Modelling Extremes of Bivariate Time-Series

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Background

In the design of offshore facilities, e.g. oil platforms or vessels, it is crucial - both for safety and reliability reasons - that they can survive the most extreme storms. Thus, we need accurate models describing the time evolution of the ocean environment during extreme events.

We focus on modelling significant wave height H_S and wind speed W_s jointly over time.

Vector autoregression

In multivariate time-series, it is common to use a VAR model for a d-dimensional time series $\boldsymbol{X}_t$ with $t \in \mathbb{Z}_{\geq 0}$. The VAR(p) is given by

$$\mathbf{X}_t = \sum_{i=1}^{p} A_i \mathbf{X}_{t-i} + \boldsymbol{\epsilon}_t,$$

where $A_i \in \mathbb{R}^{d\times d}$ and $\boldsymbol{\epsilon}_t$ is a d-dimensional random variable that is independent of $\mathbf{X}_{t-1}$ and independent of $\boldsymbol{\epsilon}_{t-k}$ for $k \geq 1$.

NB: this model is not necessarily stationary for any collection of matrices A_i, $i = 1, \ldots, p$.

Markov-Extremal model

Winter and Tawn (2016, 2017) define MEM(k): a model that captures the extremes of a stationary kth order Markov chain.

Assume that for any t, $X_{t:t+k} = (X_t, \ldots, X_{t+k})$ are identically distributed random variables with standard Laplace margins. They then model

$$X_{t+1:t+k} \mid (X_t > u) = \alpha_{1:k} X_t + X_t^{\beta_{1:k}} Z_{1:k} \qquad (1)$$

with u large, $|\alpha_i|, \beta_i \leq 1$ for $i = 1, \ldots, k$, and $Z_{1:k}$ a k dimensional random variable independent of X_t.

From stationarity, they derive the distribution of the next value conditional on the previous by integrating out all but the last term in model (1). In particular, for $j \geq 1$,

$$X_{t+k+j} \mid (X_{t+j} > u, X_{t+j+1:t+j+k-1}) = \alpha_k X_{t+j} + X_{t+j}^{\beta_k} Z_{k|1:k-1}. \qquad (2)$$

Multivariate Markov Extremal Model

The bivariate extension MMEM(k) of the MEM(k) for (X_t, Y_t) with $t \in \mathbb{Z}_{\geq 0}$ describes the dependence via

$$(X_{t+1:t+k}, Y_{t:t+k}) \mid (X_t > u) = \alpha_{1:2k+1} X_t + X_t^{\beta_{1:2k+1}} Z_{1:2k+1}.$$

Our Time-Series Models

Our idea is to combine VAR and MEM to an Extremal Vector Autoregressive (EVAR) model. In particular, let (X_t, Y_t) be a time-series with $t \in \mathbb{Z}_{\geq 0}$ on standard Laplace margins.

Then EVAR(1,1) is

$$\begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} \mid (X_t > u, Y_t) = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} + X_t^{\beta_{1:2}} Z_{1:2}.$$

Similar techniques to those in equation (2) are used to describe the distribution of the next value conditional on the past.

In general, EVAR(p,1) extends EVAR(1,1) using p previous terms on the right-hand side:

$$\begin{pmatrix} X_{t+p} \\ Y_{t+p} \end{pmatrix} = \sum_{i=1}^{p} \begin{pmatrix} \alpha_{11,i} & \alpha_{12,i} \\ \alpha_{21,i} & \alpha_{22,i} \end{pmatrix} \begin{pmatrix} X_{t+p-i} \\ Y_{t+p-i} \end{pmatrix} + X_t^{\beta_{1:2}} Z_{1:2}$$

and EVAR(1,k) extends EVAR(1,1) using k terms on the left-hand side

$$\begin{pmatrix} X_{t+1:t+k} \\ Y_{t+1:t+k} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \vdots & \vdots \\ \alpha_{2k,1} & \alpha_{2k,2} \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} + X_t^{\beta_{1:2k}} Z_{1:2k}.$$

EVAR presents a number of practical difficulties:

- Estimating the distribution of $Z_{2k|1:2k-1}$ non-parametrically is hard;
- Comparing EVAR(p,1) with EVAR(q,1) for $p \neq q$ is not as trivial as comparing VAR(p) with VAR(q) since the conditioning sets are not the same.

Findings, Simulations and more

For the latest set of information, scan the QR code. Here, you can find:
(1) discussion on model selection;
(2) which model performs best for data observed in the North Sea;
(3) a grid of plots containing data of real storms and simulated storms. Can you tell which one is which?

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