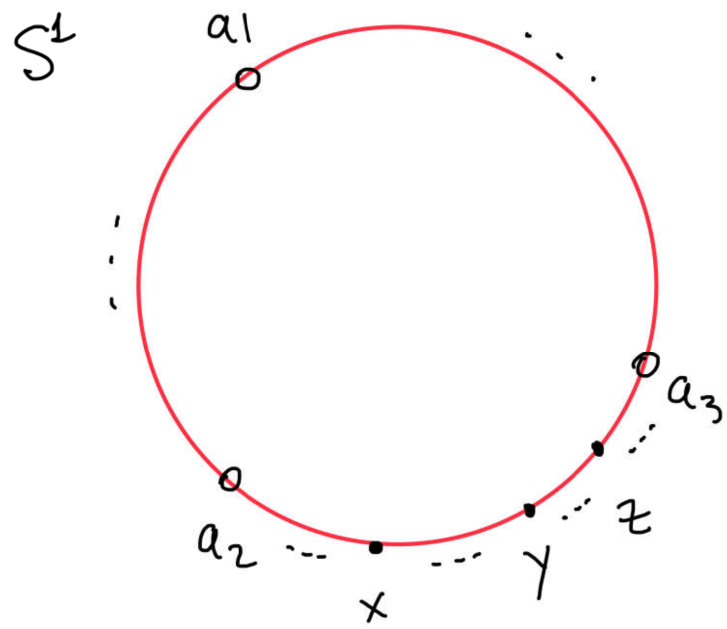
t-structures and thick subcategories of discrete cluster categories

(jt. w. Sira Gratz)

[IT] k - field



$Z \subset S^1$

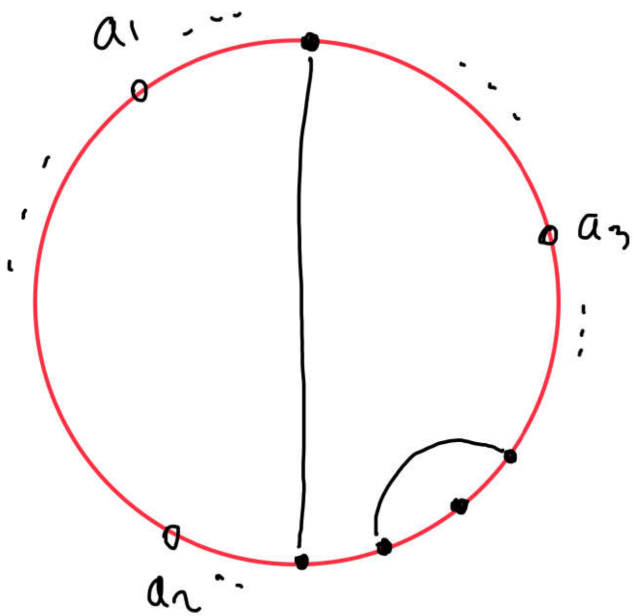
discret, infinite

$|\bar{Z} \setminus Z| = n$

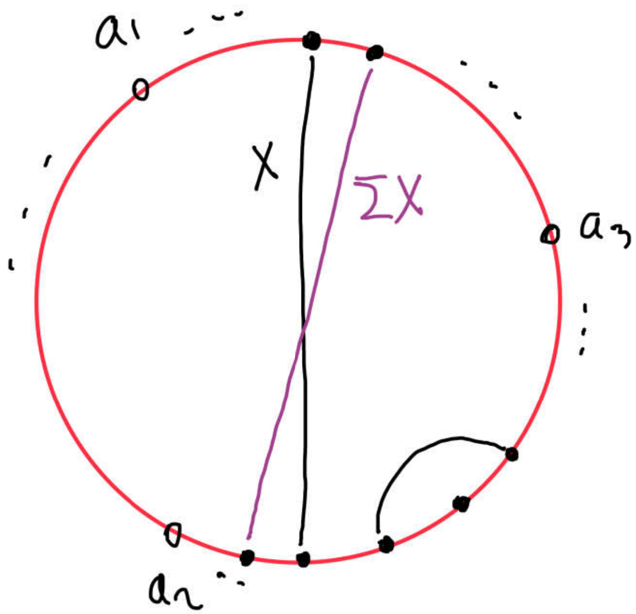
all limit points are two-sided limit points

$\bar{Z}$ has a cyclic ordering

$x < y < z$



Arcs are 2-element subsets of Z, such that the end-points are not immediate successor or predecessors



$\rightsquigarrow$ discrete cluster category $\mathcal{C}(\mathcal{Z})$

k-linear, triangulated, 2-CY, Krull-Schmidt

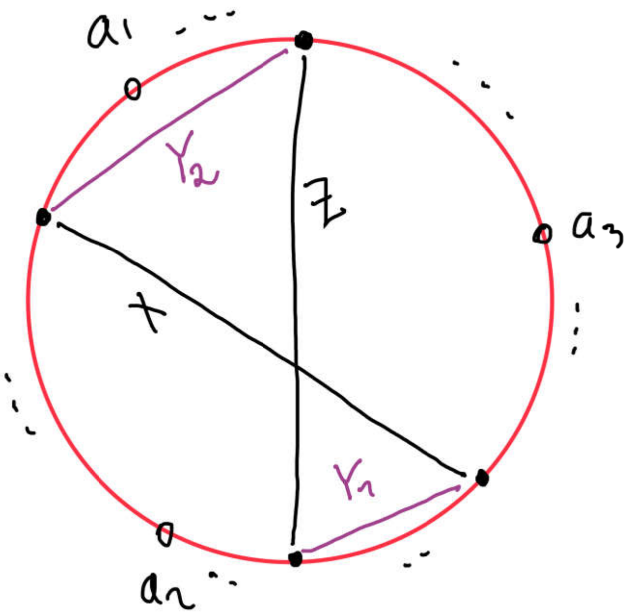
Indecomposable objects = arcs

Shift = clockwise rotation

$$\operatorname{Hom}(X, \Sigma Y) = \begin{cases} k, & X \text{ \& } Y \text{ cross} \\ 0, & \text{otherwise} \end{cases}$$

1 accumulation point $\Rightarrow$

$$\mathcal{C}(\mathcal{Z}) = D^f(k[t]), \quad \deg(t) = 1$$



$X \longrightarrow Y_1 \oplus Y_2 \longrightarrow Z \longrightarrow \Sigma X$

[GHJ] cluster-tilting subcategories ⇝
certain infinite triangulations ...

$\mathcal{T}$ – triangulated category

$\mathcal{X}$ – thick subcategory in $\mathcal{T}$, if $\mathcal{X}$ is a triangulated subcategory closed under direct summands

form a lattice under inclusion

classified in many cases:

[H, N] perf of a commut. noetherian ring

[BCR] $\underline{\text{mod}}$-kG

[B] tensor. triangulated cat

discrete classifications

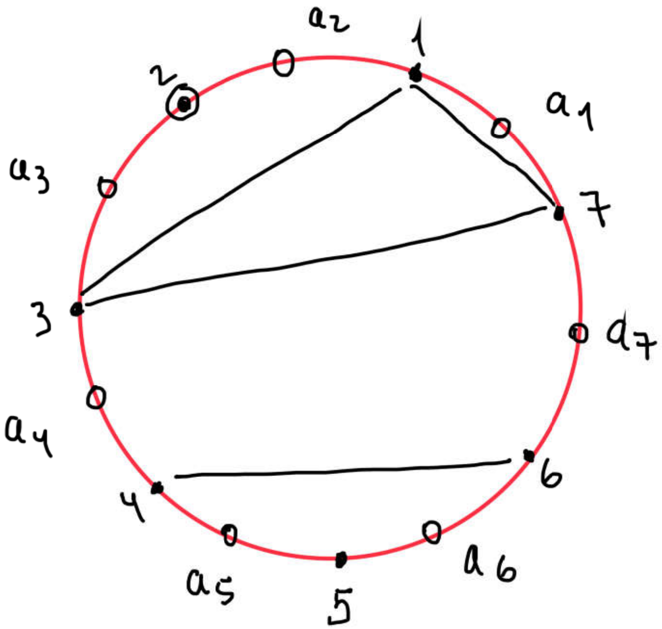
[IT] $D^b(kQ)$ Q – Dynkin

[B] $D^b(A)$ A – derived discrete

[GS] $D^b(\mathrm{gr}\, k[x]/(x^2))$

Theorem (Gratz-Z)
the lattice of thick subcategories of $C(\mathcal{Z})$ is isomorphic to the lattice of non-exhastive non-crossing partitions of $[n]$

A non-exhaustive non-crossing partition $\mathcal{P}$ of $[n]$ is a collection of blocks $\{B_i \mid i \in I\}$ such that the convex hulls of the blocks do not cross (= non-crossing partition of a subset of $[n]$ with the induced linear order, the subset may be empty)

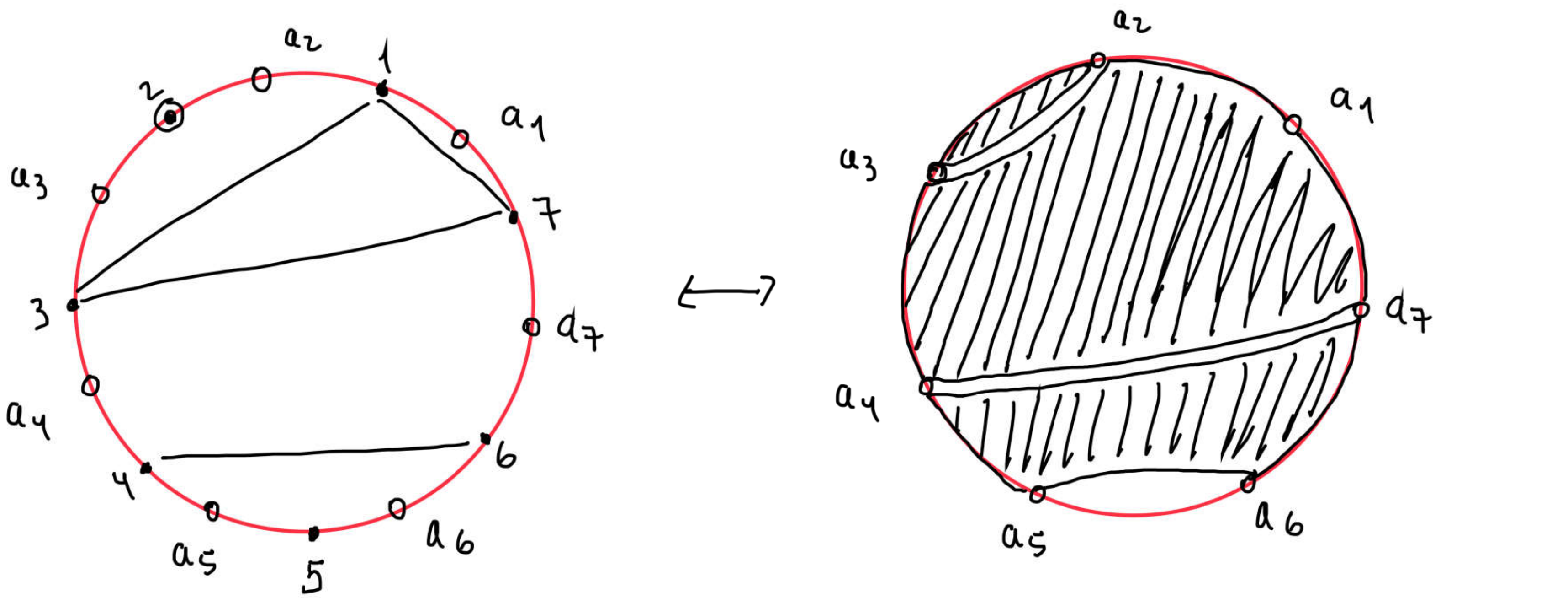


order is the same as for non-crossing partitions:

$$\mathcal{P} = \{ B_m \mid m \in I \}, \quad \mathcal{P}' = \{ B'_m \mid m \in I' \}$$

$$\mathcal{P} \leq \mathcal{P}', \quad \text{if} \quad \forall \; B_m \in \mathcal{P} \;\; \exists \;\; B'_\ell \in \mathcal{P}' : \; B_m \leq B'_\ell$$

$$\mathcal{P} \wedge \mathcal{P}' = \{ B_m \cap B'_\ell \mid m \in I, \; \ell \in I', \; B_m \cap B_\ell \neq \emptyset \}.$$



Blocks = collections of intervals connected by arcs in $\mathfrak{X}$

All arcs with endpoints in the intervals cor. to some block t-structures:

$(\mathcal{X}, \mathcal{Y})$ - pair of full subcategories of $\mathcal{T}$

- $\mathrm{Hom}(\mathcal{X}, \mathcal{Y}) = 0$
- $\mathcal{X} * \mathcal{Y} = \mathcal{T}$ $\quad \big(\forall\ Z \in \mathcal{T}\ \exists\ X \in \mathcal{X}, Y \in \mathcal{Y} :\ X \to Z \to Y \to \big)$
- $\Sigma \mathcal{X} \subseteq \mathcal{X}$

approximations w.r.to $\mathcal{X}$ & $\mathcal{Y}$

$\mathcal{H} := \mathcal{X} \cap \Sigma \mathcal{Y}$ is an abelian category

Ex: $\mathcal{T} = D^b(A)$, A - fin. dim. k-algebra, $(D^{\leq 0}, D^{\geq 1})$

standard t-structure

$$D^{\leq 0} = \{ X \in D^b(A) \mid H^i(X) = 0\ \forall\ i > 0 \}$$
$$D^{\geq 1} = \{ X \in D^b(A) \mid H^i(X) = 0\ \forall\ i < 1 \}$$

$$\begin{tikzcd}
\cdots \arrow[r] & X^{-2} \arrow[r] \arrow[d] & X^{-1} \arrow[r] \arrow[d] & \ker d^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots & \in D^{\leq 0} \\
\cdots \arrow[r] & X^{-2} \arrow[r] \arrow[d] & X^{-1} \arrow[r] \arrow[d] & X^0 \arrow[r, "d^0"] \arrow[d] & X^1 \arrow[r] \arrow[d] & X^2 \arrow[r] \arrow[d] & \cdots & \in D^b(A) \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X^0/\ker d^0 \arrow[r] & X^1 \arrow[r] & X^2 \arrow[r] & \cdots & \in D^{\geq 1}
\end{tikzcd}$$

$\mathcal{H} \cong \mathrm{mod}\text{-}A$.

For simplicity assume $(\mathcal{X}, \mathcal{Y})$ is non-degenerate, i.e.

$$\bigcap_{n\in\mathbb{Z}} \Sigma^n \mathcal{X} = 0 = \bigcap_{n\in\mathbb{Z}} \Sigma^n \mathcal{Y}$$

Theorem (Gratz - Z): there is a bijection

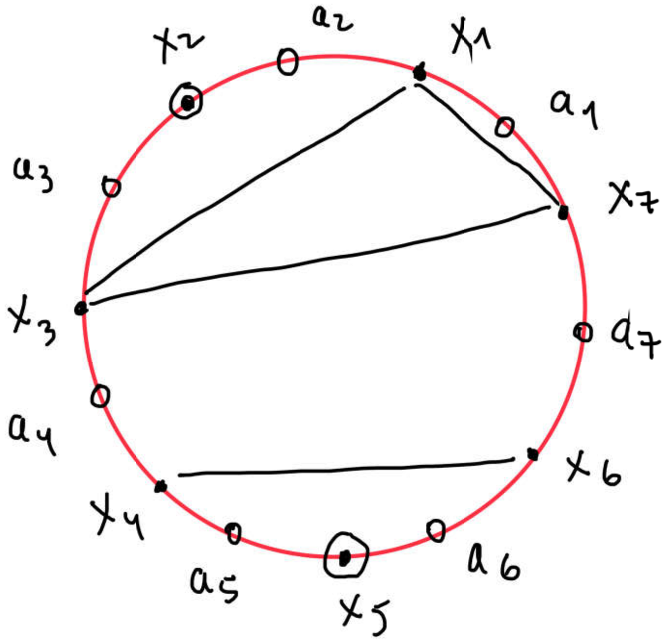
$$\left\{ \begin{array}{c} \text{non-degenerate} \\ \text{t-structures on} \\ C(\mathcal{Z}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathcal{Z}\text{-decorated} \\ \text{non-crossing partitions} \\ \text{of } [n] \end{array} \right\}$$

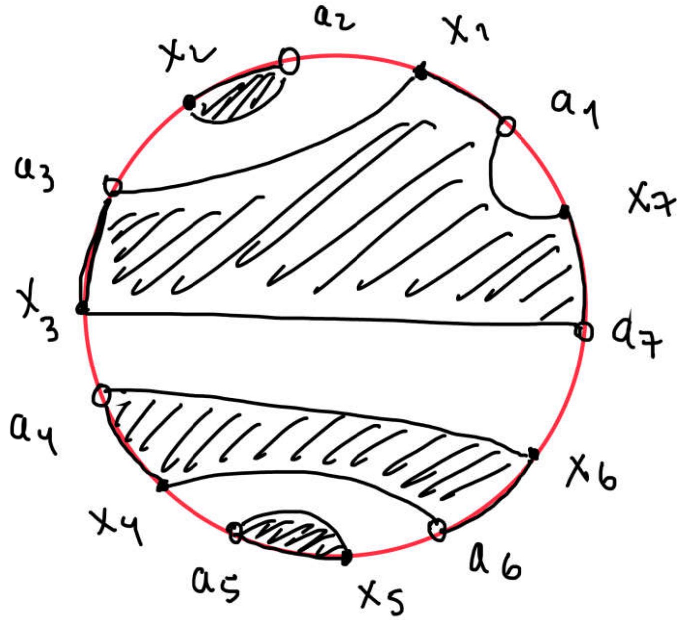
A $\mathcal{Z}$-decorated non-crossing partition of $[n]$ is a non-crossing partition of $[n]$ together with $x = (x_1, \dots x_n)$, $a_i < x_i < a_{i+1}$

Rem: • More general version with all t-structures is true, some decorations can be in $\overline{\mathcal{Z}}$ depending on the partition.

• $n = 1$, the classif was obtained by Ng

$(\mathcal{P}, x)$ $\longleftrightarrow$ $(\mathcal{X}, \mathcal{Y})$



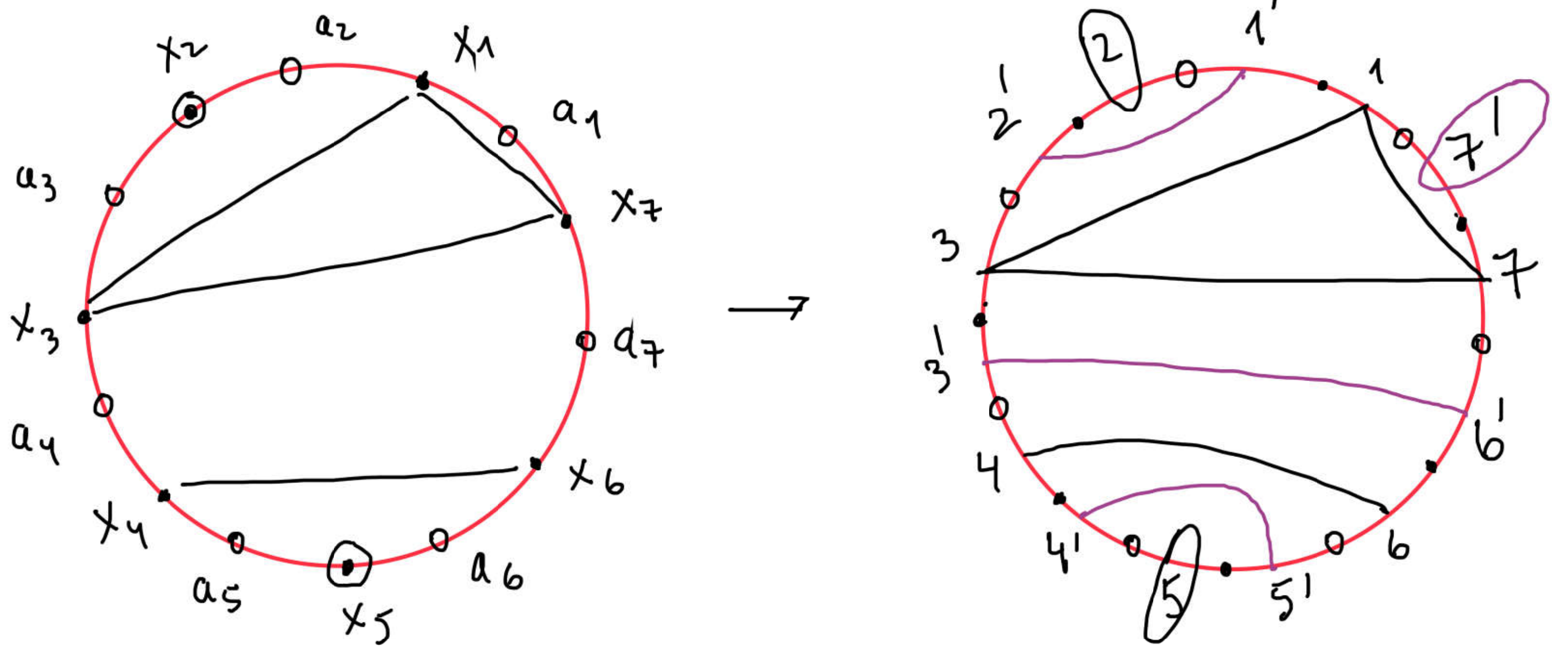


Blocks = collections of intervals connected by arcs in $\mathcal{X}$; x_i = max. elem in (a_i, a_{i+1}) with an arc ending in x_i

Arcs between the intervals $(a_i, x_i]$ for each block give an aisle $\mathcal{X}$ of a non. deg. t-st. $(\mathcal{X}, \mathcal{Y})$

Rem: proof uses [GHJ]

Kreweras complement – antiisomorphism of the lattice of non-crossing partitions

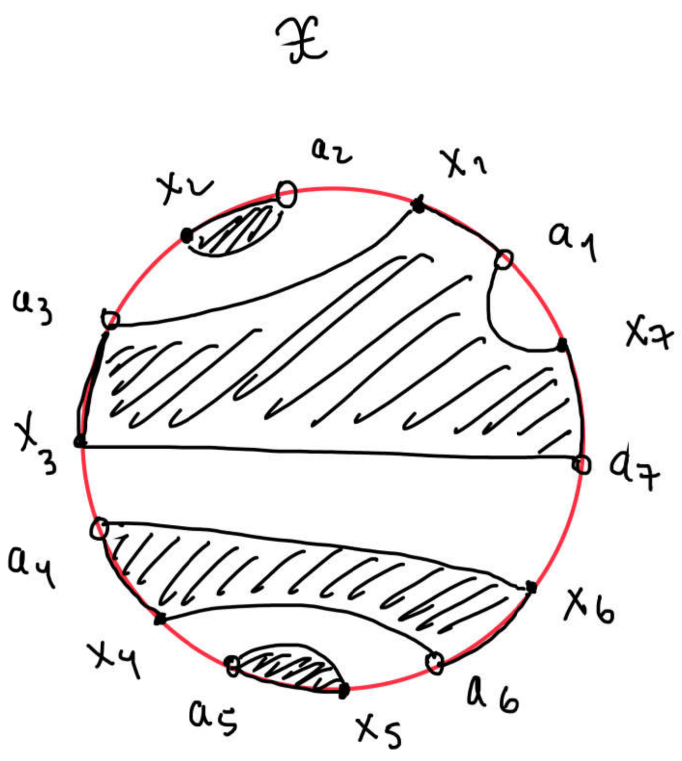


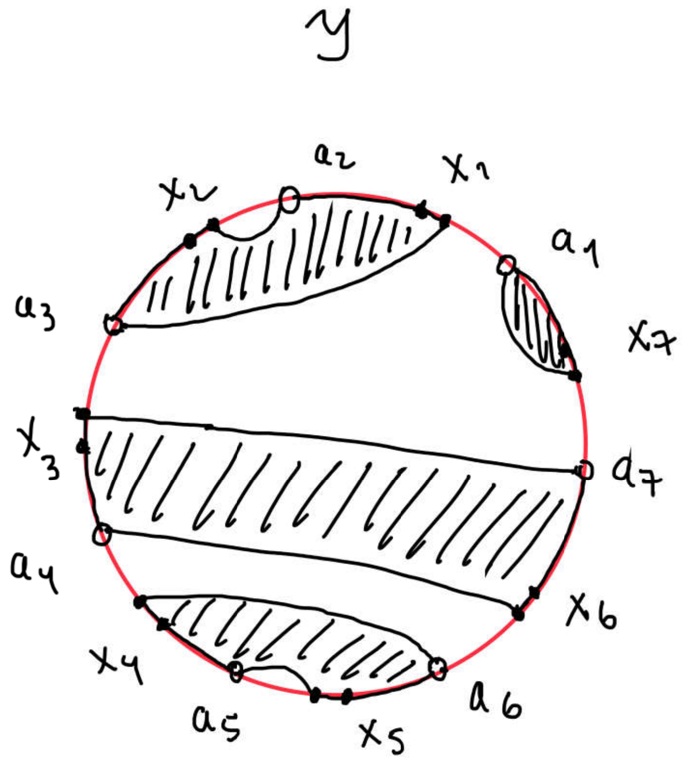
$P = \{ \{1,3,7\}, \{2\}, \{4,6\}, \{5\} \}$ $\qquad$ $P^c = \{ \{1',2'\}, \{3',6'\}, \{4',5'\}, \{7'\} \}$

unique maximal noncrossing partition of $\{1',2',\dots,n'\}$:

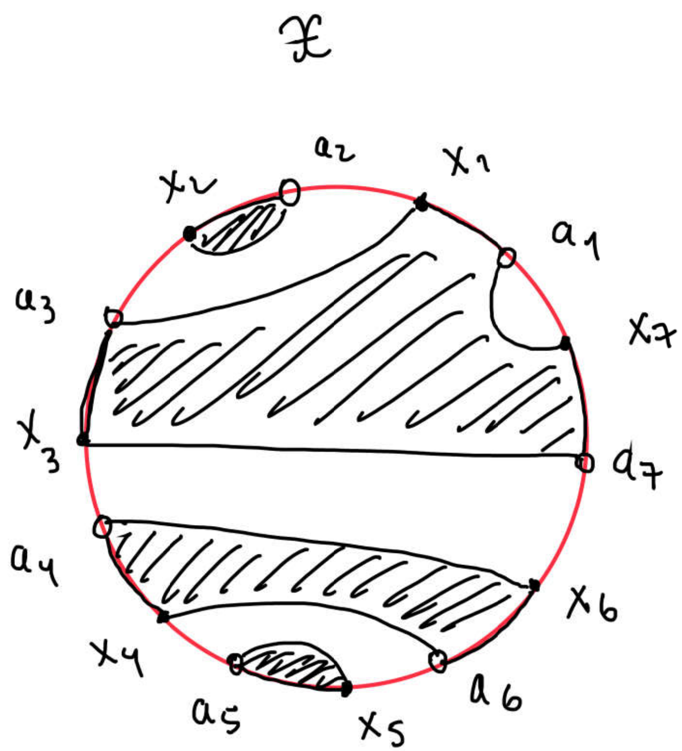
$P \cup P^c$ is a non-crossing part. of $[2n]$

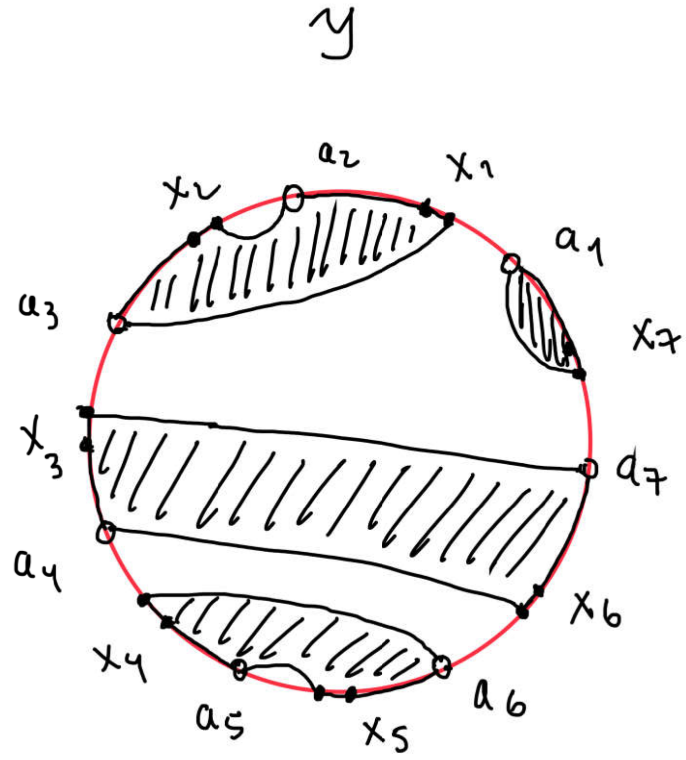
Proposition (Gratz - Z): Let $(\mathcal{X}, \mathcal{Y})$ be a non-degenerate t-structure in $C(\mathcal{Z})$, with the aisle $\mathcal{X}$ corresponding to $(\mathcal{P}, x)$, then its co-aisle $\mathcal{Y}$ corresponds to $(\mathcal{P}^c, y)$, $y_i = x_i^-$ in the following way:





Arcs with endpoints in intervals $[y_i, a_{i+1})$ for each block of $\mathcal{P}^c$

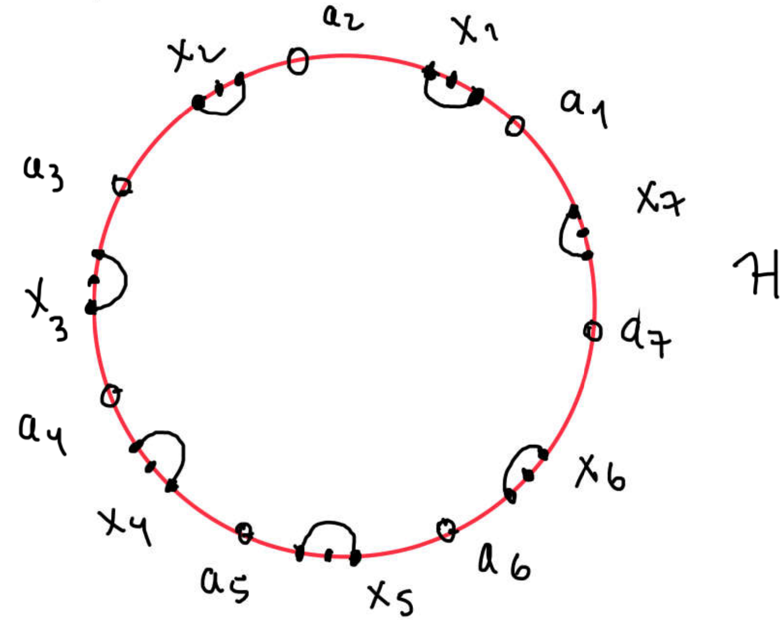


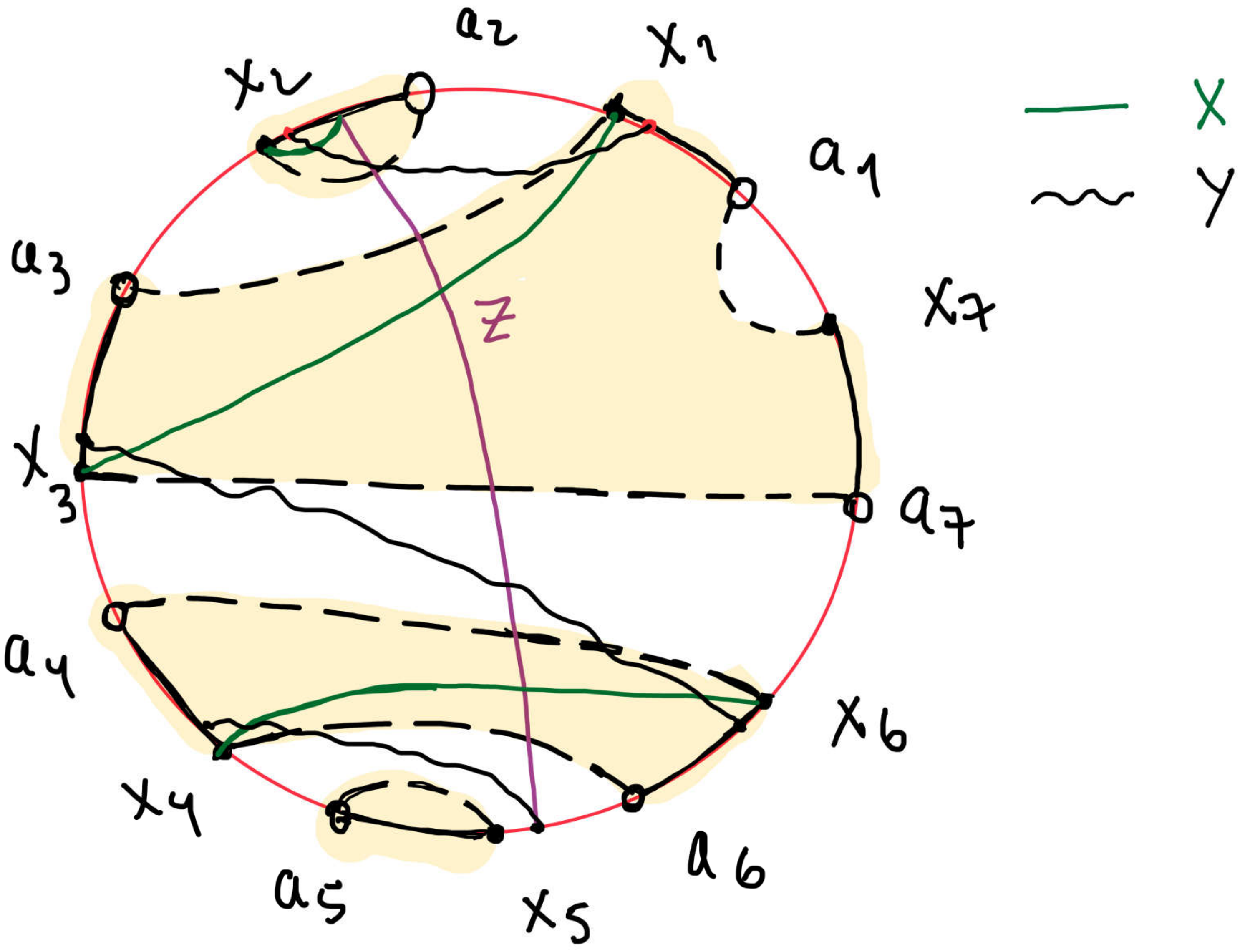


the heart $\mathcal{H} = \mathfrak{X} \cap \Sigma \mathcal{Y}$ correspond to the arcs

$\{x_i, x_i^{(-2)}\}$, $i = 1, \cdots n$

$$\mathcal{H} \cong \mathrm{mod}(\underbrace{k \times \cdots \times k}_{n\text{-times}})$$





Approximation triangles $X \to Z \to Y \to$

w.r.t. $(\mathcal{X}, \mathcal{Y})$ $\forall$ indecomposable $Z \in \mathcal{C}(\mathcal{Z})$

Theorem (Gratz–Z): the set of t-structures of $C(\mathbb{Z})$ forms a lattice under inclusion of aisles

$(\mathcal{X}, \mathcal{Y}), (\mathcal{X}', \mathcal{Y}')$ - two (non-degenerate) t-structures corresponding to $(\mathcal{P}, x)$ and $(\mathcal{P}', x')$

$\mathcal{X} \subseteq \mathcal{X}' \iff \mathcal{P} \leq \mathcal{P}'$ and $a_i \leq x_i \leq x_i' \leq a_{i+1}$

$\mathcal{X} \cap \mathcal{X}'$ is the aisle of the t-structure, corresponding to $(\mathcal{P} \wedge \mathcal{P}', \min\{x, x'\})$

$\min\{x, x'\} = \min\{x_1, x_1'\}, \dots, \min\{x_n, x_n'\}$

This situation is quite rare

Some conditions were studied in [B, BPP]