LECTURE 2: SINGLE VARIABLE OPTIMIZATION

ANSWERS AND SOLUTIONS

Answers to True/False Questions

False_ The profit maximization model of the firm is invalidated if managers are able to maximize profits using rules of thumb and professional experience rather than explicit optimization.

False_ The first order conditions of optimization provide the conditions for a maximum, while the second order conditions of optimization provide the conditions for a minimum.

True_ A manager who operates a firm to maximize value of his compensation is behaving according to the optimization hypothesis.

False_ The optimization hypothesis says that the firm's management will choose its actions so as to maximize the firm's profits.

True_ The only function with a derivative equal to itself is the exponential function.

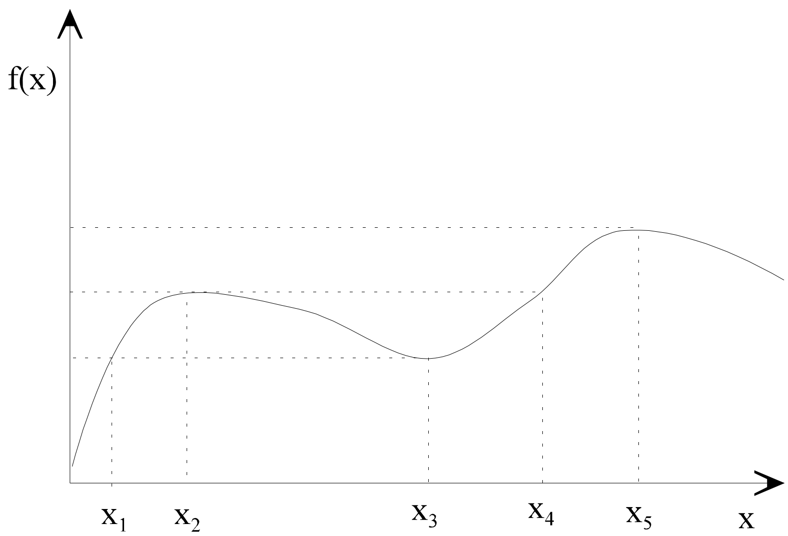
True_ If managers are able to profit maximize by trial and error, one can use optimization based models to describe firm behavior.

False_ The first order conditions of optimization provide the necessary conditions for a minimum, while the second order conditions of optimization provide the necessary conditions for a maximum.

False_ If a manager of a firm makes his managerial decisions to maximize the firm's profit per unit output, then this manager does not behave in accordance with the optimization hypothesis.

Answers to Short Questions

1. Consider the function graphed below.



a. Which of the values of *x* shown above would satisfy the first order conditions of maximization of the function *f*(*x*)?

They are x_2, x_3, and x_5.

b. Which of these values of *x* would satisfy both the first and second order conditions of maximization?

They are x_2 and x_5.

c. Which of the values of *x* shown above would satisfy the first order condition of minimization of the function *f*(*x*)?

The first order condition of minimization is the same as the same as the first order condition of maximization. Therefore, the values of *x* that satisfy the first order condition of minimization are x_2, x_3, and x_5.

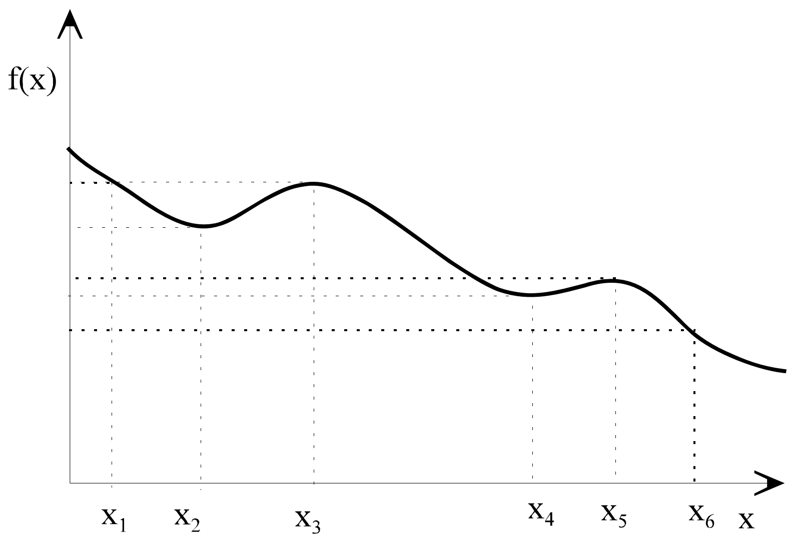
d. Which of these values of *x* would satisfy both the first and second order conditions of minimization?

There is only one such value of *x* and it is x_3.

2. Give two reasons that have led to the widespread use of optimization-based economics.

The three main reasons for why optimization-based economics is prevalent are: (i) the assumption of optimal behavior is a reasonably good approximation for average behavior in most economic settings, (ii) the concept of optimization is very precise, and (iii) there is a large number of available tools that can be used to analyze problems that are posed as optimization problems.

3. Consider the function graphed below.



a. Which of the values of x shown above would satisfy the first order condition of minimization of the function $f(x)$?

They are x_2, x_3, x_4 and x_5.

b. Which of these values of x would satisfy both the first and second order conditions of minimization?

They are x_2 and x_4.

c. Which of the values of x shown above would satisfy the first order condition of maximization of the function $f(x)$?

The first order condition of maximization is the same as the first order condition for minimization. Therefore, the values of x that satisfy the first order conditions of maximization are x_2, x_3, x_4 and x_5.

d. Which of these values of x would satisfy both the first and second order conditions of maximization?

They are x_3 and x_5.

4. What is the derivative of $5x^2+2\log(1+x)$?

It is $10\,x + \dfrac{2}{1+x}$

5. What is the derivative of $(3r^2+5)r^3$?

It is equal to

$$\begin{aligned} 6\,r\,r^3 + 3\,r^2\,(3\,r^2+5) &= 6\,r^4 + 9\,r^4 + 15\,r^2 \\ &= 15\,r^4 + 15\,r^2 \\ &= 15\,r^2\,(r^2+1) \end{aligned}$$

6. Consider the function $f(y) = 3\sqrt{y}$. Suppose that y is given by the function

$$y(z) = 4\log(1+z)$$

What is the derivative of $f(y(z))$ with respect to z? [Hint: you will need to use the chain rule.]

The derivative of $f(y(z))$ with respect to z is given by the chain rule as:

$$\frac{df(y(z))}{dz} = \frac{df}{dy}\frac{dy}{dz}$$

We will first evaluate separately each derivative on the right hand side of the above expression.

$$\frac{df}{dy} = 3 \ \frac{1}{2} \ y^{-1/2}$$

$$\frac{dy}{dz} = \frac{4}{1 + z}$$

Therefore,

$$\frac{df(y(z))}{dz} = \frac{3}{2} \ y(z)^{-1/2} \ \frac{4}{1 + z}$$

$$= 6 \ \frac{y(z)^{-1/2}}{1 + z}$$

Substituting the expression for $y(z)$ above yields:

$$\frac{df(y(z))}{dz} = 6 \ \frac{(4 \ \log(1 + z)^{-1/2})}{1 + z}$$

This in turn simplifies to:

$$\frac{df(y(z))}{dz} = \frac{6}{\sqrt{4} \ \sqrt{\log(1+z)} \ (1 + z)}$$

$$= \frac{3}{\sqrt{\log(1+z)} \ (1 + z)}$$

Solutions to Problems

1. A firm's profit function is given by $\Pi(q) = 10+2q-4q^2$, where q is the firm's output.

a. How does an increase in output affect profits, i.e., what is the derivative of the profit function with respect to q?

The derivative of the profit function with respect to output is

$$\frac{d\Pi(q)}{dq} = 2 - 8q$$

b. For what range of output, does an increase in output increase profits?

The derivative is positive if

$$2 - 8q > 0 \quad \Rightarrow$$

$$1 > 4q \quad \Rightarrow$$

$$q < \frac{1}{4}$$

Therefore, an increase in output increases profits if output is less than 1/4.

2. A consumer's willingness to pay for a basket of oranges is $WTP(x) = 7 \log(1+2x)$ where x is the weight of the basket. [All logs in this course refer to the natural log].

a. What is the per-unit-of-weight willingness of the consumer to pay for oranges. That is, what is the ratio of willingness to pay for a basket of oranges divided by the basket's weight?

The per-unit-of-weight willingness to pay for oranges is

$$\frac{WTP(x)}{x} = \frac{7\log(1+2x)}{x}$$

b. How does an increase in the weight of the basket affect the per-unit-of-weight willingness to pay for oranges? Show your answer using calculus.

Using the ratio rule, we can calculate the derivative of the per-unit-of-weight willingness to pay for oranges as follows:

$$\frac{d\left[\frac{WTP(x)}{x}\right]}{dx} = \frac{7\,\frac{1}{1+2\,x}\,2\,x - 7\log(1+2x)}{x^2}$$

$$= \frac{14}{x\,(1+2x)} - 7\,\frac{\log(1+2x)}{x^2}$$

3. A firm is contemplating a national advertizing campaign by airing television commercials on a papular network series. The per minute cost of advertising on this series is 1 million dollars, and the advertising will air throughout the country.

The firm obtains revenue from 3 regions: The North, the South, and the West. The revenue (in millions of dollars) that the firm obtains from these regions depends on the number of minutes, *A*, consumers are exposed to this advertising and is given by

$$R_{North} = 10 + 2\sqrt{A},$$

$$R_{South} = 5 + \frac{3}{4}A$$

and

$$R_{West} = 7 + \sqrt{A} - \frac{1}{2}A$$

It is clear that greater exposure to the commercial, the greater the revenue in the North and the South, but in the West, too much advertising eventually turns consumers off (you can verify this claim by calculating the derivative of Π_{West} with respect to *A* and showing that it is negative for very high values of *A*). Given that the advertising is broadcast nationally, the value of *A* will be the same for all three regions.

The firm wants to determine what is the level of advertising (in minutes) that will maximize its profit. Assume that there are no other costs; thus profit equals the sum of revenue from the three markets minus the cost of the advertising.

a. What are the profits of the firm as a function of the minutes of advertising bought?

Adding up the revenues from the three regions and subtracting the advertising costs, we obtain the profit function of the firm in millions of dollars to be

$$\Pi(A) = 10 + 2\sqrt{A} + 5 + \frac{3}{4}A + 7 + \sqrt{A} - \frac{1}{2}A - A$$

which simplifies to

$$\Pi(A) = 22 + 3\sqrt{A} - \frac{3}{4}A$$

b. What is the optimal amount of money spent on this campaign?

Maximizing $\Pi(A)$ with respect to A yields the First Order Condition

$$\frac{d\Pi(A)}{dA} = 0 \quad \Rightarrow \quad 3\,\frac{1}{2}A^{-\frac{1}{2}} - \frac{3}{4} = 0$$

Simplifying and solving for A, we obtain

$$A^{-\frac{1}{2}} - \frac{1}{2} = 0 \quad \Rightarrow$$

$$A^{-\frac{1}{2}} = \frac{1}{2} \quad \Rightarrow$$

$$\frac{1}{A} = \frac{1}{4} \quad \Rightarrow$$

$$A = 4$$

This is the optimal amount of advertising in minutes. Given that each minute costs \$1 million, the optimal expenditure in millions of dollars is 4.

c. What is the revenue from each region at the optimal advertising expenditure?

Substituting $A = 4$ in the three revenue expressions above we obtain

$$R_{North} = 10 + 2\sqrt{4}$$

$$= 14$$

$$R_{South} = 5 + \frac{3}{4}4$$

$$= 8$$

$$R_{West} = 7 + \sqrt{4} - \frac{1}{2}4$$

$$= 7$$

d. Suppose that the firm could ask the television network not to air some of the commercials in the West even though the firm paid for them. Would the firm choose to it? Why?

To answer this question, we need to evaluate the slope of the revenue in the West at $A = 4$. If the slope of R_{West} at $A = 4$ is positive or zero, then it would not pay for the firm to "through away" advertising that it has paid for. If it is negative, then it would be preferable to through away advertising in the West, even though the firm has paid for it. That is, if $dR_{West}/dA < 0$, then airing less advertising would increase revenue in the West.

The derivative of R_{West} with respect to A is

$$\frac{dR_{West}}{dA} = \frac{1}{2}\frac{1}{\sqrt{A}} - \frac{1}{2}$$

At $A = 4$, the derivative takes the value of $1/4 - 1/2 = -1/4$. Therefore, not airing in the West some of the already paid for advertising would increase revenues there.

4. A firm's profit as a function of its physical capital (i.e., tangible assets) is given by

$$\Pi(K) = 5\sqrt{K} - 1$$

The rate of return on physical assets is the ratio of profits $\Pi(K)$ to the level of physical assets, K

$$RoR(K) = \frac{\Pi(K)}{K} = \frac{5\sqrt{K} - 1}{K}$$

a. What is the value of K that maximizes the rate of return on physical assets?

The first order condition of maximizing the rate of return with respect to K is

$$\frac{d\,RoR(K)}{dK} = 0 \quad \Rightarrow \quad \frac{5\,\frac{1}{2}\,K^{-\frac{1}{2}}\,K \,-\, 5\,K^{\frac{1}{2}} \,+\, 1}{K^2} = 0 \quad \Rightarrow$$

$$5\,\frac{1}{2}\,K^{-\frac{1}{2}}\,K \,-\, 5\,K^{\frac{1}{2}} \,+\, 1 = 0 \quad \Rightarrow$$

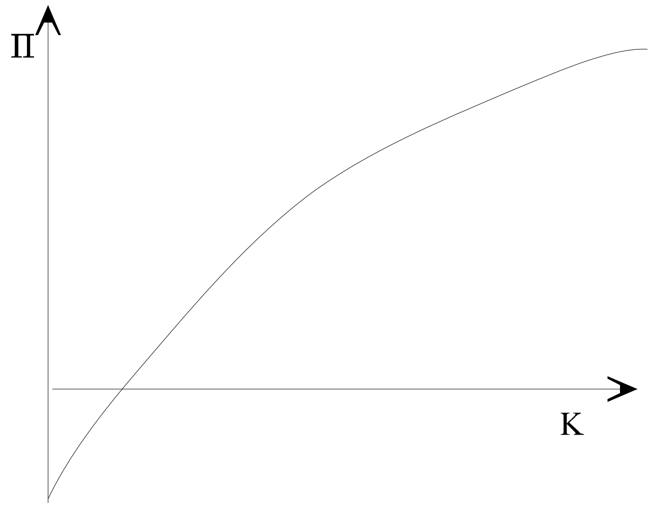
$$5\,\frac{1}{2}\,K^{\frac{1}{2}} \,-\, 5\,K^{\frac{1}{2}} \,+\, 1 = 0 \quad \Rightarrow$$

$$\frac{5}{2}\,K^{\frac{1}{2}} = 1 \quad \Rightarrow$$

$$K = \frac{4}{25}$$

b. Would a manager that wants to maximize the firm's profits increase the size of the firm (i.e., choose a higher value of K) or decrease it (i.e., choose a smaller value of K)? [Hint: Plot the profit function to see what it looks like before you take any derivatives.]

There are two ways to answer this question. The first is to graph the profit function. It looks like this:



The function is monotonically increasing in *K*. There simply is no finite optimal size: the bigger the firm the better. Therefore, a manager that limits the size of the firm to 4/25 (which is the solution in part (a) above) is certainly keeping the firm too small.

The other way to answer this question is to evaluate the derivative of the profit function at *K*=4/25. The derivative of the profit function is

$$\frac{d\,\Pi(K)}{d\,K} = 5\ \frac{1}{2}\ K^{-\frac{1}{2}}$$

This is positive for every value of *K*, including for *K*=4/25. Therefore, not only is a firm with a size of capital stock of 4/25 too small, but a manager should expand the size of this firm as much as possible.

5. A worker total satisfaction from the income, *I*, he earn's is given by

$$f(I) = 2\ I^{2/3}$$

The income he earns is a function of the effort, *E*, be puts into his work. This function is given by

$$I(E) = 3\ E^{1/2}$$

However, the more effort he puts, the more stress and pressure he feels, which reduces his satisfaction. Therefore, effort increases his satisfaction indirectly through his income, but decreases it directly through an increase in stress and pressure. The combined effect is given by the utility function $U(E)$:

$$\begin{aligned} U(E) &= 2\ I(E)^{2/3} - E \\ &= 2\ (3E^{1/2})^{2/3} - E \end{aligned}$$

a. What is the derivative of $U(E)$ with respect to E? [Hint: you may use the chain rule, but you may also simplify the first term of $U(E)$ to avoid using it.]

In order to make the differentiation easier, we will first simplify the expression for the worker's utility by observing that

$$\begin{aligned} U(E) &= 2\ (3\ E^{1/2})^{2/3} - E \\ &= 2\ 3^{2/3}\ E^{2/6} - E \\ &= 2\ 3^{2/3}\ E^{1/3} - E \end{aligned}$$

The derivative of this expression with respect to E is

$$\frac{d\,U(E)}{d\,E} = \frac{1}{3}\ 2\ 3^{2/3}\ E^{-2/3} - 1$$

b. What effort level maximizes the worker's utility, $U(E)$?

The effort level that maximizes the worker's utility satisfies the first order condition

$$\frac{d\,U(E)}{d\,E} = 0 \quad \Rightarrow$$

$$\frac{1}{3}\ 2\ 3^{2/3}\ E^{-2/3} - 1 = 0 \quad \Rightarrow$$

$$\frac{1}{3}\ 2\ 3^{2/3}\ E^{-2/3} = 1 \quad \Rightarrow$$

$$\frac{2\ 3^{2/3}}{3\ E^{2/3}} = 1 \quad \Rightarrow$$

$$E^{2/3} = \frac{2\ 3^{2/3}}{3} \quad \Rightarrow$$

$$E = \left(\frac{2}{3}\right)^{3/2} 3$$

6. Suppose a firm can sell its output at a price P. Then, its revenue is equal to P times q, where q is its output level. [P is the market price and is assumed to be beyond the control of the firm. The firm only controls its output.] Also, suppose that the costs of the firm are equal to $5 + q^2$, where the first term captures the fixed costs and the second term the variable costs.

The firm's profits are equal to its revenues minus its costs.

a. Write down the firm's profit function (i.e, write the expression for its profits).

$$\Pi(q) = P\,q - q^2 - 5$$

b. What choice of output maximizes the firm's profits ? How does this choice respond to an increase in the market price ?

Maximizing the profit function with respect to output yields the First Order Condition

$$\frac{d\,\Pi(q)}{d\,q} = 0 \quad \Rightarrow \quad P - 2\,q^* = 0$$

which solving for q^* yields

$$q^* = \frac{P}{2}$$

The higher the market price, the higher the profit maximizing choice of output.

c. Derive the firm's profits at the profit maximizing choice of output.

The profit function, when the firm is choosing the optimal level of output, is given by:

$$\Pi(q^*(P)) = P\,\frac{P}{2} - \left(\frac{P}{2}\right)^2 - 5$$

$$= \frac{P^2}{2} - \frac{P^2}{4} - 5$$

$$= \frac{P^2}{4} - 5$$

Notice that in denoting the profit function by $\Pi(q^*(P))$ we make explicit the dependence of the profits (evaluated at the optimal choice of output) on the market price.

Also note that the profits at the optimal level of output do *not* depend on the firm's output since that decision has been already *solved out* in terms of price.

d. How high must the price be for the firm's profits to be positive. [That is, compute the lowest price at which the firm's profits will not be negative.]

For $\Pi(p^*) > 0$ we need

$$\frac{P^2}{4} - 5 > 0 \quad \Rightarrow$$

$$P^2 > 20 \quad \Rightarrow$$

$$P > \sqrt{20}$$

When the price equals $\sqrt{20}$, the profits of the firm drop to zero. For prices that are even lower, profits turn negative.

7. Suppose the profits of a firm, as a function of installed capital, are given by

$$\Pi(K) = 10\ \log(1+K) - K$$

a. What are the firm's profits if the firm has no capital ?

When $K = 0$, the profits of the firm are

$$\Pi(0) = 10 \log(1+0) - 0$$
$$= 10 \log(1)$$
$$= 0$$

b. What capital level would maximize the firm's profits ?

The First Order Condition of profit maximization with respect to capital yields

$$\frac{d\,\Pi(K)}{d\,K} = 0 \quad \Rightarrow \quad 10 \frac{1}{1+K^*} - 1 = 0$$

Solving for K^* we get

$$\frac{10}{1 + K^*} = 1 \quad \Rightarrow$$

$$10 = 1 + K^* \quad \Rightarrow$$

$$K^* = 9$$

c. Compute the firm's profits at the optimal level of capital.

Substituting K^* into the profit function we get:

$$\Pi(K^*) = 10 \log(1 + 9) - 9$$
$$= 10 \log(10) - 9$$
$$= 14.026$$