Interdependent Reciprocation*

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Abstract. From work groups to file sharing systems, agents interact repeatedly. Often, they exhibit reciprocating behavior. We need to understand this behavior better before thinking about how to improve individual results or collective reciprocal behavior. To this end, we need a model for such interactions that is simple enough to enable formal analysis, but sufficiently accurate to explain our observations. Inspired by psychology and game theory, we consider two intuitive reciprocating attitudes where an agent's action is a weighted combination of the others' last actions and either i) her innate kindness, or ii) her own last action. We analyze a network of repeatedly interacting agents, each having one of these attitudes, and prove that their actions converge to specific limits. For the case of two agents, we describe the interaction process and give the exact limit values. For a general connected network, we find these limit values if all the agents employ the second attitude, and show that agents' actions then all become equal. We discuss how well this describes observations found in behavioral economics and social sciences.

1 Introduction

Interaction is central in human behavior, e.g., at school, when sharing files over networks, in business cooperation. Instead of being economically rational, people tend to adopt other ways of behavior [22], not necessarily maximizing some utility function. Furthermore, people tend to reciprocate, i.e., react on the past actions of others [9, 10, 13]. Since reciprocation is ubiquitous, we aim to explain this behavior. Therefore, we need a model for reciprocating agents that is simple enough for an analytical analysis and precise enough to predict such interactions.

Extant models of reciprocation (sometimes repeated) include it in the utility function of rational agents [7, 9, 19], for example as in the well-known iterative prisoner's dilemma [2, 24]. Reciprocation is also described in qualitative research [18]. Several other studies deal with reciprocation without considering the extent of actions, see e.g., Gintis [11, Chapter 11].

We are interested in the complementary approach of formal modeling and analysis of repeated reciprocation, where the weight of a reaction is the main aspect, and the reciprocation is intrinsic, rather than maximizing some utility function.

* This work is not intended for the proceedings.

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We represent actions by *weight*, where a higher value means a more desirable contribution. Agents reciprocate both to the agent they are acting on and to their whole neighborhood. We were mainly inspired by Trivers [[24]] (a psychologist), who describes a balance between an inner quality (kindness) and costs/benefits when determining an action. We model this by two reciprocation attitudes, being a convex combination between i) one's own kindness or ii) one's own last action, and the other's and neighborhood's last actions. Trivers also talks about a naturally selected complicated balance between altruistic and cheating tendencies, which is modeled as kindness, which represents the inherent inclination to contribute. The balance between complying and not complying is mentioned in the conclusion of [25], motivating the convex combination between own kindness or action and others' actions. The idea of humans behaving according to a convex combination was inspired by the altruistic extension in several papers, like [5, 6, 14, 20], and Chapter iii.2 in [17], and by modeling antisocial behavior [3]. Defining action (or how much it changes) or state by a linear combination of the other side's actions and own actions and qualities was also used to analyze arms race [8, 26] and spouses' interaction [12] (piecewise linear in this case). Attitude i) depending on the (fixed) kindness is called *fixed*, and ii) depending on one's own last action is called *floating*. Additional motivation stems from the bargaining and negotiation realm, where Pruitt [18] mentions that in negotiation, cooperation often takes place in the form of reciprocation and that personal traits influence the way of cooperation, which corresponds in our model to the personal kindness and reciprocation coefficients. We study actions in the limit of time approaching infinity, to model the unbounded future.

Example 1. Consider n colleagues $1, 2, \ldots, n$, who can help or harm each other. Let the possible actions be: giving bad work, showing much contempt, showing little contempt, supporting emotionally a little, supporting emotionally a lot, advising, and let their respective weight be a point in $[-1, -0.5)$, $[-0.5, -0.2)$, $[-0.2.0)$, $(0, 0.4)$, $[0.4, 0.7)$, $[0.7, 1]$. Assume that each person knows what the other has done to him last time. The work climate also influences behavior. However, we may just concentrate on a single pair of even-tempered colleagues who reciprocate regardless others.

We first consider two agents, assuming their interaction is independent of other agents, or that the total influence of the others on the pair is negligible. Then, we treat convergence for interaction of many agents, and find the limit when all the agents have the *floating* reciprocation attitude. These results explain interactions and lay the foundation for further analysis of interaction.

2 Model of Interdependent Reciprocation

Let $N = \{1, 2, \ldots, n\}$ be $n \geq 2$ interacting agents. We assume that possible actions are described by an undirected interaction graph $G = (N, E)$, such that agent i can act on j and vice versa if and only if $(i, j) \in E$. Denote the degree of agent $i \in N$ in G by $d(i)$. To be able to mention directed edges, we shall treat this graph as a directed one, where for every $(i, j) \in E$, we have $(j, i) \in E$. Time is modeled by a set of discrete moments $t \in T \triangleq \{0, 1, 2, \ldots\}$, defining a time slot whenever at least one agent acts. Agent i acts at times $T_i \triangleq \{t_{i,0} = 0, t_{i,1}, t_{i,2}, \ldots\} \subseteq T$, and $\cup_{i \in N} T_i = T$. We assume that all agents act at $t = 0$, since otherwise we cannot sometimes consider the last action of another agent, which would force us to complicate the model and render it even harder for theory. It would be interesting to model the case of non-simultaneous starts. When all agents always act at the same times ($T_1 = T_2 = \ldots = T_n = T$), we say they act *synchronously*. For the sake of asymptotic analysis, we assume that each agent gets to act an infinite number of times; that is, T_i is infinite for every $i \in N$. Any real application will, of course, realize only a finite part of it, and infinity models the unboundedness of the process in time.

When (i, j) is in E, we denote the weight of an action by agent $i \in N$ on another agent $j \in N$ at moment t by $\mathrm{act}_{i,j}(t) \colon T_i \to \mathbb{R}$. We extend $\mathrm{act}_{i,j}$ to T by assuming that at $t \in T \setminus T_i$, we have $\mathrm{act}_{i,j}(t) = 0$. Since only the weight of an action is relevant, we usually write "action" while referring to its weight. For example, when interacting by file sharing, sending a valid file, nothing, or a file with a virus has a positive, zero, or a negative weight, respectively.

For $t \in T$, we define *the last action time* $s_i(t) \colon T \to T_i$ *of agent* i as the largest $t' \in T_i$ that is at most t. Since $0 \in T_i$, this is well defined. The last action of agent i on (another) agent j is given by $x_{i,j}(t) \triangleq \mathrm{act}_{i,j}(s_i(t))$. Thus, we have defined $x_{i,j}(t) \colon T \to \mathbb{R}$. We denote the total received contribution from all the neighbors $\mathrm{N}(i)$ at the last action times not later than t by $\mathrm{got}_i(t) \colon T \to \mathbb{R}$; formally, $\mathrm{got}_i(t) \triangleq \sum_{j \in \mathrm{N}(i)} \mathrm{act}_{j,i}(s^j(t))$.

We now define two reciprocation attitudes. The kindness of agent i is denoted by $k_i \in \mathbb{R}$; w.l.o.g., $k_n \geq \ldots \geq k_2 \geq k_1$ throughout the paper. Kindness models inherent inclination to help others; in particular, it determines the first action of an agent, before others have acted. We model agent i's inclination to mimic a neighboring agent's action and the actions of the whole neighborhood in G by reciprocation coefficients $r_i \in [0, 1]$ and $r'_i \in [0, 1]$ respectively, such that $r_i + r'_i \leq 1$. r_i is the fraction of $\mathrm{act}_{i,j}(t)$ that is determined by the last action of j upon i, and r'_i is the fraction that is determined by $\frac{1}{|\mathrm{N}(i)|}$th of the total contribution to i from all the neighbors at the last time. Intuitively, the *fixed* attitude depends on the agent's kindness at every action, while the *floating* one is loose, moving freely in the reciprocation process, and kindness directly influences such behavior only at $t = 0$. In both cases $\mathrm{act}_{i,j}(0) \triangleq k_i$.

Definition 1 *For fixed reciprocation attitude, agent i's reaction on the other agent j and on the neighborhood is determined by the agent's kindness weighted by $1 - r_i - r'_i$, by the other agent's action weighted by r_i and by the total action of the neighbors weighted by r'_i and divided over all the neighbors:* $\mathrm{act}_{i,j}(t) \triangleq$

$$(1 - r_i - r'_i) \cdot k_i + r_i \cdot \mathop{\mathrm{act}}_{j,i}(s_j(t-1)) + r'_i \cdot \frac{\mathrm{got}_i(t-1)}{|\mathrm{N}(i)|}.$$

Definition 2 *In the floating reciprocation attitude, agent i's action is a weighted average of her own last action, of that of the other agent j and of the total action of the neighbors divided over all the neighbors:* $\mathrm{act}_{i,j}(t) \triangleq$

$$(1 - r_i - r'_i) \cdot \underset{i,j}{\mathrm{act}}(s_i(t-1)) + r_i \cdot \underset{j,i}{\mathrm{act}}(s_j(t-1)) + r'_i \cdot \frac{\mathrm{got}_i(t-1)}{|\mathrm{N}(i)|}$$

These neighborhood models are identical to a pairwise interaction when $r'_i = 0$ or when there are no neighbors besides the other agent in the considered pair.

In Example 1, let (just here) $n = 3$ and the reciprocation coefficients be $r_1 = r_2 = 0.5, r'_1 = r'_2 = 0.3, r_3 = 0.8, r'_3 = 0.1$. Assume the kindness to be $k_1 = 0, k_2 = 0.5$ and $k_3 = 1$. Since this is a small group, all the colleagues may interact, so the graph is a clique[1]. At $t = 0$, every agent's action on every other agent is equal to her kindness value, so agent 1 does nothing, agent 2 supports emotionally a lot, and 3 provides advice. If all agents act at all times, and all get carried away by the process so that they forget the kindness in the sense of employing *floating* reciprocation, then, at $t = 1$ they act as follows: $\mathrm{act}_{1,2}(1) = (1-0.5-0.3)\cdot 0 + 0.5\cdot 0.5 + 0.3\cdot \frac{0.5+1}{2} = 0.475$ (supports emotionally a lot), $\mathrm{act}_{1,3}(1) = (1-0.5-0.3)\cdot 0 + 0.5\cdot 1 + 0.3\cdot \frac{0.5+1}{2} = 0.975$ (providing advice), $\mathrm{act}_{2,1}(1) = (1-0.5-0.3)\cdot 0.5 + 0.5\cdot 0 + 0.3\cdot \frac{0+1}{2} = 0.25$ (supports emotionally a little), and so on.

3 Dynamics of Pairwise Interaction

We now consider the interaction of only two agents, 1 and 2. When T_1 contains precisely all the even numbered slots and T_2 zero and all the odd ones, we say they are *alternating*. Since agent 1 can only act on agent 2 and vice versa, we write $\mathrm{act}_i(t)$ instead for $\mathrm{act}_{i,j}(t)$, $x(t)$ instead of $x_{1,2}(t)$ and $y(t)$ instead $x_{2,1}(t)$. We often write $v(t)$ for $\mathrm{act}_1(t)$ and $w(t)$ for $\mathrm{act}_2(t)$. W.l.o.g, we assume here that $r'_i = 0$ for all agents i.

We analyze reciprocation in the above model. We first look at the behavior in the limit for *fixed* reciprocation, then for *floating*, and investigate the process itself. To formally discuss the actions after the interaction has settled down, we consider the limits (if exist) $\lim_{p\to\infty} v(t_{1,p})$, or $\lim_{t\to\infty} x(t)$, for agent 1, and $\lim_{p\to\infty} w(t_{2,p})$ or $\lim_{t\to\infty} y(t)$ for agent 2. Since the sequence $\{x(t)\}$ is $\{v(t_{1,p})\}$ with finite repetitions, the limit $\lim_{t\to\infty} x(t)$ exists if and only if $\lim_{p\to\infty} v(t_{1,p})$ does. If they exist, they are equal; the same holds for $\lim_{t\to\infty} y(t)$ and $\lim_{p\to\infty} w(t_{2,p})$. Denote $L_x \triangleq \lim_{t\to\infty} x(t)$ and $L_y \triangleq \lim_{t\to\infty} y(t)$.

3.1 *Fixed* Reciprocation

Here we prove that both action sequences converge.

[1] A clique is a fully connected graph.

Theorem 1. *If the reciprocation coefficients are not both* 1*, which means* $r_1 r_2 < 1$*, then we have, for* $i \in N$*:* $\lim_{p\to\infty} \text{act}_i(t_{i,p}) = \frac{(1-r_i)k_i + r_i(1-r_j)k_j}{1-r_i r_j}$.

The limits of these actions are shown in Figures 1 and 2. In Example 1, if agents

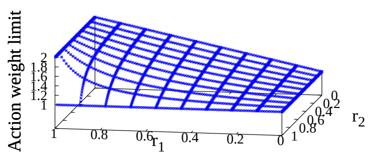

Fig. 1. The limit of the actions of agent 1 as a function of the reciprocity coefficients. *Fixed* - *fixed* reciprocation, $k_1 = 1, k_2 = 2$. Given r_1, agent 2 receives most when $r_2 = 0$.

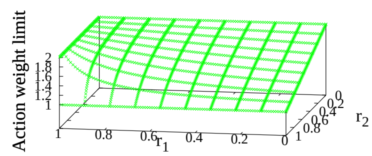

Fig. 2. The limit of the actions of agent 2 as a function of the reciprocity coefficients. *Fixed* - *fixed* reciprocation, $k_1 = 1, k_2 = 2$. Given r_2, agent 1 receives most when $r_1 = 1$.

1 and 2 employ *fixed* reciprocation, $r_1 = r_2 = 0.5, r'_1 = r'_2 = 0.0$ and $k_1 = 0, k_2 = 0.5$, then we obtain $L_x = \frac{0.5\cdot(1-0.5)0.5}{1-0.5\cdot 0.5} = 1/6$ and $L_y = \frac{(1-0.5)\cdot 0.5}{1-0.5\cdot 0.5} = 1/3$.

In order to prove this theorem, we first show that it is sufficient to analyze the synchronous case; i.e., $T_1 = T_2 = T$.

Lemma 1. *Consider a pair of interacting agents. Denote the action sequences in case both agents acted at the same time, (i.e.,* $T_1 = T_2 = T$*), by* $\{v'(t)\}_{t\in T}$ *and* $\{w'(t)\}_{t\in T}$*, respectively. Then the action sequences*[2] $\{v(t_{1,p})\}_{p\in\mathbb{N}}, \{w(t_{2,p})\}_{p\in\mathbb{N}}$ *are subsequences of* $\{v'(t)\}_{t\in T}$ *and* $\{w'(t)\}_{t\in T}$*, respectively.*

The proof follows from Definition 1 by induction. (Straightforward proofs in this paper have been replaced by their general ideas due to lack of space). Using this lemma, it is sufficient to further assume the synchronous case.

Lemma 2. *In the synchronous case, for every* $t > 0 : x(2t-1) \geq x(2t+1)$*, and for every* $t \geq 0 : x(2t) \leq x(2t+2) \leq x(2t+1)$*. By analogy,* $\forall t > 0 : y(2t-1) \leq y(2t+1)$*, and* $\forall t \geq 0 : y(2t) \geq y(2t+2) \geq y(2t+1)$*. All the inequations are strict if and only if* $0 < r_1, r_2 < 1, k_2 > k_1$*.*[3]

Since we also have $t \geq 0 : x(2t) \leq x(2t+1)$, we obtain $t > 0 : x(2t-1) \geq x(2t+1) \geq x(2t)$, and for every $t \geq 0 : x(2t) \leq x(2t+2) \leq x(2t+1)$. By analogy, $\forall t > 0 : y(2t-1) \leq y(2t+1) \leq y(2t)$, and $\forall t \geq 0 : y(2t) \geq y(2t+2) \geq y(2t+1)$. Intuitively, this means that the sequence $\{x(t)\}$ is alternating while its amplitude is getting smaller, and the same holds for the sequence $\{y(t)\}$, with another alternation direction. The intuitive reasons are that first, agent 1 increases her action, while 2 decreases it. Then, since 2 has decreased her action, so does 1, while since 1 has increased hers, so does 2. We now formally prove the lemma.

[2] When agent i acts at times in T_i.

[3] We always assume that $k_2 \geq k_1$.

Proof. We employ induction. For $t = 0$, we need to show that $x(0) \leq x(2) \leq x(1)$ and $y(0) \geq y(2) \geq y(1)$. We know that $x(0) = k_1$, $x(1) = (1-r_1)\cdot k_1 + r_1 \cdot k_2$, and $y(0) = k_2$, $y(1) = (1-r_2)\cdot k_2 + r_2 \cdot k_1$. Since $y(1) \leq k_2$, we have $x(2) = (1-r_1)\cdot k_1 + r_1 \cdot y(1) \leq x(1)$. Since $y(1) \geq k_1$, we also have $x(2) = (1-r_1)\cdot k_1 + r_1 \cdot y(1) \geq x(0)$. The proof for ys is analogous.

For the induction step, for any $t > 0$, assume that the lemma holds for $t - 1$, which means $x(2t-3) \geq x(2t-1)$ (for $t > 1$), $x(2t-2) \leq x(2t) \leq x(2t-1)$, and $y(2t-3) \leq y(2t-1)$ (for $t > 1$), $y(2t-2) \geq y(2t) \geq y(2t-1)$.

We now prove the lemma for t. By Definition 1, $x(2t-1) = (1-r_1)k_1 + r_1 y(2t-2)$ and $x(2t+1) = (1-r_1)k_1 + r_1 y(2t)$. Since $y(2t-2) \geq y(2t)$, we have $x(2t-1) \geq x(2t+1)$. By analogy, we can prove that $y(2t-1) \leq y(2t+1)$.

Also by definition, $x(2t) = (1-r_1)k_1 + r_1 y(2t-1)$ and $x(2t+2) = (1 - r_1)k_1 + r_1 y(2t+1)$. Since $y(2t-1) \leq y(2t+1)$, we have $x(2t) \leq x(2t+2)$. By definition, $x(2t+1) = (1-r_1)k_1 + r_1 y(2t)$. Since $y(2t) \geq y(2t-1)$, we conclude that $x(2t+1) \geq x(2t)$. By analogy, we prove that $y(2t+1) \leq y(2t)$. From this, we conclude that $x(2t+2) \leq x(2t+1)$, and we have shown that $x(2t) \leq x(2t+2) \leq x(2t+1)$. By analogy, we prove that $y(2t) \geq y(2t+2) \geq y(2t+1)$.

The equivalence of strictness in all the inequations to $0 < r_1, r_2 < 1, k_2 > k_1$ is proven by repeating the proof with strict inequalities in one direction, and by noticing that each case of one of the conditions $0 < r_1, r_2 < 1, k_2 > k_1$ failing to hold implies equality in at least one of the statements on the lemma.

With these results we now prove Theorem 1.

Proof. Using Lemma 1, we assume the synchronous case. We first prove convergence, and then find its limit. For each agent, Lemma 2 implies that the even actions form a monotone sequence, and so do the odd ones. Both sequences are bounded, which can be easily proven by induction, and therefore each one converges. The whole sequence converges if and only if both limits are the same. We now show that they are indeed the same for the sequence $\{x(t)\}$; the proof for $\{y(t)\}$ is analogous. $x(t+1) - x(t)$
$= (1-r_1)k_1 + r_1 y(t) - (1-r_1)k_1 - r_1 y(t-1)$
$= r_1(y(t) - y(t-1)) = r_1 r_2(x(t-1) - x(t-2)) = \ldots$
$= (r_1 r_2)^{\lfloor t/2 \rfloor} \begin{cases} x(1) - x(0) & t = 2s, s \in \mathbb{N} \\ x(2) - x(1) & t = 2s+1, s \in \mathbb{N}. \end{cases}$ As $r_1 r_2 < 1$, this difference goes to 0 as t goes to ∞. Thus, $x(t)$ converges (and so does $y(t)$). To find the limits $L_x = \lim_{t\to\infty} x(t)$ and $L_y = \lim_{t\to\infty} y(t)$, notice that in the limit we have $(1-r_1)k_1 + r_1 L_y = L_x$ and $(1-r_2)k_2 + r_2 L_x = L_y$ with the unique solution: $L_x = \frac{(1-r_1)k_1 + r_1(1-r_2)k_2}{1-r_1 r_2}$ and $L_y = \frac{(1-r_2)k_2 + r_2(1-r_1)k_1}{1-r_1 r_2}$.

We see that $L_x \leq L_y$, which is intuitive, since the agents are always considering their kindness, so the kinder one acts with a bigger weight also in the limit. In the simulation results in Figure 3, in one example, $y(t)$ is always larger than $x(t)$, and in the other, they alternate several times before $y(t)$ gets larger.

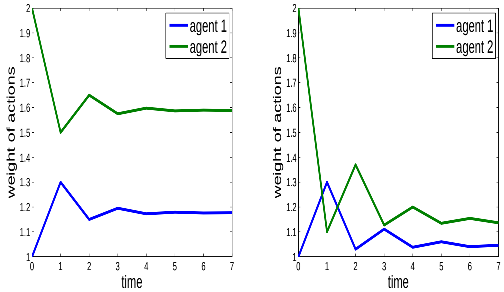

Fig. 3. Simulation of actions for the synchronous case, $r_1 + r_2 < 1$ (left) and $r_1 + r_2 > 1$ (right). *Fixed* - *fixed* reciprocation, $k_1 = 1, k_2 = 2, r_1 = 0.3$. In the left graph, $r_2 = 0.5$, while in the right one, $r_2 = 0.9$. Each agent's actions go up and down while converging to her own limit.

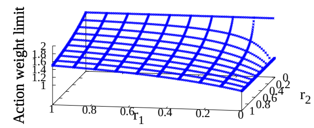

Fig. 4. The common limit of the actions as a function of the reciprocity coefficients. *Floating* - *floating* reciprocation, $k_1 = 1, k_2 = 2$. Given r_2, agent 1 receives most when $r_1 = 1$, and given r_1, agent 2 receives most when $r_2 = 0$.

3.2 *Floating* Reciprocation

If both agents have a *floating* reciprocation attitude, their action sequences converge to a common limit.

Theorem 2. *If the reciprocation coefficients are neither both* 0 *and nor both* 1*, which means* $0 < r_1 + r_2 < 2$*, then, as* $t \to \infty$*,* $x(t)$ *and* $y(t)$ *converge to the same limit. In the synchronous case* ($T_1 = T_2 = T$)*, they both approach*

$$\frac{1}{2}\left(k_1 + k_2 + (k_2 - k_1)\frac{r_1 - r_2}{r_1 + r_2}\right) = \frac{r_2}{r_1 + r_2}k_1 + \frac{r_1}{r_1 + r_2}k_2.$$

The common limit of the actions is shown in Figure 4. In Example 1, if agents 1 and 2 employ *fixed* reciprocation, $r_1 = r_2 = 0.5, r'_1 = r'_2 = 0.0$ and $k_1 = 0, k_2 = 0.5$, then we obtain $L_x = L_y = (1/2) \cdot 0 + (1/2) \cdot 0.5 = 0.25$.

Throughout the paper, whenever we need concrete T_1, T_2, we consider the synchronous case. The alternative case is omitted due to lack of space.

Proof. We first prove that the convergence takes place.

If both agents act at time $t > 0$, then $y(t) - x(t)$

$$\begin{aligned} &= x(t-1)(r_2 - 1 + r_1) + y(t-1)(1 - r_2 - r_1) \\ &= (1 - r_1 - r_2)(y(t-1) - x(t-1)). \end{aligned}$$

Since $0 < r_1 + r_2 < 2$, we have $|(1 - r_1 - r_2)| < 1$.

If only agent 1 acts at time $t > 0$, then $y(t) - x(t)$

$$= y(t-1)(1 - r_1) - x(t-1)(1 - r_1) = (1 - r_1)(y(t-1) - x(t-1)).$$

If $r_1 > 0$, then $|(1 - r_1)| < 1$. Similarly, if only agent 2 acts, then $y(t) - x(t) = (1 - r_2)(y(t-1) - x(t-1))$. Since $r_1 + r_2 > 0$, either r_1 or r_2 is greater than 0, and since each agent acts an infinite number of times, we obtain $\lim_{t\to\infty} |y(t) - x(t)| = 0$. Since $\forall t > 0 : x(t), y(t) \in [\min\{x(t-1), y(t-1)\}, \max\{x(t-1), y(t-1)\}]$, we have a nested sequence of segments, which lengths approach zero, thus $x(t)$ and $y(t)$ both converge, and to the same limit.

Assume $T_1 = T_2 = T$ now, to find the common limit. For all $t > 0$,

$$\begin{aligned} x(t) + y(t) &= x(t-1)(1 - r_1 + r_2) + y(t-1)(r_1 + 1 - r_2) \\ &= x(t-1) + y(t-1) + (r_1 - r_2)(y(t-1) - x(t-1)) \\ \Rightarrow \lim_{t\to\infty} x(y) + y(t) &= k_1 + k_2 + \sum_{t=0}^{\infty} (r_1 - r_2)(y(t) - x(t)) \\ \underbrace{\overset{\text{geom. series}}{\rightarrow}}_{t\to\infty} k_1 + k_2 + (r_1 - r_2)\frac{k_2 - k_1}{r_1 + r_2} &= k_1 + k_2 + (k_2 - k_1)\frac{r_1 - r_2}{r_1 + r_2}. \end{aligned}$$

Since we have shown that both limits exist and are equal, each is equal to half of $k_1 + k_2 + (k_2 - k_1)\frac{r_1 - r_2}{r_1 + r_2}$.

The relation between the sequences of xs and ys is given by the following.

Proposition 1. *If $r_1 + r_2 \leq 1$, then, for every $t \geq 0 : y(t) \geq x(t)$. If $r_1 + r_2 \geq 1$, then, $y(0) \geq x(0)$. For every $t > 0, t \in T_1 \cap T_2$, we have $y(t-1) \geq x(t-1) \Rightarrow y(t) \leq x(t)$, and $y(t-1) \leq x(t-1) \Rightarrow y(t) \geq x(t)$. For any other t, we have $y(t-1) \geq x(t-1) \Rightarrow y(t) \geq x(t)$, and $y(t-1) \leq x(t-1) \Rightarrow y(t) \leq x(t)$. In words, xs and ys alter their relative positions if and only if both act.*

Proof. Consider the case $r_1 + r_2 \leq 1$ first. We employ induction. The basis is $t = 0$, where $y(0) = k_2 \geq k_1 = x(0)$.

For the induction step, assume the proposition for all the times smaller than $t > 0$ and prove it for t. If only 1 acts at t, then $y(t) = y(t-1)$ and $x(t) = (1-r_1)x(t-1) + r_1 y(t-1)$. Therefore, $y(t) \geq x(t) \iff y(t-1) \geq (1-r_1)x(t-1) + r_1 y(t-1)$, which is equivalent to $(1-r_1)y(t-1) \geq (1-r_1)x(t-1)$, which holds by the induction hypothesis. The case where only 2 acts at t is similar.

If both agents act at t, then $x(t) = (1-r_1)x(t-1) + r_1 y(t-1)$ and $y(t) = (1-r_2)y(t-1) + r_2 x(t-1)$. Therefore, $y(t) \geq x(t) \iff (1-r_2)y(t-1) + r_2 x(t-1) \geq (1-r_1)x(t-1) + r_1 y(t-1) \iff (1-r_1-r_2)y(t-1) \geq (1-r_1-r_2)x(t-1)$, which is true by the induction hypothesis and using the assumption $r_1 + r_2 \leq 1$.

Consider the case $r_1 + r_2 \geq 1$ now. We employ induction again. The basis is $t = 0$, where $y(0) = k_2 \geq k_1 = x(0)$.

For the induction step, assume the proposition for all values smaller than $t > 0$ and prove it for t. The cases where only agent 1 acts at t and where only 2 acts at t are shown analogously to how they are shown for the case $r_1 + r_2 \leq 1$. If both agents act at t, then we have shown that $y(t) \geq x(t)$ $\iff (1 - r_1 - r_2)y(t-1) \geq (1 - r_1 - r_2)x(t-1)$, which means that $y(t-1) \geq x(t-1) \Rightarrow y(t) \leq x(t)$ and $y(t-1) \leq x(t-1) \Rightarrow y(t) \geq x(t)$, assuming $r_1 + r_2 \geq 1$.

The proposition implies that if $r_1 + r_2 \leq 1$, then $\{x(t)\}$ do not decrease and $\{y(t)\}$ do not increase, since the next $x(t)$ (or $y(t)$) is either the same of a combination of the last one with a higher value (lower value, for $y(t)$).

For $r_1 + r_2 > 1$, both $\{x(t)\}$ and $\{y(t)\}$ are not monotonic, unless $T_1 \cap T_2 = \{0\}$, in which case they are monotonic, for the reason above (in this case we always have $y(t) \geq x(t)$). For $T_1 \cap T_2 \neq \{0\}$, take any positive t in $T_1 \cap T_2$. Then the larger value at $t - 1$ becomes the smaller one at t, thereby getting smaller, and the smaller value gets larger analogously. In the future, the smaller will only grow and the larger will decrease, thereby behaving non-monotonically. This discussion assumes $r_1 < 1, r_2 < 1$, to avoid getting $x(t) = y(t)$ when a single player acts. Some examples are simulated in Figure 5.

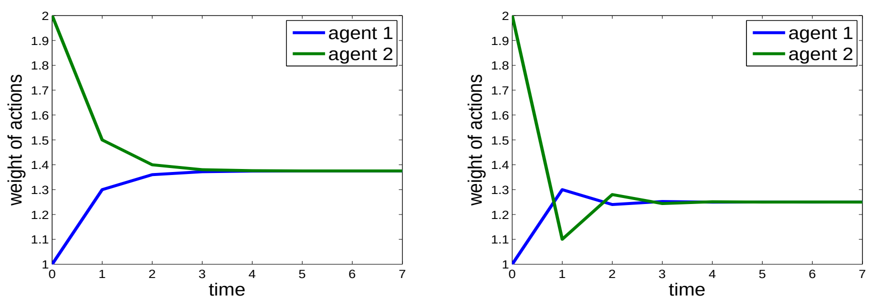

Fig. 5. Simulation results for the synchronous case, $r_1 + r_2 < 1$ (left) and $r_1 + r_2 > 1$ (right). *Floating* - *floating* reciprocation, $k_1 = 1, k_2 = 2, r_1 = 0.3$. In the left graph, $r_2 = 0.5$, while in the right one, $r_2 = 0.9$. In the left graph, agent 1's actions are smaller than those of 2; agent 1's actions increase, while those of agent 2 decrease. In the right graph, the actions of the agents alter their relative positions at each time step; each agent's actions go up and down.

3.3 *Fixed* and *Floating* Reciprocation

Assume that agent 1 employs the *fixed* reciprocation attitude, while 2 acts by the *floating* reciprocation. We can show Theorem 3, using the following lemma.

Lemma 3. *If $r_2 > 0$ and $r_1 + r_2 \leq 1$, then, for every $t \geq t_{1,1} : x(t+1) \leq x(t)$, and for every $t \geq 0 : y(t+1) \leq y(t)$.*

The proof is by induction on t, using the definitions of reciprocation. With this lemma, we can prove the following.

Theorem 3. *If $r_2 > 0$ and $r_1 + r_2 \leq 1$, then,* $\lim_{t\to\infty} x(y) = \lim_{t\to\infty} y(t) = k_1$.

Proof. We first prove that the convergence takes place, and then find its limit. For each agent, Lemma 3 implies that her actions are monotonically non-increasing. Since the actions are bounded below by k_1, which can be easily proven by induction, they both converge.

To find the limits, notice that in the limit we have

$$(1 - r_1)k_1 + r_1 L_y = L_x \tag{1}$$

$$(1 - r_2)L_y + r_2 L_x = L_y. \tag{2}$$

From (2), we conclude that $L_x = L_y$, since $r_2 > 0$. Substituting this to (1) gives us $L_x = L_y = k_1$, since $r_2 > 0$ and $r_1 + r_2 \leq 1$ imply $r_1 < 1$.

The relation between the sequences of xs and ys is given by the following proposition (also covering the case $r_1 + r_2 \geq 1$).

Proposition 2. *If $r_1 + r_2 \leq 1$, then for every $t \geq 0 : y(t) \geq x(t)$. If $r_1 + r_2 \geq 1$, then $y(0) \geq x(0)$. For every $t \geq 0$ such that $t \in T_1 \cap T_2$, we have $y(t-1) \leq x(t-1) \Rightarrow y(t) \geq x(t)$. For any other $t \in T_1$, we have $y(t) \geq x(t)$, and for any other $t \in T_2$, we have $y(t-1) \geq x(t-1) \Rightarrow y(t) \geq x(t)$, and $y(t-1) \leq x(t-1) \Rightarrow y(t) \leq x(t)$.*

The proof employs induction on t.

We note that although we do not know whether Theorem 3 holds for $r_1 + r_2 > 1$, we do know that neither monotonicity (Lemma 3) nor $y(t)$ being always at least as large as $x(t)$ or the other way around holds in this case. As a counterexample for both of them, consider the case of $r_2 = 1, 0 < r_1 < 1, k_2 > k_1$. One can readily prove by induction that for all t we have $x(2t+1) > x(2t) = x(2t+2)$ and $y(2t) > y(2t-1) = y(2t+1)$, and thus both sequences are not monotonic. In addition, one can inductively prove that $x(2t+1) > y(2t+1), x(2t) < y(2t)$, and therefore no sequence is always larger than the other one.

Both cases are simulated in Figure 6. The actions seem to converge also in the unproven case $r_1 + r_2 > 1$.

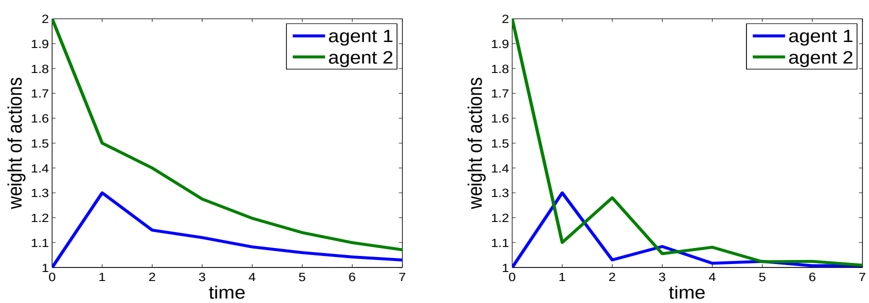

Fig. 6. Simulation results for the synchronous case, $r_1 + r_2 < 1$ (left) and $r_1 + r_2 > 1$ (right). *Fixed - floating* reciprocation, $k_1 = 1, k_2 = 2, r_1 = 0.3$. In the left graph, $r_2 = 0.5$, while in the right one, $r_2 = 0.9$. In the left graph, agent 1's actions are smaller than those of 2; agent 1's actions decrease after $t = 1$, while those of agent 2 decrease all the time. The common limits' value fits the theorem's prediction. In the right graph, The actions of the agents alter their relative positions at each time step; each agent's actions go up and down. The apparent common limits' value equals k_1.

In the case of the mirroring assumption that agent 1 acts according to the *floating* reciprocation attitude, while 2 acts according to the *fixed* reciprocation, we can obtain the following similar results by analogy.

Theorem 4. *If $r_1 > 0$ and $r_1 + r_2 \leq 1$, then,* $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = k_2$.

Regarding the relation between xs and ys, we prove the following, by analogy to how Proposition 2 is proven:

Proposition 3. *If $r_1 + r_2 \leq 1$, then for every $t \geq 0 : y(t) \geq x(t)$. If $r_1 + r_2 \geq 1$, then, $y(0) \geq x(0)$. For every $t \geq 0$, such that $t \in T_1 \cap T_2$, we have $y(t-1) \leq x(t-1) \Rightarrow y(t) \geq x(t)$. For any other $t \in T_2$, we have $y(t) \geq x(t)$, and for any other $t \in T_1$, we have $y(t-1) \geq x(t-1) \Rightarrow y(t) \geq x(t)$ and $y(t-1) \leq x(t-1) \Rightarrow y(t) \leq x(t)$.*

Using Perron-Frobenius Theorem, we can prove a convergence theorem with

Corollary 1. *Consider synchronous pairwise interaction, where agent i employs* fixed *reciprocation and the other agent j employs the* floating *one. Assume that $r_i < 1$ and $r_j > 0$. Then, both actions sequences converge geometrically to k_i.*

For all the considered cases, we conclude the following

Proposition 4. *Unless we have a non-synchronous case with $r_1 = 0$ or $r_2 = 0$, or $r_1 + r_2 > 1$, if both L_x and L_y exist, then $L_x \leq L_y$.*

4 Dynamics of Interdependent Interaction

We now analyze the general interdependent interaction, when agents interact with many agents. To formally discuss the actions after the interaction has settled down, we consider the limits (if exist) $\lim_{p\to\infty} \text{act}_{i,j}(t_{1,p})$, or $\lim_{t\to\infty} x_{i,j}(t)$, for agents i and j. Since the sequence $\{x_{i,j}(t)\}$ is $\{\text{act}_{i,j}(t_{1,p})\}$ with finite repetitions, the limit $\lim_{p\to\infty} \text{act}_{i,j}(t_{1,p})$ exists if and only if $\lim_{t\to\infty} x_{i,j}(t)$ does. If they exist, they are equal. Denote $L_{i,j} \triangleq \lim_{t\to\infty} x_{i,j}(t)$.

We show that in the synchronous case, for every two agents i, j such that $(i, j) \in E$, actions $x_{i,j}(t)$ converge to a strictly positive combination of all the kindness values. The rate of convergence is geometric.

If all agents employ *fixed* reciprocation, we can prove that the action in any other case are subsequences of the actions in the synchronous case, so the synchronous case represents all the cases in the limit (a straightforward generalization of Lemma 1.) We now prove the main convergence result.

Theorem 5. *Given a connected interaction graph, consider the synchronous case where for all agents i, $r'_i > 0$. If there exists a circle of an odd length in the graph (or at least one agent i employs* floating *reciprocation and has $r_i + r'_i < 1$), then, for all pairs of agents $i \neq j$ such that $(i, j) \in E$, the limit $L_{i,j}$ exists and it is a positive combination of all the kindness values $k_1, \ldots, k_n$. The convergence is geometrically fast. Moreover, if all agents employ* floating *reciprocation, then* *all these limits are equal to each other and it is a convex combination of the kindness values, namely*

$$L = \frac{\sum_{i \in N} \left(\frac{d(i)}{r_i + r'_i} \cdot k_i \right)}{\sum_{i \in N} \left(\frac{d(i)}{r_i + r'_i} \right)}. \tag{3}$$

Proof. We first prove the case where all agents use *floating* reciprocation, when the ambivalent case of $r_i + r'_i = 1$ is taken to be *floating*. We express how each action depends on the actions in the previous time in a matrix, and prove the theorem by applying the famous Perron-Frobenius theorem [23, Theorem 1.1, 1.2] to this matrix. We now define the dynamics matrix $A \in \mathbb{R}_+^{|E| \times |E|}$:

$$A((i,j),(k,l)) \triangleq \begin{cases} (1 - r_i - r'_i) & k = i, l = j; \\ r_i + r'_i \frac{1}{|\mathrm{N}^+(i)|} & k = j, l = i; \\ r'_i \frac{1}{|\mathrm{N}^+(i)|} & k \neq j, l = i; \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

According to the definition of *floating* reciprocation, if for each time $t \in T$ the column vector $\boldsymbol{p(t)} \in \mathbb{R}_+^{|E|}$ describes the actions at time t, in the sense that its (i,j)th coordinate contains $\mathrm{act}_{i,j}(t)$ (for $(i,j) \in E$), then $\boldsymbol{p}(t+1) = A\boldsymbol{p}(t)$. We then call $\boldsymbol{p(t)}$ an action vector. Initially, $\boldsymbol{p}_{(i,j)}(0) = k_i$.

Further, we shall need to use the Perron-Frobenius theorem for primitive matrices. We now prepare to use it, and first we show that A is primitive. First, A is irreducible since we can move from any $(i,j) \in E$ to any $(k,l) \in E$ as follows. We can move from an action to its reverse, since if $k = j, l = i$, then $A((i,j),(k,l)) = r_i + r'_i \frac{1}{|\mathrm{N}^+(i)|} > 0$. We can also move from an action to another action by the same agent, since we can move to any action on the same agent and then to its reverse. To move to an action on the same agent, notice that if $l = i$, then $A((i,j),(k,l)) \geq r'_i \frac{1}{|\mathrm{N}^+(i)|} > 0$. Now, we can move from any action (i,j) to any other action (k,l) by moving to the reverse action (j,i) (if $k = j, l = i$, we are done). Then, follow a path from j to k in graph G by moving to the appropriate action by an agent and then to the reverse, as many times as needed till we are at the action (k,j) and finally to the action (k,l). Thus, A is irreducible.

By definition, A is non-negative. Next, we notice that A is aperiodic, since either at least one agent i has $r_i + r_j < 1$ and thus the diagonal contains non-zero elements, or there exists a circle of an odd length in the interaction graph G. In the latter case, let the circle be $i_1, i_2, \ldots, i_p$ for an odd p. Consider the following cycles between the index set of the matrix: $(i,j),(j,i),(i,j)$ for any $(i,j) \in E$ and $(i_2,i_1),(i_3,i_2),\ldots,(i_p,i_{p-1}),(i_1,i_p),(i_2,i_1)$. Their lengths are 2 and p, respectively, which greatest common divisor is 1, implying aperiodicity. Being irreducible and aperiodic, A is primitive by [23, Theorem 1.4]. Since the sum of every row is 1, the spectral radius is 1.

According to the Perron-Frobenius theorem for primitive matrices [23, Theorem 1.1], the absolute values of all eigenvalues except one eigenvalue of 1 are strictly less than 1. The eigenvalue 1 has unique right and left eigenvectors, up to a constant factor. Both these eigenvectors are strictly positive. Therefore, [23, Theorem 1.2] implies that $\lim_{t \to \infty} A^t = \mathbf{1}\boldsymbol{v'}$, where $\boldsymbol{v'}$ is the left eigenvector of the value 1, normalized such that $\boldsymbol{v'}\mathbf{1} = 1$, and the approach rate is geometric. Therefore, we obtain $\lim_{t \to \infty} \boldsymbol{p(t)} = \lim_{t \to \infty} A^t \boldsymbol{p}(0) = \mathbf{1}\boldsymbol{v'}\boldsymbol{p}(0) = \mathbf{1} \sum_{(i,j) \in E} v'((i,j))k_i$. Thus actions converge to $\mathbf{1}$ times $\sum_{(i,j) \in E} v'((i,j))k_i$.

To find this limit, consider the vector v' defined by $v'((i,j)) = \frac{1}{r_i + r'_i}$. Substitution shows it is a left eigenvector of A. To normalize it such that $\boldsymbol{v'}\mathbf{1} = 1$, divide this vector by the sum of its coordinates, which is $\sum_{i \in N} \frac{d(i)}{r_i + r'_i}$, obtaining $v'((i,j)) = \frac{1}{\sum_{i \in N} \frac{d(i)}{r_i + r'_i}} \cdot \frac{1}{r_i + r'_i}$. Therefore, the common limit is $\frac{\sum_{i \in N} \left(\frac{d(i)}{r_i + r'_i} \cdot k_i \right)}{\sum_{i \in N} \left(\frac{d(i)}{r_i + r'_i} \right)}$.

We now prove the case where at least one agent employs *fixed* reciprocation. We define the dynamics matrix A analogously to the previous case, besides that the first line from (4) is missing, since own behavior does not matter. In this case, we have $\boldsymbol{p}(t+1) = A\boldsymbol{p}(t) + \boldsymbol{k'}$, where $\boldsymbol{k'}$ is the relevant kindness vector, formally defined as $k'((i,j)) \stackrel{\Delta}{=} (1 - r_i - r'_i)k_i$. By induction, we obtain $\boldsymbol{p}(t) = A^t \boldsymbol{p}(0) + \left(\sum_{l=0}^{t-1} A^l \right) \boldsymbol{k'}$.

Analogically to the previous case, A is irreducible and non-negative. As shown above, A is aperiodic. Since A is also irreducible, we conclude that is is primitive. Since at least one agent employs *fixed* reciprocation, at least one line of A sums to less than 1, and therefore the spectral radius of A is strictly less than 1.

Now, the Perron–Frobenius implies that all the eigenvalues are strictly smaller than 1. Since we have $\lim_{t \to \infty} \boldsymbol{p}(t) = \lim_{t \to \infty} A^t \boldsymbol{p}(0) + \left(\lim_{t \to \infty} \sum_{l=0}^{t-1} A^l \right) \boldsymbol{k'}$, [23, Theorem 1.2] implies that this limits exist (the first part converges to zero, while the second one is a series of geometrically decreasing elements.) Since A is primitive, $\left(\lim_{t \to \infty} \sum_{l=0}^{t-1} A^l \right) > 0$.

As an immediate conclusion of this theorem, we can finally generalize Theorem 3 to the case $r_1 + r_2 > 1$ as follows.

Corollary 2. *Consider synchronous pairwise interaction, where one agent i employs* fixed *reciprocation and the other agent j employs the* floating *one. Assume that $r_i < 1$ and $\min\{r_i, r_j\} > 0$. Then, both limits exist and are equal to k_i. The convergence is geometrically fast.*

Proof. The situation can be equivalently described by considering $r'_l = r_l, r_l = 0$, for all $l \in N$. Then, Theorem 5 implies geometrically fast convergence. We find the limits as in the proof of Theorem 3.

Let us consider several examples of Formula(s) (3).

Example 2. If the interaction graph is regular, that is all the degrees are equal to each other, we have $L = \frac{\sum_{i \in N} \left(\frac{k_i}{r_i + r'_i} \right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i} \right)}$. This is the case for cliques, modeling small human collectives, and for circles, modeling circular computer networks.

Example 3. If the interaction graph is a star, modeling such networks of a supervisor of several people or entities, assume w.l.o.g. that agent 1 is the center, and we have $L = \frac{\frac{n-1}{r_1+r'_1} \cdot k_1 + \sum_{i \in N \setminus \{1\}} \left(\frac{k_i}{r_i+r'_i}\right)}{\frac{n-1}{r_1+r'_1} + \sum_{i \in N \setminus \{1\}} \left(\frac{1}{r_i+r'_i}\right)}$.

We can also prove a more general convergence result, allowing agents to act in a more general way than modeled above. It is omitted due to lack of space.

5 Conclusions and Future Work

To understand reciprocation, we modeled two reciprocation attitudes where a reaction is a weighted combination of the action of the other player, the total action of the whole neighborhood and either one's own kindness or one's own last action. This combination's weights are defined by the reciprocation coefficients. For a pairwise interaction, we showed that actions converge, found the exact limits, and showed that if you consider your kindness while reciprocating (*fixed*), then, asymptotically, your actions values get closer to your kindness, than if you consider it only at the outset. For a network, we proved convergence and found the common limit if all agents consider their last own action (*floating*). We now substantiate these insights from our results, beginning from the pairwise case.

For two agents with *fixed* reciprocation, (i.e., when a reaction partly depends on one's kindness), kinder agent's action are larger in the limit. While interacting, each agent goes back and forth in her actions, monotonically narrowing to her limit. Probably, this alternating may make the process confusing for an outsider.

For two agents with *floating* reciprocation, (i.e., when a reaction partly depends on one's own last action), both agents' actions converge to a common limit, which vicinity to an agent's kindness is reversely proportional to her reciprocation coefficient. The commonality of the limit can intuitively result from an agent's next action being a combination of her last action with the other agent's last action, which makes the new action closer to the other's action, this new action to be taken into account in determining the next action.

For two agents, when one agent acts according to the *fixed* reciprocation, and the other one according to the *floating* one, both actions converge to the kindness of the agent who employs *fixed* reciprocation. This can be intuitively explained as a result of one agent always considering her kindness in determining the next action and thereby having a firm stance, while the other agent aligning himself. In Example 1 with two colleagues, the colleague who ignores her inherent inclination and remembers only the last moves will behave as the colleague who constantly considers her kindness. Another conclusion is that if the numerical parameters are set, then the kinder agent employing *fixed* attitude and the other one employing *floating* attitude is the best for the total reciprocation.

When an agent may interact with any number of agents, we have proven convergence and shown that if all agents employ *floating* reciprocation, the limit is common. This limit is a weighted average of the kindness values, the weight of an agent's kindness being her degree in the interaction graph divided by the sum of her reciprocation coefficients. Intuitively, the agents align to each other, and the more connected and the less reciprocating an agent is, the more it influences the common limit. In Example 1 with the parameters from the end of Section 2, (all the agents employ *floating* reciprocation), (3) implies that all the actions approach $\frac{\frac{2}{0.5+0.3}\cdot 0+\frac{2}{0.5+0.3}\cdot 0.5+\frac{2}{0.8+0.1}\cdot 1}{\frac{2}{0.5+0.3}+\frac{2}{0.5+0.3}+\frac{2}{0.8+0.1}} = 25/52$ in the limit, that is they all support each other emotionally a lot.

In real life, the fact that persistence makes the interaction go one's way is reflected, for example, in recommendations to reject undesired requests by firmly repeating the reasons for rejection [4, Chapter 1] and [25, Chapter 8]. Another expression of our theory is that growing up, people acquire their own style of reciprocating with their acquaintances [21], which are the limiting actions. In organizations, many styles are often very similar from person to person, though not the same, and they vary across various organizations [15].

As we saw in examples, real life situations sometimes require more complex models, motivating further research. For instance, modeling interactions with a known finite time horizon would be interesting. Since people may change while reciprocating, modeling changes in the reciprocity coefficients and/or reciprocation model would be interesting. In addition, groups of colleagues and nations get and lose people, motivating modeling a dynamically changing set of reciprocating agents. We have looked into the interaction process where agents follow predefined strategies. To predict real situations and to be able to give constructive advice about what parameters and attitudes of the agents are useful, we should define utility functions to the agents and consider the game where agents choose their parameters before the interaction commences. The agents' strategizing behavior may come at cost with respect to the social welfare, so considering price of anarchy [16] and stability [1] of such a game is in order. Considering how to influence agents to change their behavior is also relevant.

Analytically proving properties of the interaction process lays the foundation to further modeling and analysis of reciprocation, in order to predict and improve the individual utilities and the social welfare.

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