The beauty of fractals

Gwyneth Stallard

The Open University

Florence Nightingale Day
December 2015

Why maths?

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My favourite maths results

- are surprising

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No technical equipment required!

Why fractals?

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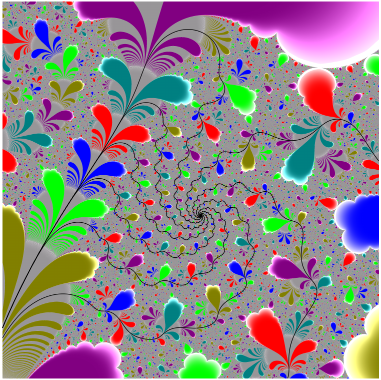
Research



Why fractals?

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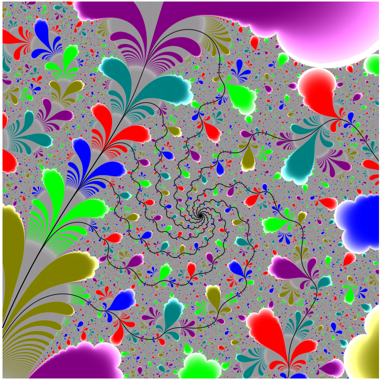
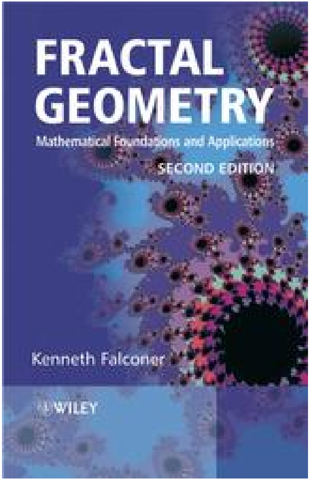


Teaching



The von Koch curve

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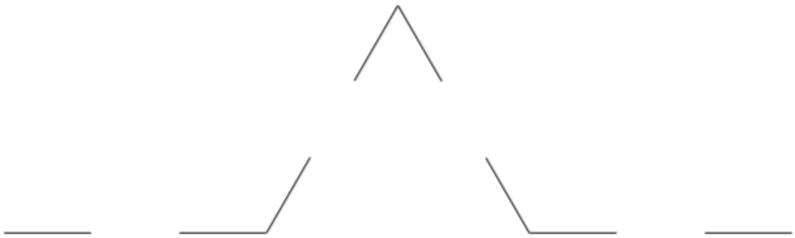

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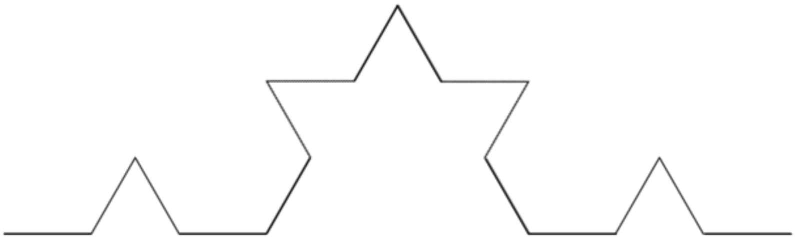

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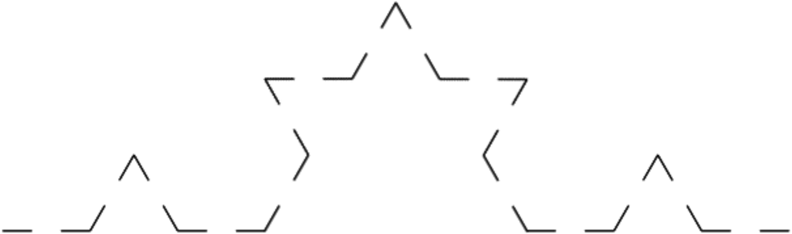

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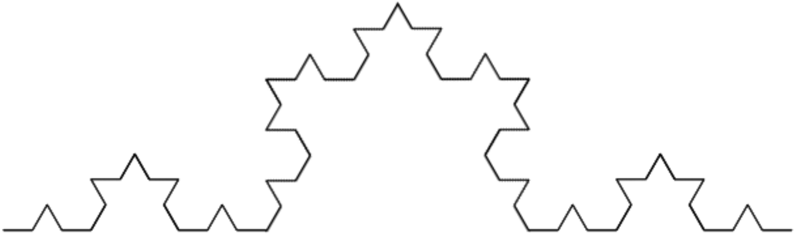

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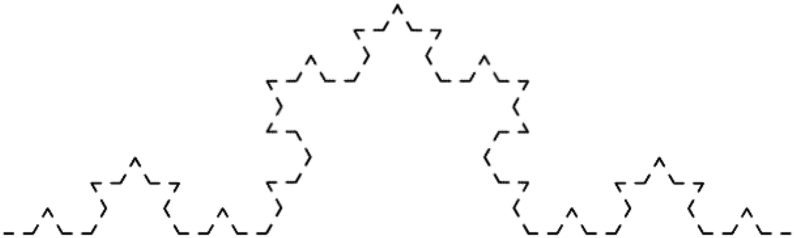

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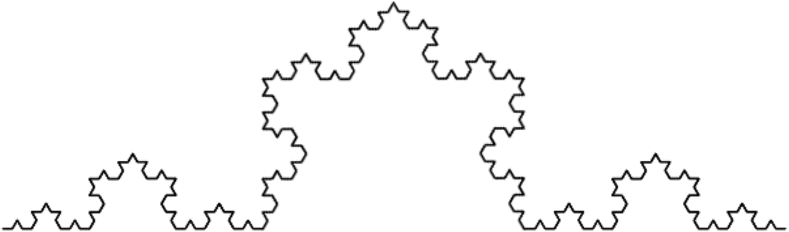

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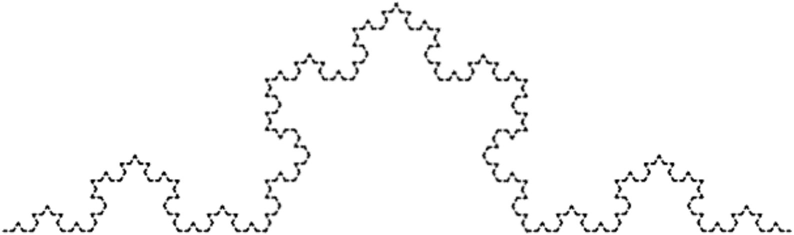

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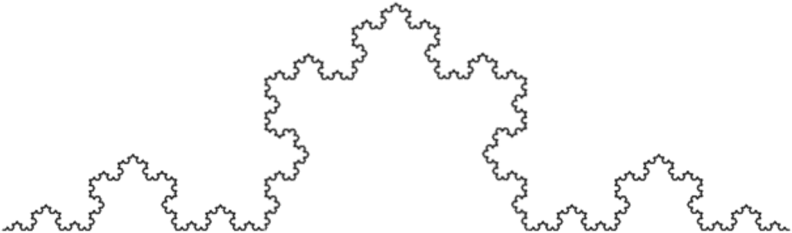
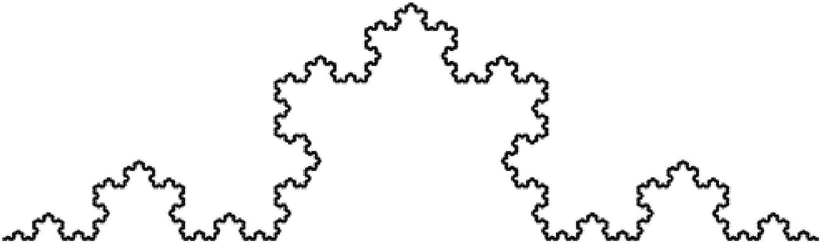


Zooming in



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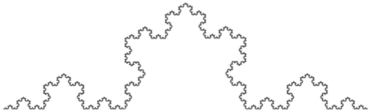

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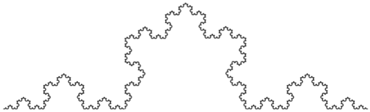

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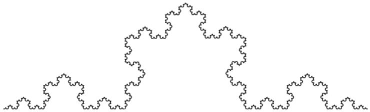

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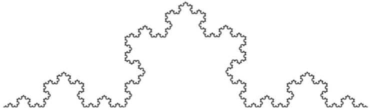

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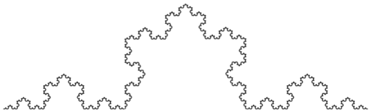

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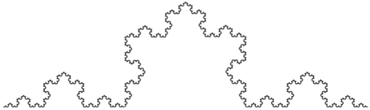

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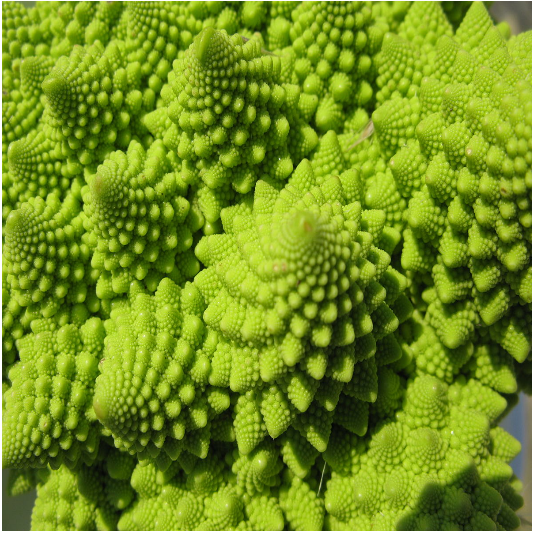
Many natural objects are fractals

fractal cauliflower



The middle third Cantor set

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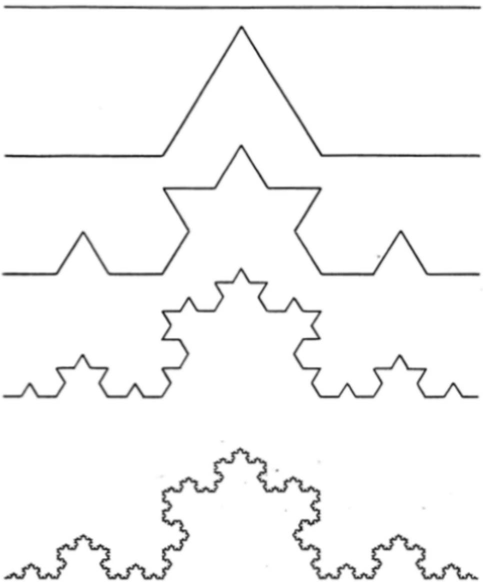

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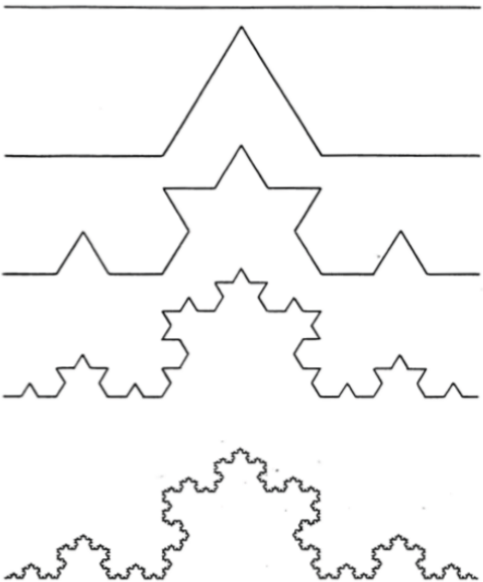

Length = 1

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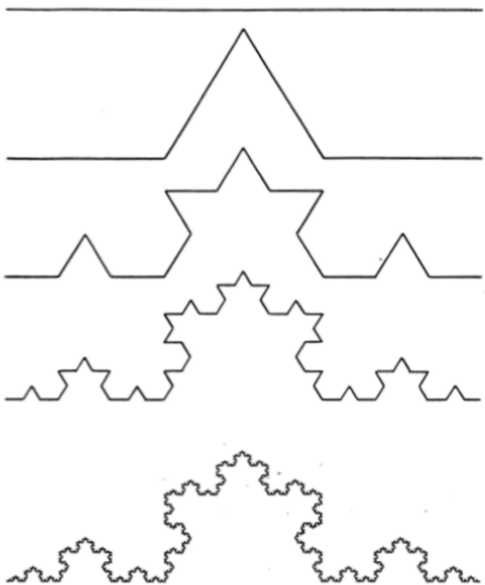
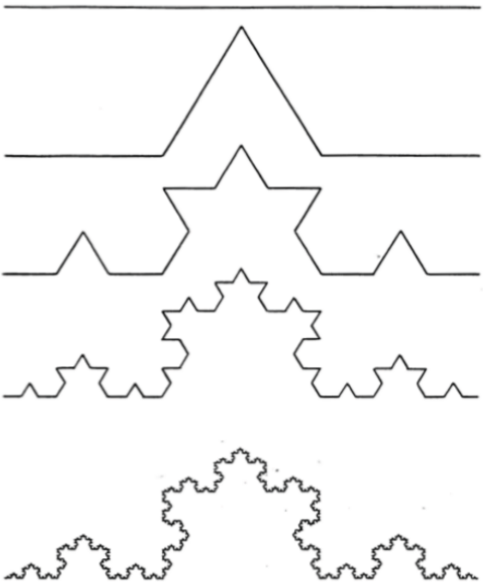


Length = 1

$4/3$

How long is the von Koch curve?



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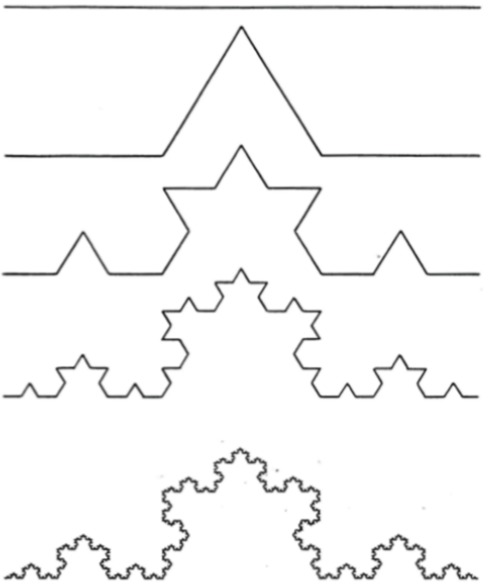

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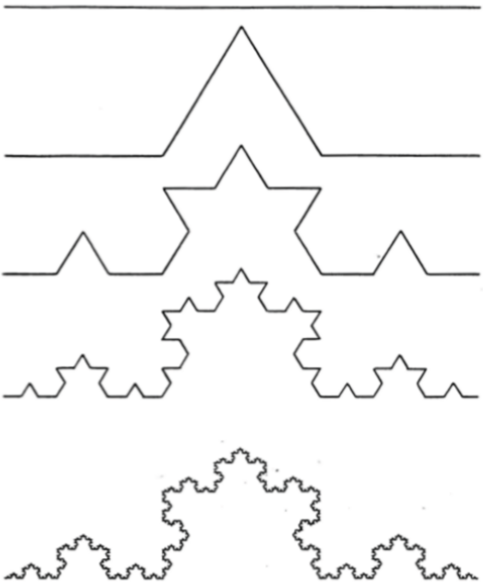

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$\vdots$ At k-th step:

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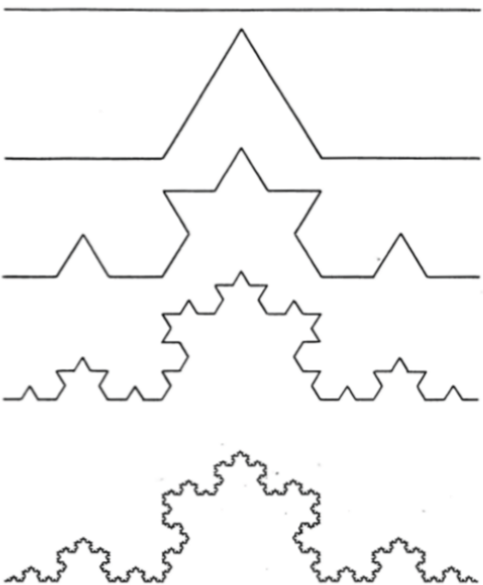

Length = 1

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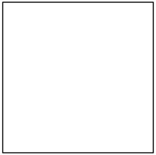
$\vdots$ At k-th step:

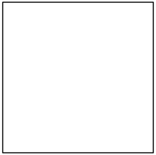
$(4/3)^k \to \infty$

Dimensions of fractals



dimension = 2

Dimensions of fractals



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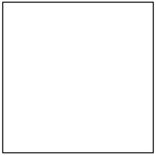

dimension = 2

dimension = 1

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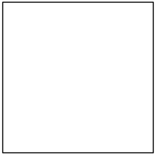

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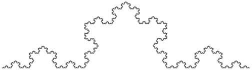
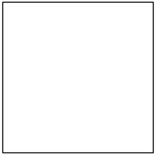


dimension = a, $1 < a < 2$

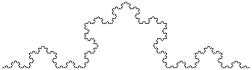
Dimensions of fractals



dimension = 2

dimension = 1

dimension = 0



dimension = a, $1 < a < 2$

dimension = b, $0 < b < 1$

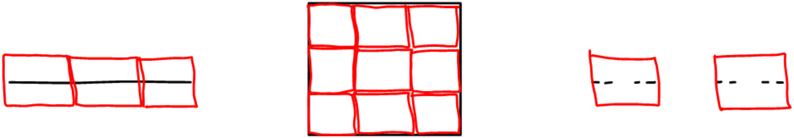
Box dimension

Box dimension

How many boxes of side d do we need to cover a set?

Box dimension

If $d = \frac{1}{3}$ then



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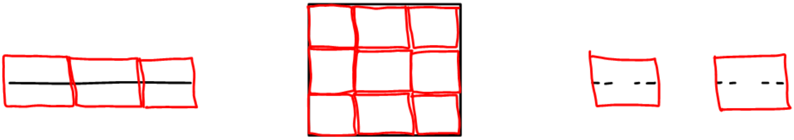

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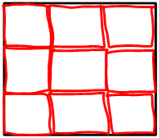

$N = 9 = 3^2$

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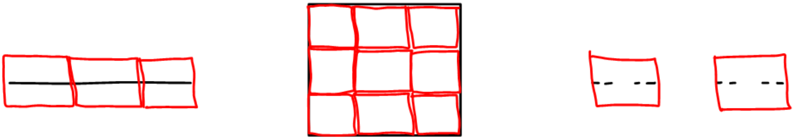

$N = 3 = 3^1$ $\quad$ $N = 9 = 3^2$ $\quad$ $N = 2 = 3^{\log 2 / \log 3}$

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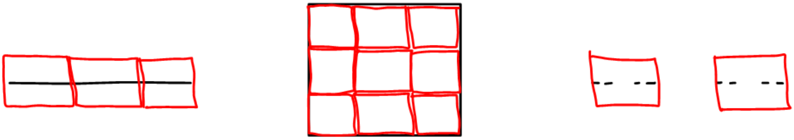

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If $d = \frac{1}{3^2} = \frac{1}{9}$ then

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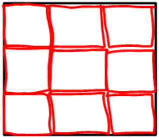

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$N = 9 = (3^2)^1$

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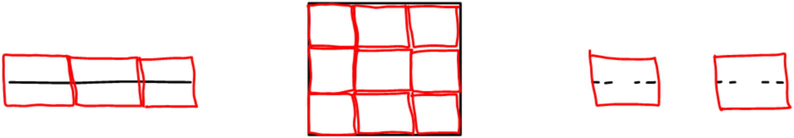

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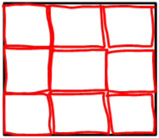

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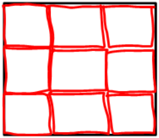

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Box dimension

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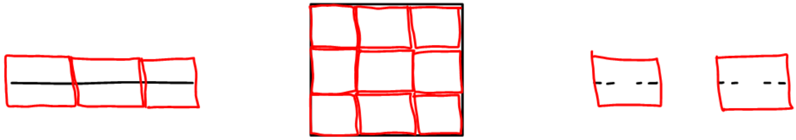

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 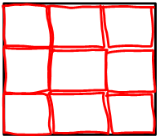 

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dimension = 1 $\quad$ dimension = 2 $\quad$ dimension = log 2/log 3

Complex numbers

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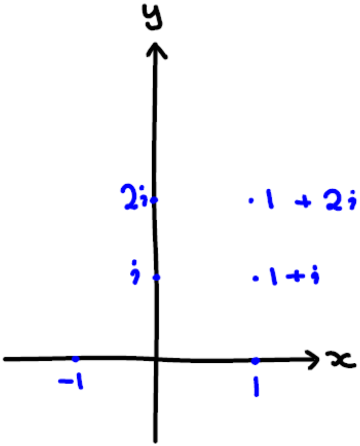

Functions of complex numbers

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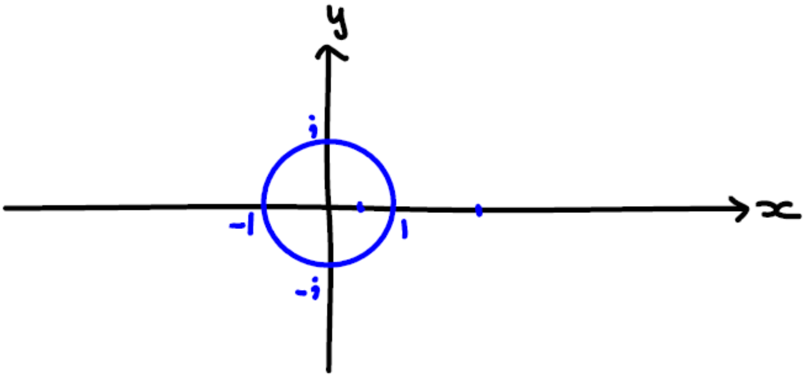
We write $f^n(2) = 2^{2^n}$.

$\frac{1}{2} \to \frac{1}{4} \to \ldots \to \frac{1}{2^{2^n}} \to \ldots \to 0$

$f(z) = z^2$



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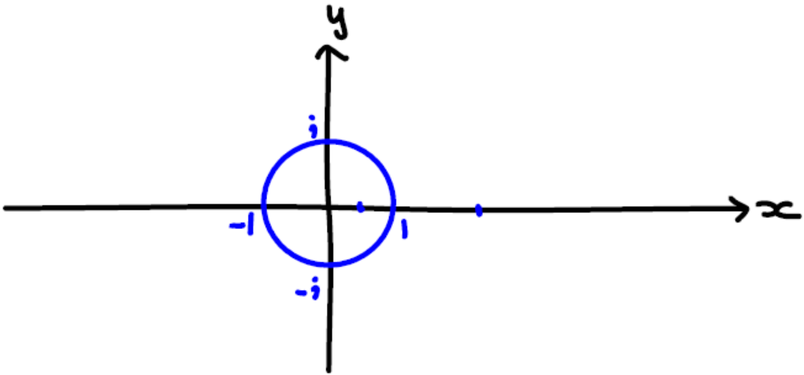

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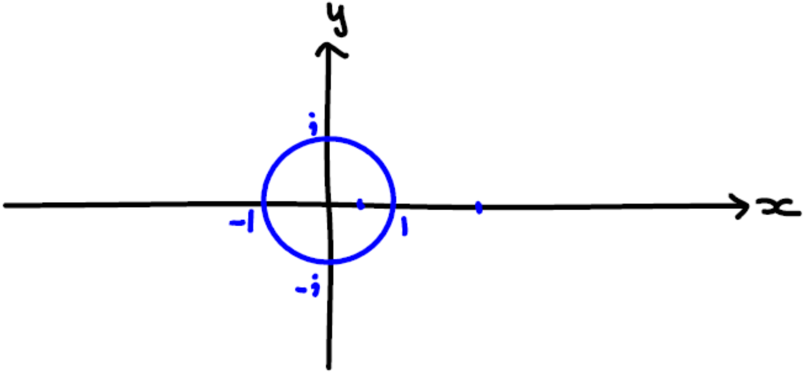

If z is outside circle then $f^n(z) \to \infty$ as $n \to \infty$.

If z is inside circle then $f^n(z) \to 0$ as $n \to \infty$.

Complex dynamics

Definition

The *escaping set* is

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

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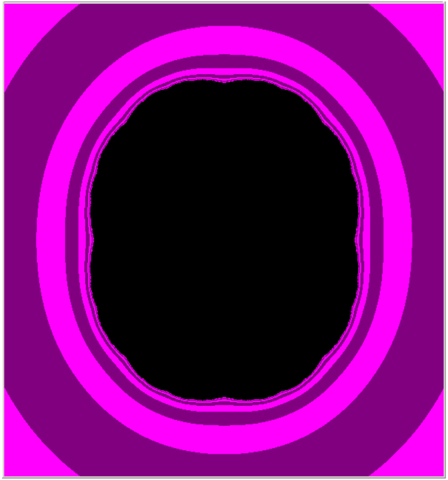
Definition

The *Julia set* $J(f)$ is the boundary of the escaping set.

Quadratic examples

$f(z) = z^2 + 0.1$



Quadratic examples

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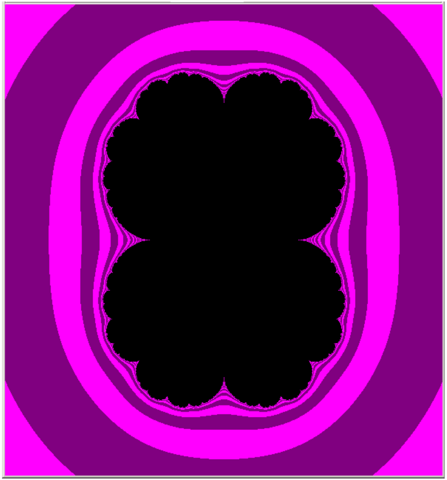
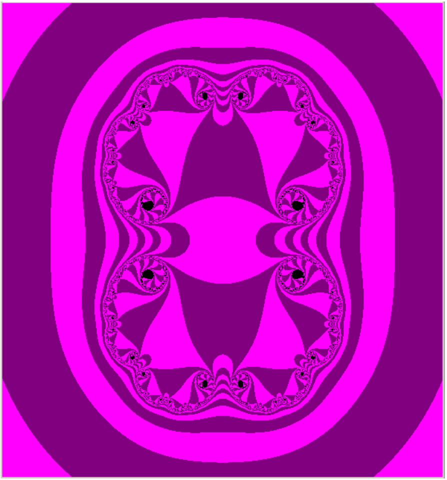


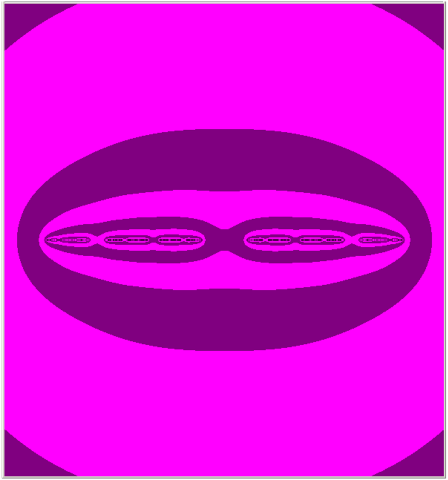
Quadratic examples

$f(z) = z^2 + 0.3$



Quadratic examples

$f(z) = z^2 - 2.1$



Exponential functions

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- $J(f) = I(f)$ plus all endpoints
- dim $J(f)$ = dim $I(f) = 2$ (McMullen)
- curves without endpoints have dimension 1; endpoints have dimension 2 (Karpińska's paradox)

Possible values of dimensions

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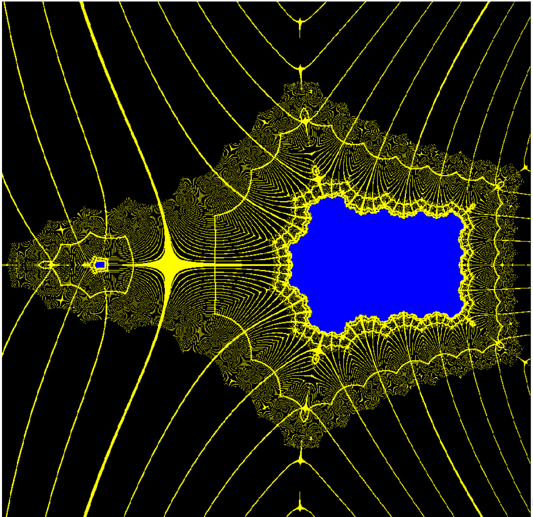
- $\dim J(f) = \dim I(f) = d$, for any d satisfying $1 < d \leq 2$

A new structure for the escaping set

$$\boxed{f(z) = 0.5(\cos z^{1/4} + \cosh z^{1/4})}$$



Happy Christmas!