added portfolio $h$ is maximised:

$$\max_{\omega} IR = \frac{\text{avg}(r_{pt})}{\text{std}(r_{pt})}$$

subject to $\sum_k \omega_k = 1$ and $0 \leq \omega_k \leq 1$

The expression for the average return can be approximated by

$$\text{avg}(r_{pt}) = \kappa(N-1)\sigma_r \omega' \sum_p \left( \delta^p \times IC_p^k \right) \quad \text{where} \quad IC_p^k = \text{avg}(IC_{pt}^k) \quad \delta_p^p \equiv \eta_p^p \beta_{p+1}^p$$

The expression for the portfolio variance is given by

$$\text{var}(r_{pt}) = [\kappa(N-1)\sigma_r]^2 \omega' \text{var}\left\{ \sum_p \left( \delta^p \times IC_p^k \right) \right\} \omega$$

To simplify, we make the following assumptions

$$\text{cov}(IC_p^k, IC_{p-j}^k) = \begin{cases} \sigma_{k,p}^2 & \forall j = 0 \\ 0 & \forall j > 0 \end{cases} \quad \text{cov}(IC_p^k, IC_{p-j}^l) = \begin{cases} \sigma_{k,l,p} & \forall j = 0 \\ 0 & \forall j > 0 \end{cases}$$

i.e. factor ICs are only contemporaneously correlated
Decomposing the ex-post IR

- The expression for the portfolio variance can therefore be approximated by

\[
\kappa(N - 1)\sigma_r^2 \omega' \left( \sum_{p=1}^{p^*} (\delta^{2p} \times \Omega_p) \right) \omega \quad \text{where} \quad \Omega_{p[i,j]} = \sigma_{i,j,p}
\]

The term \( \Omega_p \) represent the variance covariance matrix of the lagged factor ICs for lag \( p \), which we will assume can be consistently estimated using sample data, \( \hat{\Omega}_p \).

- Therefore the ex-post IR can be approximated by

\[
\max_\omega IR = \frac{\text{avg}(r_{pt})}{\text{std}(r_{pt})} = \frac{\omega' \sum_{p=1}^{p^*} (\delta^{p} \times IC_p)}{\sqrt{\left( \omega' \sum_{p=1}^{p^*} (\delta^{2p} \times \hat{\Omega}_p) \right) \omega}}
\]

which is of the form

\[
\omega' \mu_p^* = \sqrt{\omega' \Sigma_{p^*} \omega}
\]

- If the objective function is Gross IR maximisation, we could use this expression in the objective function to search for optimal factor weights, \( \omega \)

\[
\omega = \frac{\mu_{p^*}' \Sigma_{p^*}^{-1}}{\mu_{p^*}' \Sigma_{p^*}^{-1} \frac{1}{k}}
\]

- In Appendix 2, we consider the case when the objective function is Net IR
Simulation Study: Details of Data construction

- Using controlled experimental data we can better understand the impact of factor turnover on optimal alpha and portfolio construction.
- To do so, we would like to create data that possess salient characteristics of observed returns and factors typically used in the quant process.
- Key parameters

  **Market:** $N = 350, \ T = 100$

  **Risk Model:** Residual Risk estimated from a CAPM model based on FTSE 350 over 5 years ending 2008, cross sectional volatility estimated from returns

  **Alpha is constructed from 4 typical quant factors, with performance:**

<table>
<thead>
<tr>
<th>Factor (Type)</th>
<th>avg(IC)</th>
<th>std(IC)</th>
<th>1st order rho</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor 1 (Quality Type)</td>
<td>3.0%</td>
<td>4.0%</td>
<td>90.0%</td>
</tr>
<tr>
<td>Factor 2 (Earnings Revision Type)</td>
<td>4.0%</td>
<td>6.0%</td>
<td>50.0%</td>
</tr>
<tr>
<td>Factor 3 (Event Type Signal)</td>
<td>8.0%</td>
<td>6.5%</td>
<td>30.0%</td>
</tr>
<tr>
<td>Factor 4 (Momentum Type Signal)</td>
<td>5.0%</td>
<td>8.5%</td>
<td>70.0%</td>
</tr>
</tbody>
</table>

  **IC correlations 20% across factors**

  **Based on these ICs and Factors, returns for the stocks are generated**
Simulation Study: generating factors and returns

Given cross sectional return volatility, observed ICs and the 1st order correlation in factors we can produce simulated factors and returns

Generating the factors

Each period, \( t \), factors are generated using a non-linear least squares procedure which selects factor values that ensure: a cross sectional mean of zero, a cross sectional variance of 1, orthogonal factors, and the specified 1st order serial correlation

Generating the returns

Using the history of simulated ICs generated from estimated mean and covariance, cross sectional return volatility estimates and the generated factor scores; returns are generated based on the following cross-sectional regression model:

\[
\bar{r}_i = \bar{f}_i \beta_i + u_i
\]

\( \bar{f}_t \) is the NxK matrix of factor values at time, \( t \)

\[
\beta_i = (\bar{f}_t' \bar{f}_t)^{-1} \bar{f}_t' \bar{r}_t = IC_t \times \sigma_r \quad \text{since the factors are orthogonal}
\]

See appendix 3 for details on algorithm
Simulation Study: Observed Properties of Factors and Returns

- Using the calibrated parameters we can further investigate how factor turnover influences IC.

Factor Decay: Autocorrelation in Factor

- Factor 1 (Quality Type)

- Factor 2 (Earnings Revision Type)

- Factor 3 (Event Type Signal)

- Factor 4 (Momentum Type Signal)
Simulation Study: Results

- Factor decay (autocorrelation) structure has a clear impact on the IC decay

This is captured by the delta function which weights these lagged ICs according to the selected turnover penalty, \( \eta \).

It is this profile that we wish to incorporate into the factor weighting scheme when constructing alphas that account for the impact of turnover.
Simulation Study: Performance of Factor Weighting Scheme

- In understanding the impact of the weighting scheme in dealing with turnover we consider the following strategy:
  
  - We construct optimal factor weights over the sample period over a range of turnover penalties (eta) from eta = 0 (no turnover penalty) to 100 (high turnover penalty)
  - We then measure gross and net (IR) performance of the resulting alphas using the same objective function:

\[
\max_{h_t} \ h_t'\alpha_t - \frac{1}{2} \lambda h_t'\sum_i h_t
\]

for a range of transaction costs: 0bps to 150bps

- Factor weighting scheme outcomes for a range of penalties

<table>
<thead>
<tr>
<th>Factor 1 (Quality Type)</th>
<th>Eta = 0  (No Turnover Penalty)</th>
<th>Eta = 1</th>
<th>Eta = 7</th>
<th>Eta = 20</th>
<th>Eta = 50</th>
<th>Eta = 100 (High Turnover Penalty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor 1 (Quality Type)</td>
<td>11.4%</td>
<td>12.9%</td>
<td>24.1%</td>
<td>40.4%</td>
<td>47.5%</td>
<td>48.8%</td>
</tr>
<tr>
<td>Factor 2 (Earnings Revision Type)</td>
<td>28.0%</td>
<td>28.0%</td>
<td>26.0%</td>
<td>20.9%</td>
<td>18.5%</td>
<td>18.0%</td>
</tr>
<tr>
<td>Factor 3 (Event Type Signal)</td>
<td>45.2%</td>
<td>41.9%</td>
<td>25.7%</td>
<td>14.7%</td>
<td>11.0%</td>
<td>10.3%</td>
</tr>
<tr>
<td>Factor 4 (Momentum Type Signal)</td>
<td>15.4%</td>
<td>17.1%</td>
<td>24.2%</td>
<td>23.9%</td>
<td>23.0%</td>
<td>22.9%</td>
</tr>
</tbody>
</table>
Simulation Study: Performance of Factor Weighting Scheme

- Factor weighting scheme outcomes for a range of penalties

As turnover penalties increase (transaction costs matter), factors with slower decay progressively obtain a higher weight (Factor 1)

Factors with high factor decay progressively obtain a lower weight (Factor 3)
To gauge performance we look at relative net IR

In absence of any transaction costs, the most optimal scheme is one which sets eta to zero.

However as transaction costs increase, it is clearly sub-optimal to construct alphas that ignore eta;

At 100bps, the cost of ignoring turnover penalties (eta of 50) is almost 1.0 in terms of net IR
Conclusions and Extensions

Summary

- We have prescribed a framework which is an extension of earlier models designed to form optimal factor weights in the presence of turnover and transaction costs.

- By introducing turnover considerations in the portfolio construction process, we explicitly acknowledge the impact of alpha memory of the portfolios.

- In particular we find that when the penalty to turnover is high, we prefer factors that deliver not stable performance in the short run but have slow information decay as captured by the autocorrelation of the alpha factors.

- We find that failing to incorporate the impact of factor based turnover produces sub-optimal weighting schemes and alphas in realistic portfolio settings.

- The framework adopted here and applied to the problem of transaction costs have potentially many other applications where more general utility functions are warranted in order to capture realistic portfolio construction.
Appendix 1 Calculation of risk adjusted IC

- Employing the definition of sample covariance
  \[
  \text{cov} \left( \frac{\bar{\alpha}_i}{\sigma_i}, \frac{\bar{r}_i}{\sigma_i} \right) = \frac{1}{N-1} \sum_{i=1}^{N} \left( \frac{\bar{\alpha}_i}{\sigma_i} \cdot \frac{\bar{r}_i}{\sigma_i} \right) - \frac{1}{N} \sum_{i=1}^{N} \left( \frac{\bar{\alpha}_i}{\sigma_i} \right) \cdot \frac{\sum_{i=1}^{N} \left( \frac{\bar{r}_i}{\sigma_i} \right)}{N}
  \]

- Therefore
  \[
  \sum_{i=1}^{N} \left( \frac{\bar{\alpha}_i}{\sigma_i} \cdot \frac{\bar{r}_i}{\sigma_i} \right) = (N-1) \left\{ \text{cov} \left( \frac{\bar{\alpha}_i}{\sigma_i}, \frac{\bar{r}_i}{\sigma_i} \right) + \frac{1}{N} \sum_{i=1}^{N} \left( \frac{\bar{\alpha}_i}{\sigma_i} \right) \cdot \frac{\sum_{i=1}^{N} \left( \frac{\bar{r}_i}{\sigma_i} \right)}{N} \right\}
  \]

- Given that by construction through arbitrary scaling of the returns
  \[
  \sum_{i=1}^{N} \left( \frac{\bar{r}_i}{\sigma_i} \right) = 0
  \]
  \[
  \sum_{i=1}^{N} \left( \frac{\bar{\alpha}_i}{\sigma_i} \cdot \frac{\bar{r}_i}{\sigma_i} \right) = (N-1) \text{cov} \left( \frac{\bar{\alpha}_i}{\sigma_i}, \frac{\bar{r}_i}{\sigma_i} \right)
  \]
  \[
  = (N-1) \text{corr} \left( \frac{\bar{\alpha}_i}{\sigma_i}, \frac{\bar{r}_i}{\sigma_i} \right) \cdot \text{std} \left( \frac{\bar{\alpha}_i}{\sigma_i} \right) \cdot \text{std} \left( \frac{\bar{r}_i}{\sigma_i} \right)
  \]
Appendix 2: Further Extensions- Ex-post Net IR Formulation

- If transaction costs matter, then it is more natural to use Net IR rather than Gross IR in the objective function:

\[
\max_{\omega} IR = \frac{\text{avg}(r_{pt}) - \text{avg}(TC_{pt})}{\text{std}(r_{pt})}
\]

- To obtain an estimate we approximate the transaction cost of the portfolio at time \( t \) with

\[
TC_{pt} = \eta(h_t - h_{t-1})'I(h_t - h_{t-1}) = \eta \sum_i (h_{it} - h_{it-1})^2
\]

\[
= \eta \sum_i (h_{it}^2 + h_{it-1}^2 - 2h_{it}h_{it-1})
\]

\[
= 2\eta(\frac{\sigma_A^2}{\sigma^2} - \sum_i h_{it}h_{it-1}) \quad \text{since the portfolio variance} \quad \sigma_A^2 = \sum_i h_{it}^2 \sigma^2 \quad \Rightarrow \sum_i h_{it}^2 = \frac{\sigma_A^2}{\sigma^2}
\]

- Using once again the definition for the optimal portfolio, \( h \), and recursively substituting

\[
\sum_i h_{it}h_{it-1} = \sum_i \left(\frac{\alpha_{it} + \eta h_{it-1}}{\beta}\right) \left(\frac{\alpha_{it-1} + \eta h_{it-2}}{\beta}\right)
\]

\[
\approx \frac{1}{\beta^2} \sum_i (\alpha_{it} \alpha_{it-1}) \quad \text{assuming cross product terms are negligible}
\]
Appendix 2: Ex-post Net IR Formulation

- Noting again that the alpha is a composite of $K$ orthogonal factor scores:
  \[ \sum_i \alpha_{it} \alpha_{it-1} = \kappa^2 \sum_i \left( \sum_{k=1}^K \omega_k \tilde{f}_{it} \right) \left( \sum_{k=1}^K \omega_k \tilde{f}_{it-1} \right) \]

- And assuming that the factors have the following autocorrelation structure
  \[ \text{corr}(\tilde{f}_{it}^k, \tilde{f}_{it-1}^j) = \begin{cases} \rho_{kt} & \forall \ j = k \\ 0 & \forall \ j \neq 0 \end{cases} \]

- Then it can be shown that
  \[ \sum_i \alpha_{it} \alpha_{it-1} \approx \kappa^2 (N - 1) \left( \sum_{k=1}^K \omega_k^2 \rho_{kt} \right) \]

- The transaction cost of the portfolio at time $t$ can then be approximated with
  \[ TC_{pt} \approx 2\eta \left( \frac{\sigma_A^2}{\sigma} - \sum_i h_{it} h_{it-1} \right) \]
  \[ \approx 2\eta \left( \frac{\sigma_A^2}{\sigma} - \frac{\kappa^2 (N - 1)}{\beta^2} \sum_k \omega_k^2 \rho_{kt} \right) \]

- The average can be approximated by
  \[ \text{avg}(TC_{pt}) \approx 2\eta \left( \frac{\sigma_A^2}{\sigma} - \frac{\kappa^2 (N - 1)}{\beta^2} \sum_k \omega_k^2 \text{avg}(\rho_{kt}) \right) \]
Appendix 2: The Final Objective Function

- Including turnover results in an objective function with three broad components:
  \[
  \max_{\omega} IR = \frac{\text{avg}(r_{pt}) - \text{avg}(TC_{pt})}{\text{std}(r_{pt})}
  \]

- Average Realized Portfolio Alpha is an exponentially weighted average of the lagged factor IC’s
  \[
  \text{avg}(r_{pt}) = \kappa(N-1)\omega' \sum_p \left( \delta^p \times \bar{IC}_p \right)
  \]

- Average Active Risk of the Portfolio is also an exponentially weighted average of lagged factor IC variance-covariance matrices
  \[
  \text{var}(r_{pt}) = [\kappa(N-1)]^2 \omega' \left\{ \sum_{p=1}^{p^*} \left( \delta^2 \times \Omega_p \right) \right\} \omega
  \]

- Average Transaction/Market Impact Cost of the Portfolio is governed by both the active risk of the portfolio and the degree of the decay in the factor values (as captured by their first order autocorrelation)
  \[
  \text{avg}(TC_{pt}) \approx 2\eta \left( \frac{\sigma_A^2}{\sigma} - \frac{\kappa^2(N-1)}{\beta^2} \sum_k \omega_k^2 \text{avg}(\rho_{kt}) \right)
  \]
Appendix 3: Algorithm to generate factors and returns

- **Given cross sectional return volatility, observed ICs and the 1st order correlation in factors we can produce simulated factors and returns**

- **Generating the factors (given $\rho_k$ )**

  Each period, $t$, using a non-linear least squares procedure, we select the set of factor values $\tilde{f}_t^k$ for each factor $j = 1, 4$ so that the residuals from the following set of conditions are minimised:

  - Cross sectional mean is zero: $\tilde{f}_t^k \cdot 1_N$
  - Cross sectional variance is 1: $\tilde{f}_t^k \cdot \tilde{f}_t^k - (N-1)$
  - Factors are orthogonal: $\tilde{f}_t^k \cdot \tilde{f}_t^j$
  - Factor has a serial correlation: $e_t^k, \tilde{f}_t^k$ where $e_t^k = \tilde{f}_t^k - \rho_k \tilde{f}_{t-1}^k$

- **Generating the returns (given $\sigma_r, \tilde{f}_t$ and IC$_t$ )**

  Using a nonlinear least squares procedure, returns are generated each period so that residuals from the following set of conditions are minimised:

  - Cross sectional regression residual average is zero: $u_t \cdot 1_N$ where $u_t = r_t - \sigma_r \tilde{f}_t \cdot $ IC$_t$
  - The residuals are uncorrelated with the factors and their lags: $u_t \cdot \tilde{f}_{t-k}^j$
References


