Combinatorics of Body-bar-hinge Frameworks

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based on a handbook chapter with Csaba Király

Tokyo

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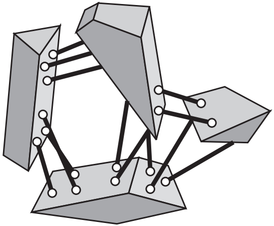
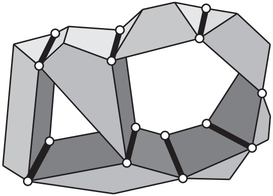
Body-bar-hinge Frameworks



body-bar framework in $\mathbb{R}^3$



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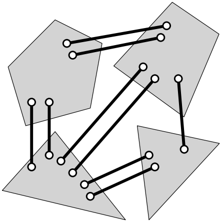
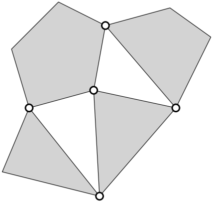


body-bar framework in $\mathbb{R}^2$



body-hinge framework in $\mathbb{R}^2$

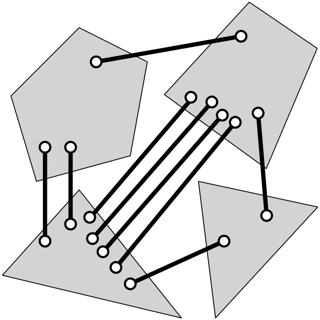
Why interesting?

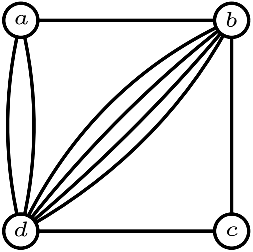
- appear in lots of real problems $\rightarrow$ Ileana's talk
- rigidity characterization problem can be solved in any dimension.

	rigidity	global rigidity
bar-joint	unsolved ($d \leq 2$: Laman)	unsolved ($d \leq 2$: Jackson-Jordán05)
body-bar	Tay84	Connelly-Jordán-Whiteley13
body-hinge	Tay89, Tay91, Whiteley88	Jordán-Király-T16

Body-bar Frameworks

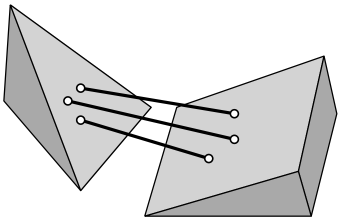
- A d-dimensional body-bar framework is a pair (G, b):
  - $G = (V, E)$: underlying graph;
  - b: a bar-configuration; $E \ni e \mapsto$ a line segment in $\mathbb{R}^d$.

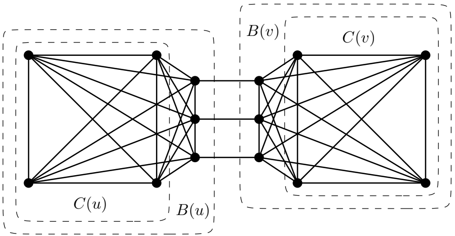




Rigidity, Infinitesimal Rigidity, Global Rigidity

- An equivalent bar-joint framework to (G, b):





- local rigidity (LR), infinitesimal rigidity (IR), global rigidity (GL) are defined through an equivalent bar-joint framework.
- All the basic results for bar-joint can be transferred
  e.g., infinitesimal rigidity $\Rightarrow$ rigidity

Maxwell and Tay

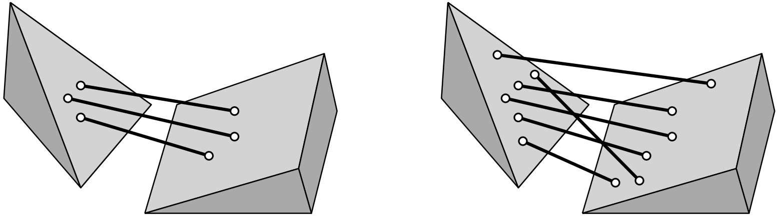
Maxwell's condition

If a d-dimensional body-bar framework (G, b) is IR, then

$$|E(G)| \geq D|V(G)| - D$$

with $D = \binom{d+1}{2}$.

for $d = 3$, $|E(G)| \geq 6|V(G)| - 6$



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with $D = \binom{d+1}{2}$.

Maxwell's condition (stronger version)

If a d-dimensional body-bar framework (G, b) is IR, then G contains a spanning subgraph H satisfying

- $|E(H)| = D|V(H)| - D$
- $\forall H' \subseteq H, |E(H')| \leq D|V(H')| - D$

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Maxwell's condition (stronger version)

If a d-dimensional body-bar framework (G, b) is IR, then G contains a spanning (D, D)-tight subgraph.

- H is (k, k)-sparse $\overset{\text{def}}{\Leftrightarrow} \forall H' \subseteq H, |E(H')| \leq k|V(H')| - k$
- H is (k, k)-tight $\overset{\text{def}}{\Leftrightarrow}$ (k, k)-sparse & $|E(H)| = k|V(H)| - k$

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- H is (k, k)-tight $\overset{\text{def}}{\Leftrightarrow}$ (k, k)-sparse & $|E(H)| = k|V(H)| - k$

Theorem (Tay84)

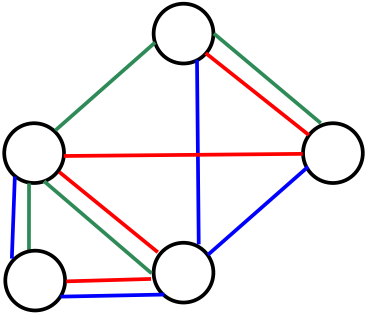
A generic d-dimensional body-bar framework (G, b) is IR (or LR) $\Leftrightarrow$ G has a spanning (D, D)-tight subgraph.

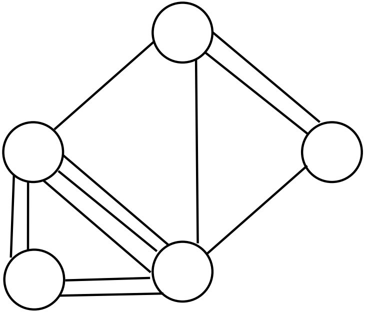
(Better) Characterizations

Theorem (Tutte61, Nash-Williams61, 64)

TFAE for a graph H:

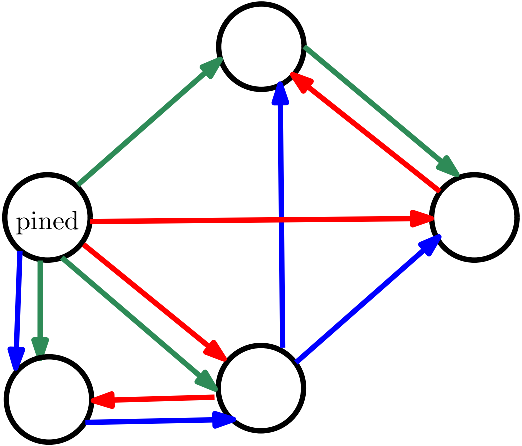
1. H contains a spanning (k, k)-tight subgraph;
2. H contains k edge-disjoint spanning trees;
3. $e_G(\mathcal{P}) \geq k|\mathcal{P}| - k$ for any partition $\mathcal{P}$ of V, where $e_G(\mathcal{P})$ denotes the number of edges connecting distinct components of $\mathcal{P}$.





Proof 1

Based on tree packing (Whiteley88):



Proof 2

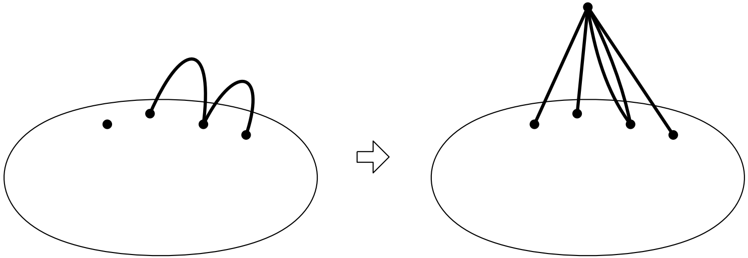
Inductive construction (Tay84):

Theorem (Tay84)

G is (k,k)-tight if and only if G can be built up from a single vertex graph by a sequence of the following operation:

- pinch i $(0 \leq i \leq k-1)$ existing edges with a new vertex v, and add $k-i$ new edges connecting v with existing vertices.

Each operation preserves rigidity.



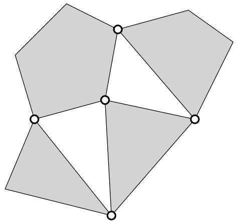
Proof 3

Quick proof (T):

- Prove: a (D, D)-sparse graph G with $|E(G)| = D|V(G)| - D - k$ has k dof.
- Take any edge $e = uv$;
- By induction, $(G - e, b)$ has $k + 1$ dof.
- Try all possible bar realizations of e
- If dof does not decrease, body u and body v behave like one body
- $\Rightarrow (G/e, b)$ has $k + 1$ dof.
- However, G/e contains a spanning (D, D)-sparse subgraph H with $|E(H)| = D|V(H)| - D - k$, whose generic body-bar realization has k dof by induction, a contradiction.

Body-hinge Frameworks

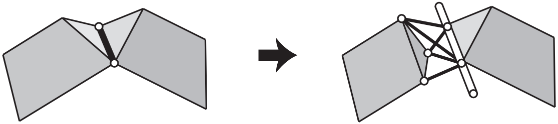
- A d-dimensional body-hinge framework is a pair (G, h):
  - $G = (V, E)$: underlying graph;
  - h: hinge-configuration; $E \ni e \mapsto$ a $(d-2)$-dimensional segment in $\mathbb{R}^d$
- LR, IR, GR are defined by an equivalent bar-joint framework.



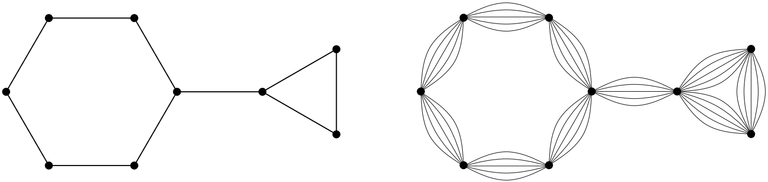
body-hinge framework in $\mathbb{R}^2$

Reduction to Body-bar (Whiteley88)

- a hinge $\approx$ five bars passing through a line

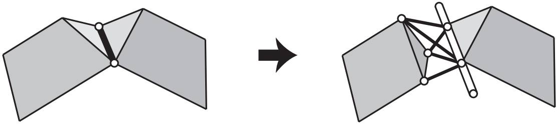


- body-hinge framework $(G, h) \approx$ body-bar framework $((D-1)G, b)$
  - kG: the graph obtained by replacing each edge with k parallel edges



Reduction to Body-bar (Whiteley88)

- a hinge ≈ five bars passing through a line



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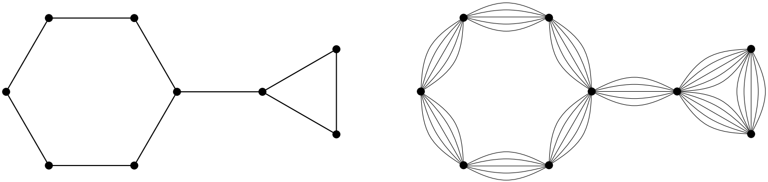


Maxwell's condition

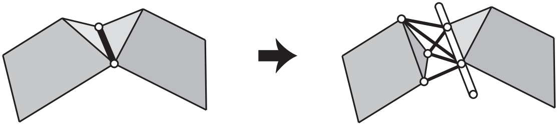
If a d-dimensional body-hinge framework (G, h) is IR, then $(D-1)G$ contains D edge-disjoint spanning trees.

Maxwell, Tay, and Whiteley

Theorem (Tay 89,91, Whiteley 88)

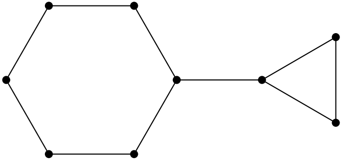
A generic d-dimensional body-hinge framework (G, b) is LR (IR) $\Leftrightarrow$ $(D-1)G$ contains D edge-disjoint spanning trees.

- Proof 1 can be applied
  - an equivalent body-bar framework is non-generic
- Body-bar-hinge frameworks (Jackson-Jordán09)
- Q. Any quick proof (without tree packing)?

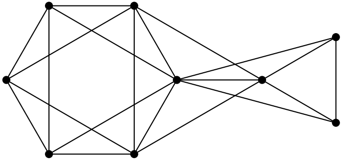


Molecular Frameworks

- square of G: $G^2 = (V(G), E(G)^2)$
  - $E(G)^2 = \{uv : d_G(u,v) \leq 2\}$



G



G^2

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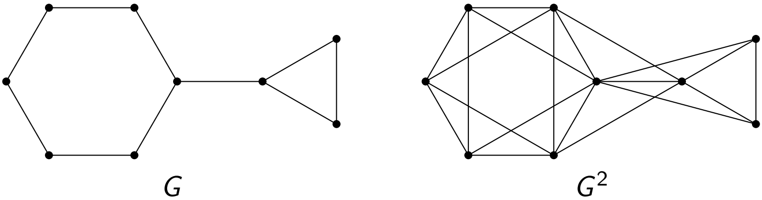

- molecular framework: a three-dimensional body-hinge framework in which hinges incident to each body are concurrent.
  - $G^2 \Leftrightarrow$ a molecular framework (G, h)

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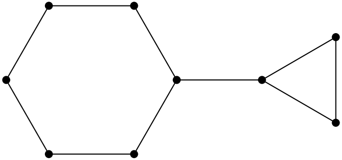
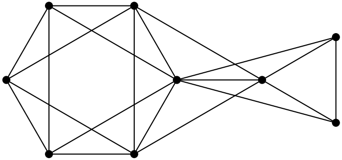


G



G^2

- molecular framework: a three-dimensional body-hinge framework in which hinges incident to each body are concurrent.
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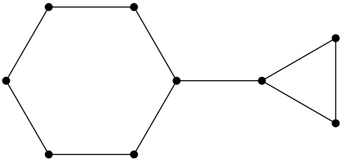
molecular framework (G, h) is LR $\Rightarrow$
$5G$ contains six edge-disjoint spanning trees.

Theorem (Katoh-T11)

generic molecular framework (G, h) is LR $\Leftrightarrow$

$5G$ contains six edge-disjoint spanning trees.

- a refined version: a characterization of rigid component decom.
  - fast algorithms for computing static properties of molecules
    - ★ Ileana's talk
  - graphical analysis of molecular mechanics
- a rank formula of G^2 in the 3-d rigidity matroid (Jackon-Jordán08)
  - Open: a rank formula of a subgraph of G^2



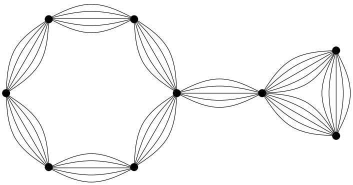
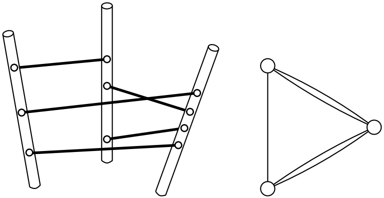


Plate-bar Frameworks

- a d-dim. k-plate-bar framework
  - vertex = k-plate (k-dim. body)
  - edge = a bar linking k-plates
- $k = d$: body-bar framework
- $k = 0$: bar-joint framework



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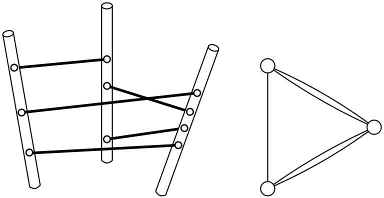


Theorem (Tay 89, 91)

A generic $(d-2)$-plate-bar framework in $\mathbb{R}^d$ is LR $\Leftrightarrow$ G contains a $(D-1, D)$-tight spanning subgraph.

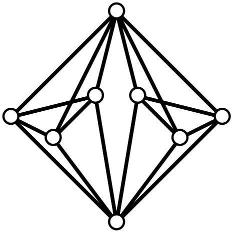
- Corollary: a characterization of identified body-hinge framework.
- Open: characterization of the rigidity of generic $(d-3)$-plate-bar framework for large d.

Body-pin Frameworks

- A d-dimensional body-pin framework is a pair (G, p):
  - G: underlying graph;
  - $p : E(G) \to \mathbb{R}^d$: a pin-configuration.
- a pin $\approx$ d bars

Maxwell's condition

If a 3-dimensional body-pin framework (G, p) is rigid, then $3G$ contains six edge-disjoint spanning trees.



Beyond Maxwell

Conjecture

A generic three-dimensional body-pin framework is rigid iff

$$\sum_{\{X,X'\}\in\binom{\mathcal{P}}{2}} h_G(X,X') \geq 6(|\mathcal{P}|-1)$$

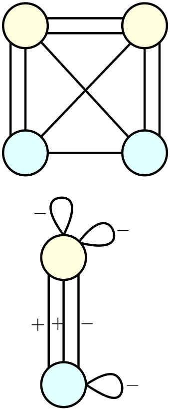
for every partition $\mathcal{P}$ of V, where $\binom{\mathcal{P}}{2}$ denotes the set of pairs of subsets in $\mathcal{P}$ and

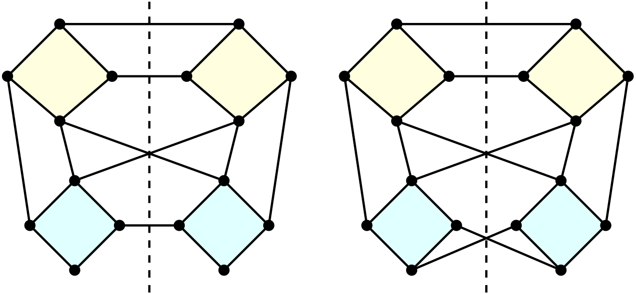
$$h_G(X,X') = \begin{cases} 6 & \text{if } d_G(X,X') \geq 3 \\ 5 & \text{if } d_G(X,X') = 2 \\ 3 & \text{if } d_G(X,X') = 1 \\ 0 & \text{if } d_G(X,X') = 0. \end{cases}$$

- If h_G were defined to be $h_G(X,X') = 6$ for $d_G(X,X') = 2$, it is Maxwell.

Symmetric Body-bar-hinge Frameworks

- $\mathcal{C}_s$: a reflection group
- A $\mathcal{C}_s$-symmetric body-bar(-hinge) framework (G, b)





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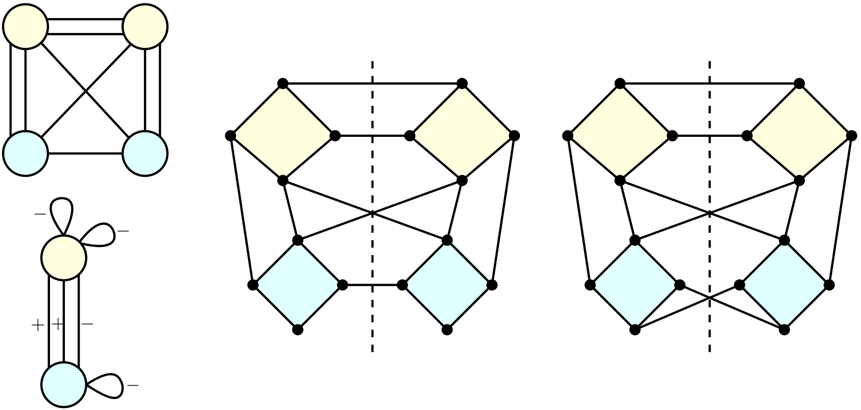


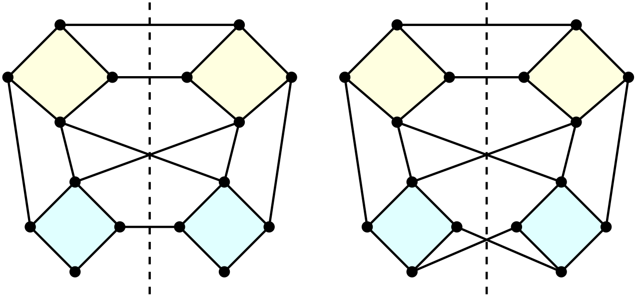
- the underlying quatiant signed graph G^σ
- L_0: the set of loops "fixed by the action"

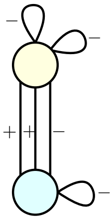
Theorem(Schulze-T14)

A "generic" body-bar (G, b) with reflection symmetry is IR in $\mathbb{R}^3 \Leftrightarrow$ $G^\sigma - L_0$ contains edge-disjoint

- three spanning trees, and
- three non-bipartite pseudo-forests.

- pseudo-tree: each connected component has exactly one cycle
- bipartite: if every cycle has even number of minus edges





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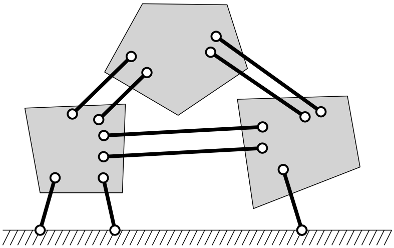
- three spanning trees, and
- three non-bipartite pseudo-forests.

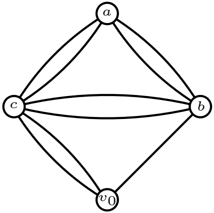
- periodic (crystallographic) infinite body-bar frameworks (Borcea-Streinu-T15, Ross14, Schulze-T14, T15)
  - Proof 1 works only if the underlying symmetry is $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.
  - Proof 3 works for any case

- body-hinge frameworks with symmetry
  - Proof 1 works if $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.
  - open for other cases

Bar-joint Frameworks with Boundaries

- body-bar framework with boundaries: some of bodies are linked by bars to the external (fixed) environment
- $=$ a body-bar framework with a designated body (corresponding to the external environment)





Characterization with non-generic boundaries

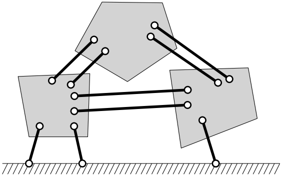
Theorem (Katoh and T13)

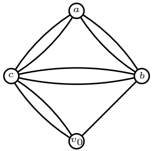
G: a graph with a designated vertex v_0;
E_0: the set of edges in G incident to v_0;
$b^0(e)$: a line segment for $e \in E_0$.
Then one can extend b^0 to b s.t. (G, b) is IR $\Leftrightarrow$

$$e_G(\mathcal{P}) \geq D|\mathcal{P}| - \sum_{X \in \mathcal{P}} \mathrm{dim\ span}\{\tilde{b}(e) : e \in E_0(X)\}$$

for every partition $\mathcal{P}$ of $V(G) \setminus \{v_0\}$, where
$E_0(X)$ is the set of edges in E_0 incident to X and
$\tilde{b}(e)$ is the Plücker coordinate of the line segment $b(e)$.

- subspace-constrained system



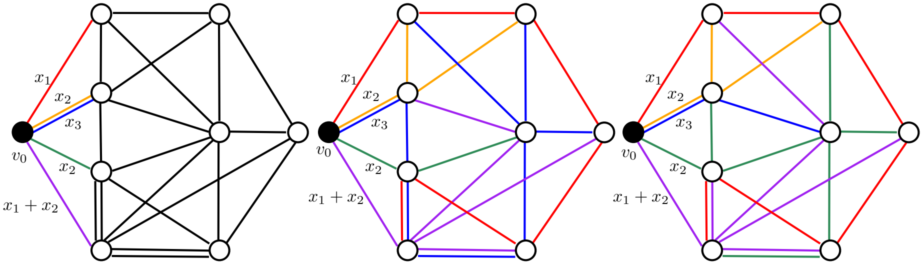


Basic Tree Packing

- $G = (V, E)$: a graph with a designated vertex v_0;
- E_0: the set of edges in G incident to v_0;
- x_e: a vector in $\mathbb{R}^k$ for each $e \in E_0$.

A packing of edge-disjoint trees $T_1, \ldots, T_s$ is basic if each $v \in V \setminus \{v_0\}$ receives a base of $\mathbb{R}^k$ from v_0 through $T_1, \ldots, T_s$.

Theorem(Katoh-T13)
$\exists$ a basic packing $\Leftrightarrow$ $e_G(\mathcal{P}) \geq k|\mathcal{P}| - \sum_{X \in \mathcal{P}} \dim \mathrm{sp}\{x_e : e \in E_0(X)\}$ $(\forall \mathcal{P})$



Other Variants

- generic infinite frameworks (Kiston-Power13)
- different normed space (Kiston-Power13)
- body-bar frameworks with direction-length constraints (Jackson-Nguyen15)
  - a characterization is still open
- angle constrained (Haller et al.12)

Global Rigidity

Theorem (Hendrickson92)

If a generic bar-joint framework is globally rigid in $\mathbb{R}^d$, then the underlying graph is a complete graph, or $(d+1)$-connected and redundantly rigid.

- sufficient in $d \leq 2$ (Jackson-Jordán05)
- may not in $d \geq 3$ (Connelly)

Connelly, Jordán, and Whiteley

Theorem (Connelly, Jordán, and Whiteley13)

A generic d-dimensional body-bar framework (G, b) is GR $\Leftrightarrow$
$\forall e \in E(G)$, $G - e$ contains D edge-disjoint spanning trees.

- Proof 1: Inductive construction (Frank and Szegö03)
- Proof 2: The underlying graph of an equivalent bar-joint framework is vertex-redundantly rigid.
  - A generic bar-joint framework is GR if the underlying graph is vertex-redundantly rigid. (T15)
- Proof 3: the same approach as Proof 3 for IR

Orientation Theorem

A characterization of ℓ-edge-redundantly rigid body-bar frameworks.

Theorem (Frank80)

TFAE for a graph.

- After deleting any ℓ edges it contains k edge-disjoint spanning trees
- it admits an r-rooted (k,ℓ)-edge-connected orientation for $r \in V(G)$.

A digraph D is r-rooted (k,ℓ)-edge-connected $\overset{\text{def}}{\Leftrightarrow}$ for any $v \in V(G)$,

- there are k arc-disjoint paths from r to v;
- there are ℓ arc-disjoint paths from v to r.

Body-hinge

Theorem (Jordán, Király, T16)

A generic d-dimensional body-hinge framework (G, b) is GR $\Leftrightarrow$
$\forall e \in E(DG)$, $DG - e$ contains D edge-disjoint spanning trees.

Body-hinge

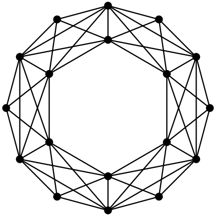
Theorem (Jordán, Király, T16)

A generic d-dimensional body-hinge framework (G, b) is GR $\Leftrightarrow$
$\forall e \in E(DG)$, $DG - e$ contains D edge-disjoint spanning trees.

Corollary

a family of graphs which satisfy Hendrickson's condition but are not GR

- Take a graph H that contains six edge-disjoint spanning trees but $H - e$ does not for some $e \in E(H)$.
- Construct an equivalent bar-joint framework by replacing each body with a dense subgraph.



Open: Global Rigidity of G^2