Enumeration of nilpotent associative algebras of class 2 over arbitrary finite fields

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In 1960 Higman introduced the notion of PORC functions (polynomials on residue classes). He proved for example that the number of isomorphism types of algebras over a finite field with $q$ elements can be given, considered as a function in $q$, by a PORC function.

We use this theory and show that the number $N_{d,r}(q)$ of isomorphism types of nilpotent associative algebras of dimension $d$, rank $r$ and class 2 over a finite field with $q$ elements, considered as a function in $q$, can be described by a PORC function in $q$. We describe an algorithm that, given a rank $r$, determines such polynomials for $N_{d,r}(q)$ for all dimensions $d$. Based on this, we determine $N_{d,r}(q)$ for $r \in \{1, \ldots, 5\}$ and arbitrary $d$.

The motivation for our work comes from a structure theorem for associative algebras by Wedderburn which states that an associative algebra can be given as a direct sum of a nilpotent and a semisimple associative algebra.

Our work can be extended to further algebraic objects. The number $L_{d,r}(q)$ of isomorphism types of class 2 Lie algebras of rank $r$ and dimension $d$ over a finite field with $q$ elements considered as a function in $q$ is PORC. Further, the number $f_{n,r}(p)$ of isomorphism types of class 2 $p$-groups of order $p^n$ with $r$ generators considered as a function in $p$ is PORC.

B. Eick extended our algorithm such that it can be used to determine the latter PORC functions as well.