OPEN PROBLEMS IN RIGIDITY

GEOMETRIC RIGIDITY WORKSHOP, 5TH-7TH JUNE, 2016

Abstract. A collection of open problems that were described in or arose from the problem session in Lancaster on June 5th 2016. The problems are listed in the order they were sent to me. If you would like to add a problem to the list then please e-mail me.

1. Stephen Power

1. Is there a nontrivial continuous flex of the kagome framework which is bounded in the sense that each joint moves no further than a fixed distance (say 1 unit) from its starting position ? Note that in view of the linear subframeworks no position in such a motion can be periodic. (Stephen Power. Posed in a 4 minute talk (and video), Banff, July 2012.)

2. Does the Sierpinski string-node mesh have a continuous, string-length preserving non-crossing flex ? (Stephen Power and Bernd Schulze [13])

2. Robert Connelly

General problem: Determine when a bar and joint framework in $\mathbb{R}^1$ is globally rigid in $\mathbb{R}^2$.

Specific problem: Find a better proof that $K_{3,3}$ in $\mathbb{R}^1$ is globally rigid in $\mathbb{R}^2$. The proof in [1] uses the topological non-embeddability of $K_{3,3}$, which is strange.

3. Derek Kitson

1. *Rigidity for the ℓ^∞-norm.* Let G be a simple graph which is an edge-disjoint union of d spanning trees $T_1, \dots, T_d$.
   (a) Does there exist a placement of G in $(\mathbb{R}^d, \|\cdot\|_\infty)$ such that the induced monochrome subgraphs of G are precisely $T_1, \dots, T_d$?
   (b) Does there exist an isostatic placement of G in $(\mathbb{R}^d, \|\cdot\|_\infty)$?

   A positive answer to (a) would imply a positive answer to (b). If $d = 2$ then the answer to both questions is "yes".
2. *Rigidity for the cylinder norm.* Let G be a simple graph which is an edge-disjoint union of two spanning subgraphs T and L where T is a tree and L is a Laman graph.
   (a) Does there exist a placement of G in $(\mathbb{R}^3, \|\cdot\|_{cyl})$ such that the induced monochrome subgraphs of G are precisely T and L?
   (b) Does there exist an isostatic placement of G in $(\mathbb{R}^3, \|\cdot\|_{cyl})$?

   Again, a positive answer to (a) would imply a positive answer to (b). The smallest graph in this class is $K_6 - e$, obtained by removing a single edge from the complete graph K_6, and this graph does admit an isostatic placement in $(\mathbb{R}^3, \|\cdot\|_{cyl})$.

For background see [11, 12].

4. Bill Jackson *Coincident realisations and vertex splitting*

Let $G = (V, E)$ be a graph and $vv' \in E$. Fekete, Jordán and Kaszanitzky [8] showed that G can be realised as an infinitesimally rigid bar-joint framework (G, p) in $\mathbb{R}^2$ with $p(v) = p(v')$ if and only if $G - vv'$ and G/vv' are both generically rigid in $\mathbb{R}^2$ (where $G - vv'$ and G/vv' are obtained from G by, respectively, deleting and contracting the edge vv'). We conjecture that the same result holds in $\mathbb{R}^d$.

Conjecture 0.1. *Let $G = (V, E)$ be a graph and $vv' \in E$. Then G can be realised as an infinitesimally rigid bar-joint framework (G, p) in $\mathbb{R}^d$ with $p(v) = p(v')$ if and only if $G - vv'$ and G/vv' are both generically rigid in $\mathbb{R}^d$.*

The proof of the 2-dimensional case given in [8] is based on a characterisation of independence in the '2-dimensional generic vv'-coincident rigidity matroid'. It is unlikely that a similar approach will work in $\mathbb{R}^d$ since it is notoriously difficult to characterise independence in the d-dimensional generic rigidity matroid for $d \geq 3$. But it is conceivable that there may be a geometric argument which uses the generic rigidity of $G - vv'$ and G/vv' to construct an infinitesimally rigid vv'-coincident realisation of G.

Conjecture 0.1 would imply the following weak version of a conjecture of Whiteley [3] on vertex splitting, see [10, Conjecture 1.3] and [4, Conjecture 2]. (Given a graph $H = (V, E)$ and $v \in V$ with $N(v) = \{v_1, v_2, \dots, v_m\}$ for some $m \geq d$, the *(d-dimensional) vertex splitting operation at v* produces a new graph G by deleting $k \geq 0$ edges $vv_1, vv_2, \dots, vv_k$ from H, and then adding a new vertex v' and $k + d$ new edges $v'v, v'v_1, v'v_2, \dots, v'v_{k+d-1}$.)

Conjecture 0.2. *Let $H = (V, E)$ be a graph which is generically globally rigid in $\mathbb{R}^d$ and $v \in V$. Suppose that G is obtained from H by a vertex split at v and that $G - vv'$ is generically rigid in $\mathbb{R}^d$. Then G is generically globally rigid in $\mathbb{R}^d$.*

In Whiteley's original conjecture, the hypothesis that $G - vv'$ is generically rigid in $\mathbb{R}^d$ is replaced by the weaker hypothesis that both v and v' have degree at least $d + 1$ in G. It has been verified for the case when $d = 2$ by Jordán and Szabadka [10].

To see that Conjecture 0.1 implies Conjecture 0.2 we may proceed as follows. Suppose that Conjecture 0.1 is true, $H = (V, E)$ is generically globally rigid in $\mathbb{R}^d$, $v \in V$, G is obtained from H by a vertex split at v and $G - vv'$ is generically rigid in $\mathbb{R}^d$. Since $G/vv' = H$ is globally rigid, G/vv' is rigid. Since $G - vv'$ is also rigid, Conjecture 0.1 implies that G has a generic vv'-coincident infinitesimally rigid realisation (G, p) i.e. we have $p(v') = p(v)$ and $p|_V$ is generic. Since $(H, p|_V)$ is globally rigid, (G, p) is also globally rigid. We can now use a result of Connelly and Whiteley [6, Theorem 13] to deduce that (G, q) is globally rigid for all q sufficiently close to p. In particular (G, q) is globally rigid for some generic q.

Note 1 Connelly [4, Theorem 29] used stress matrices to obtain the following result, which is closely related to Conjectures 0.1 and 0.2. (In particular it immediately implies that Conjecture 0.2 would follow from Conjecture 0.1.)

Theorem 0.3. *Let $H = (V, E)$ be a graph which is generically globally rigid in $\mathbb{R}^d$ and $v \in V$. Suppose that G is obtained from H by a vertex split at v and that G has an* *infinitesimally rigid vv'-coincident realisation in $\mathbb{R}^d$. Then G is generically globally rigid in $\mathbb{R}^d$.*

Note 2 Jordán, Király and Tanigawa [9, Theorem 4.3] state Conjecture 0.2 as a result of Connelly but this is false, they are in fact misquoting Theorem 0.3.

Note 3 - Added December 2016 The following counterexample to Conjecture 0.1 came out of discussions with Shin-Ichi Tanigawa in October 2016. Suppose $G = K_{5,5}$ and u, v are two vertices on different sides of the bipartition of G. Then $G - uv$ and G/uv are both generically rigid in $\mathbb{R}^3$, but no generic uv-coincident realisation of G in $\mathbb{R}^3$ is infinitesimally rigid. The last assertion follows from the following result of Bolker and Roth, and the fact that any nine points in $\mathbb{R}^3$ lie on a conic.

Theorem 0.4. [2] *Suppose that $(K_{m,n}, p)$ is a realisation of $K_{m,n}$ in general position in $\mathbb{R}^d$. Then $(K_{m,n}, p)$ is infinitesimally rigid if and only if the points in each side of the bipartition affinely span $\mathbb{R}^d$ and the complete set of points do not lie on a quadric surface in $\mathbb{R}^d$.*

5. Shin-ichi Tanigawa

Background: The singularity degree of a spherical framework is defined as the size of a smallest iterated self-stress certificate for the dimensional rigidity (or dimensional flexibility). See [5] for the iterated self-stress certificates. We define the singularity degree of a graph G as the largest singularity degree of frameworks whose underlying graph is equal to G. In the same manner, the singularity degree of signed graphs can be defined by looking at the dimensional rigidity of tensegrities.

Questions:
- Characterize the class of graphs with singularity degree at most two (or three, four, five, ...).
- Characterize the class of signed graphs with singularity degree equal to one.
- Relate the singularity degree to other graph parameter. Is there any interesting combinatorial parameter which bounds the singularity degree?

Remarks:
- A graph has singularity degree equal to one if and only if it is chordal [15]. See also [7].
- A graph has singularity degree at most two if and only if it is the clique sum of chordal graphs and K_4-minor free graphs.
- For each integer n, there is a graph with n vertices whose treewidth is three and singularity degree is $n/3 - 1$.

6. Elad Hahn

Does contracting the pinned vertices of a pinned 3D isostatic bar-joint framework into exactly two pinned vertices and adding a bar between these two vertices always produce a 3D rigidity circuit? (Offer Shai [14] presented this conjecture in 2010.)

References

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Dept. Math. Stats., Lancaster University, Lancaster LA1 4YF, U.K.
E-mail address: `a.nixon@lancaster.ac.uk`