Extensions in Jacobian algebras via punctured skein relations

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① Cluster algebras of surface type and skein relations

Fomin Shapiro Thurston defined cluster algebras of surface type 2008

Def 1: (S,M) marked surface

S compact Riemann surface, oriented (pos.) remove some disks (boundaries)
Finite set of marked points (each boundary contains at least one) M
$P \subset M$ marked points in S° → punctures

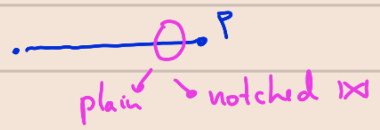
Def 2: (gen) tagged arcs

$m, m' \in M$

arc   isotopy class of a curve
γ not self crossing, not contractible to a boundary segment or a marked point
self crossings

generalized arc

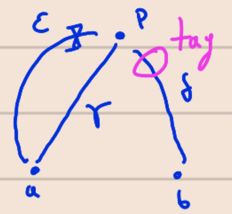
tagged arcs



Def 3: crossing number $e(\alpha,\beta)$

α,β "cross" in S° then $e(\alpha,\beta) = \#$ crossings (in minimal pos)

α,β meet at a puncture



$$e(\gamma,\delta) = \begin{cases} 0 & \text{same tag at } p \\ 1 & \neq \text{tag at } p \end{cases}$$

$\gamma \not\sim \delta \quad \gamma \sim \varepsilon$

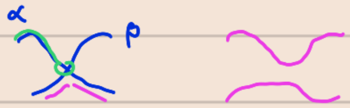
Def 4: Triangulation, cluster algebra $\mathcal{A}(S,M)$

T a triangulation is a maximal set of tagged arcs $T = \{\gamma_1, \dots \gamma_n\}$
st $e(\gamma_i,\gamma_j) = 0$ $i \neq j$. The cluster algebra $\mathcal{A}(S,M)$ is defined from an initial set $x_{\gamma_1} \dots x_{\gamma_n}$ → all the other variables are obtained by flips.

Remark: We will be in the trivial coeff. setting

Def 5: (Some) skein relations: $\underline{\alpha},\underline{\beta}$ (gen) tagged arcs

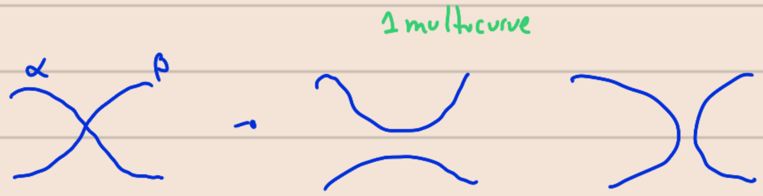
NOTATION: $\alpha^+\beta$



go through α and turn left at crossing.

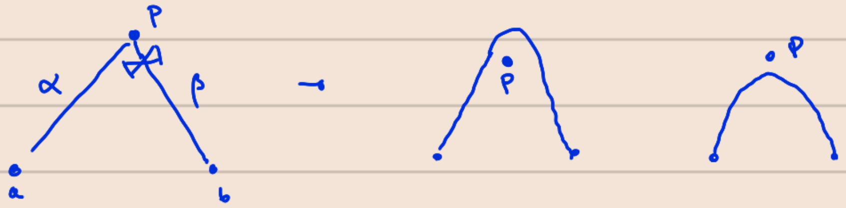
$e(\alpha,\beta)=1$

(Sk 1)



$$x_\alpha \cdot x_\beta = \underline{x_{\alpha^+\beta}} + x_{\beta^+\alpha}$$

(Sk 2)



$$x_\alpha \cdot x_\beta = x_{\alpha^+\beta} + x_{\beta^+\alpha}$$

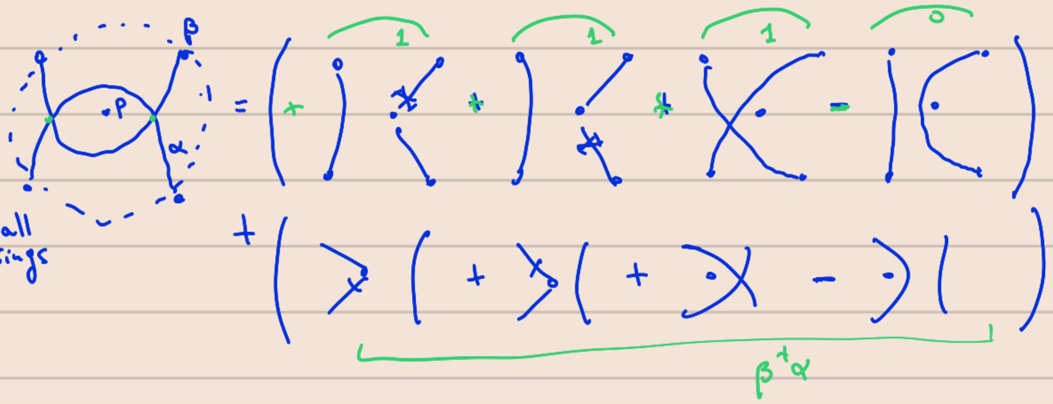
1 multicurve

4 multicurves

$\alpha^+\beta = \{c_1,\dots,c_4\}$

$e(\alpha,\beta)=2$

(Sk 3)



zoom contains all crossings

Remark: (a) these relations were studied by Musiker-Schiffler-Williams, prove positivity for $\mathcal{A}(S,M)$, find basis.

(b) it makes sense the notation $\alpha^+\beta$ also when $e(\alpha,\beta)=2$

(c) $\alpha^+\beta$ is a <u>set</u> of multicurves (just one multicurve in the unpunctured setting)

(d) skein relations are ... uncomplete description of skein rel. in the coeff (two basis proposed by MSW) // see Daniel.

② Cluster category $\mathcal{C}(S,M)$, Jacobian algebras and a theorem by Çanakçi-Schroll

$\mathcal{C}(S,M)$ in surface case BZ unpunctured, QZ punctured case

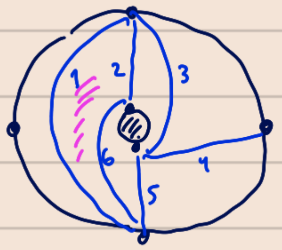
triangulated 2-CY category, shift [1]

Correspondences I

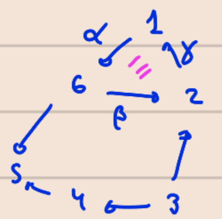
- gen tagged arcs (loops) γ $\longleftrightarrow$ ind. objects X_γ
- $e(\alpha,\beta)$ $\longleftrightarrow$ $\dim_{\mathbb{C}} \mathrm{Ext}^1(X_\alpha, X_\beta)$
- T triangulation $\longleftrightarrow$ cluster tilting obj
- $C = \{\gamma_1, \dots, \gamma_t\}$ $\longleftrightarrow$ $X_{\gamma_1} \oplus \dots \oplus X_{\gamma_t}$.

Examples I

(a)



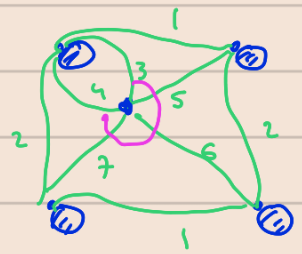
Q_T



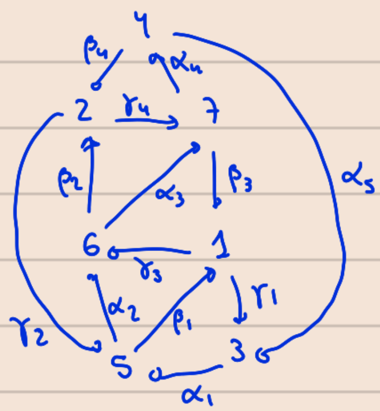
$\mathrm{Jac}(Q_T, W_T)$: $\mathbb{C}Q_T \big/ \langle \alpha\beta, \beta\gamma, \gamma\alpha \rangle$

(b)

$\rightsquigarrow$



$W_T = \circlearrowleft - \sum \Delta$



$$\mathrm{Jac}(Q_T, W_T) = \frac{\mathbb{C}Q_T}{\langle \alpha_{i+1} \cdots \alpha_{i+3} - \beta_i \gamma_i,\ \alpha_1 \cdots \alpha_3 \rangle}$$

$i = 1, \dots, 4$

Remark: well studied by Fragoso,

unpunctured case gentle algebras

punctured case not gentle. (nor skew gentle most of them)

- $T \to Jac(Q_T, \omega_T)$

Correspondence II:

- (gen) tagg arcs (loops) $\gamma \longrightarrow$ indec. rep M_γ (modules)

- multicurve $C = \{\gamma_1, \ldots, \gamma_t\} \to M_{\gamma_1} \oplus \cdots \oplus M_{\gamma_t}$.

Theorems [CS - '17] Unpunctured case

(a) let α, β be (gen) arcs s.t. $e(\alpha,\beta) = 1$
then $\circledast$ $X_\alpha \to X_{\alpha+\beta} \to X_\beta \to X_\alpha[1]$
is a basis for $Ext^1_{\mathcal{C}}(X_\beta, X_\alpha)$

$\Bigg\}$ α, β as in (Sk 1)

(b) given a T, $\alpha, \beta \notin T$ M_α crosses M_α in $\left\{\begin{array}{l} \text{- module} \\ \text{- arrow} \\ \text{- 3 cycle} \end{array}\right.$ $\circledast$ induces a non-split extension.
does not

$\circledast$ preserve dim

Examples II:

③ Main results

(S,M) with punctures, α,β ~~(gen)~~ tagged arcs, in position as in (Sk1 - Sk3)

Theorems [D- , 2021]

(a) there are non split Δ's in $\mathcal{C}(S,M)$ of the form $\underline{e(\alpha,\beta)=2}$

$$\heartsuit \quad X_\alpha \to X_c \to X_\beta \to X_\alpha[1]$$

for all c in $\underline{\alpha^+\beta}$

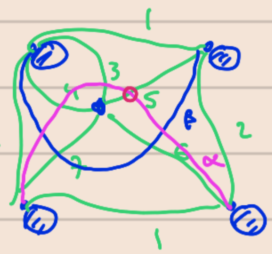
$|\alpha^+\beta|=4$, any 2 give $\underline{\text{a basis}}$

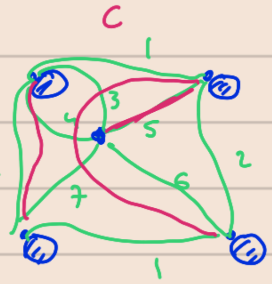
(b) Given a triangulation T, $\mathrm{Jac}(Q_T, W_T)$ the tuple $\heartsuit$ induces a $\underline{\text{non-split ext}}$. iff.

$$\underline{\dim}\, M_\alpha + \underline{\dim}\, M_\beta = \underline{\dim}\, M_c$$

Example II (b):

looks like (Sk3)





X_c is a middle term for

$$X_\alpha \to X_c \to X_\beta \to X_\alpha[1]$$

but. $\not\exists \quad 0 \to M_\alpha \to M_c \to M_\beta \to 0$

lose crossings

Final remark (?): "all" crossings. → classifying this!

non gentle setting → using "Arc reps"

Dowel Labardini 2009

Domínguez →

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