LECTURE 6: MULTI-INPUT PRODUCTION FUNCTIONS

ANSWERS AND SOLUTIONS

True/False Questions

True_ The returns to scale of a multi-input production function depend on the composition of its inputs.

False_ If the isoquants of the production function $f(K, L)$ touch the L-axis, then this means that input K is essential for production, i.e., that you need at least some of input K to produce a positive output.

False_ If the isoquants of the production function $f(K, L)$ touch the K-axis, then this means that input K is essential for production, i.e., that you need at least some of input K to produce a positive output.

True_ The *MRTS* gives the trade off between two inputs holding output constant, i.e., the amount of one input that would necessary to make up for the reduction of another input so that output remains constant.

False_ The concept of the *MRTS* is defined only for production functions of two inputs, because the concept involves two inputs.

True_ The concept of MRTS is defined even for production functions of more than two inputs.

False_ The *MRTS* between two inputs is independent of the usage level of any additional inputs, because the *MRTS* is equal to the ratio of the marginal products of these two inputs.

True_ If the isoquants of the production function $f(K, L)$ touch the L-axis, then this means that one can produce a given level of output using only labor inputs.

Short Questions

1. Consider the production function

$$q = 3\ L^{0.7}\ K^{0.1} + 2\ L^{0.3}\ E$$

where L is the level of labor inputs, K the level of capital inputs, and E the level of energy inputs.

a. What is the marginal product of labor?

The marginal product of labor is given by the partial derivative of output with respect to labor. This is given by

$$\frac{\partial q}{\partial L} = 2.1\ L^{-0.3}\ K^{0.1} + 0.6\ L^{-0.7}\ E$$

b. Does the marginal product of labor depend on the level of capital employed by the firm?

Yes, it clearly does because K appears in the expression for the marginal product of L.

c. Does more capital make labor more productive? (in other words, does the marginal product of labor increase as capital increases?)

Yes, it does. This can be seen by simply observing the expression for the marginal product of labor, because K is multiplied by a positive quantity. It can also be inferred by taking the cross partial derivative of q with respect to L and K, which is

$$\frac{\partial^2 q}{\partial L \partial K} = 0.21\ L^{-0.3}\ K^{-0.9} > 0$$

2. Consider the following production function

$$f(x,y,z) = x^2 + y^2 + x\,z$$

A. Graph the isoquant of y and z for this production function for output equal to 20 and $x = 2$.

Put z on the vertical axis and y in the horizontal axis. Label any intercepts. Show your work.

Setting output to 20 and x to 2 in the above production function we obtain

$$20 = 2^2 + y^2 + 2\,z \quad \Rightarrow$$

$$16 = y^2 + 2\,z$$

This is the equation for the isoquant involving input y and z for q=20 and x=2. To plot it with z in the vertical axis, we need to solve the above equation for z. This gives:

$$z = 8 - \frac{1}{2}\,y^2$$

This quadratic equation has an inverted U-shape. It's maximum is at y=0, at which point it takes the value of 8. When z=0, the value of y is given by

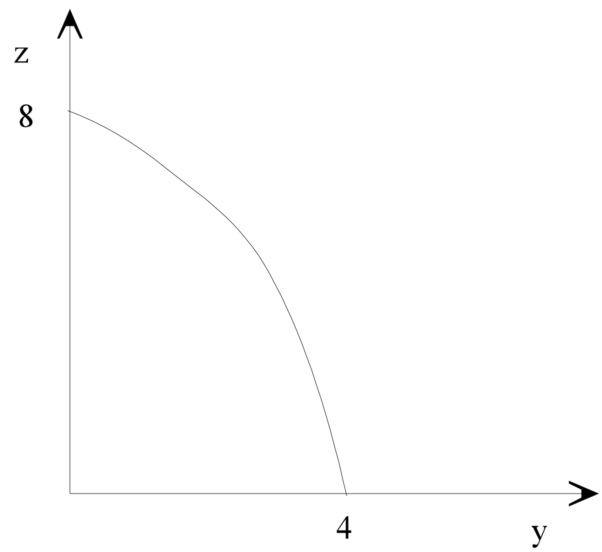
$$0 = 8 - \frac{1}{2}\,y^2 \quad \Rightarrow$$

$$8 = \frac{1}{2}\,y^2 \quad \Rightarrow$$

$$y^2 = 16 \quad \Rightarrow$$

$$y = 4$$

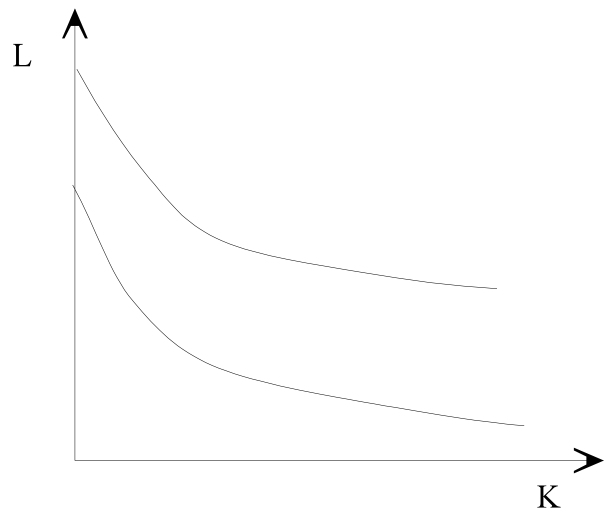
Thus, the shape of the isoquant is given below



B. Does this isoquant exhibit decreasing *MRTS*? Why or why not?

It does not. The slope of the isoquant is *increasing* as we move from left to right. As more y is used to produce 20 units of output, the marginal product of *y*, relative to that of *z*, goes up, *not* down.

3. Consider the following two isoquants of a production function:



A. Does this production function exhibit diminishing *MRTS*? Why or why not?

Yes, it does. Diminishing *MRTS* means that as you increase the ratio of one input over another, holding output constant, the marginal product of this input *relative* to that of the other input is decreasing. This manifests itself in the steepening of the slope of the isoquant as K approaches zero (which means that capital becomes more valuable relative to labor as less capital is used), or equivalently, in the flattening of the slope as K become large (which means that capital becomes less valuable relative to labor as more capital is used).

B. Is capital (K) an essential input for this production process? Explain your answer.

No it is not. One can produce the required level of output given by the two isoquants using only labor. This can be seen from the fact that the isoquants touch the L-axis, which means it is possible to produce the required output even if $K=0$.

4. Consider the production function

$$f(K,L) = \alpha \, (2 \, K^2 + \beta \, L^2)$$

where K is a measure of capital inputs, L is a measure of labor inputs, the parameter α reflects overall productivity and the parameter β reflects productivity specific to labor (e.g., increased ability or education).

What is the $MRTS_{K,L}$? Does it depend on the overall productivity parameter α? Does it depend on the labor specific productivity parameter β?

The *MRTS* of capital for labor is given by

$$MRTS_{K,L} = \frac{MP_K}{MP_L} = \frac{\alpha \, 4 \, K}{\alpha \, \beta \, 2 \, L} = \frac{2}{\beta} \, \frac{K}{L}$$

It appears that from the above that $MRTS_{K,L}$ does not depend on the overall productivity, α, of the production function. This is not surprising: increasing overall productivity does not affect the relative productivity of capital and labor, and hence it does not affect the rate at which labor can substitute for capital.

It also appears from the above expression that $MRTS_{K,L}$ does depend on the labor specific productivity parameter β. In fact, the higher the labor productivity, the fewer units of labor are needed to make-up for the loss of one unit of capital, something that also makes intuitive sense.

5. Consider the production function

$$Q = L + \log(1+K)$$

a. Write the equation for the isoquant of this production function that corresponds to an output level of $Q = 10$.

Replacing Q by 10 yields the isoquant for Q=10. Thus, the equation for this isoquant is

$$10 = L + \log(1+K)$$

b. Show that this production function is characterized by diminishing *MRTS*. [You can do this using the above isoquant, if you like, or using the expression for the production function given above.]

$$MRTS_{K,L} \equiv -\left.\frac{dL}{dK}\right|_{q=constant} = \frac{MP_K}{MP_L}$$

and

$$MRTS_{L,K} \equiv -\left.\frac{dK}{dL}\right|_{q=constant} = \frac{MP_L}{MP_K}$$

I can use either $MRTS_{K,L}$ or $MRTS_{L,K}$ to demonstrate whether there is diminishing *MRTS* (as we have seen in the notes, the one is the inverse of the other). I'll take $MRTS_{K,L}$.

One approach is to solve the isoquant for *L* as a function of *K*, then compute its slope. Solving the isoquant for L we get

$$L = 10 - \log(1+K)$$

Therefore,

$$-\left.\frac{dL}{dK}\right|_{Q=10} = -\left(-\frac{1}{1+K}\right) = \frac{1}{1+K}$$

The production function exhibits diminishing *MRTS* because the higher the use of capital, holding output constant, the more valuable labor is relative to capital.

The second approach is to use the ratio of marginal products. This is given by

$$\frac{MP_K}{MP_L} = \frac{\frac{1}{1+K}}{1} = \frac{1}{1+K}$$

This is the same answer as that obtained by the first approach and shows that the value of capital relative to the value of labor goes down the greater the use of capital is (holding output constant). In other words, the production function is characterized by diminishing *MRTS*.

Problems

1. The production function for trucks in a single plant is given by

$$Q = \alpha\ K^{0.3}\ L^{0.6}$$

where K is the amount of capital in the production process, L the amount of labor employed (in thousands), and α a parameter that measures the efficiency of production. It is immediately apparent that the greater the capital and labor inputs, the greater the output. Also, an increase in production efficiency increases output proportionately.

Suppose that in the US the value of α is equal to 2, while in China it is equal to 1, that is, US plants are twice as efficient as Chinese plants. Also suppose that Chinese plants have only a fifth as much capital per worker as US plants (simply because capital is relatively scarce in China). In particular, a US plant that employs 2,000 workers has 5 units of capital, while its Chinese counterpart (which also employees 2,000 workers) only has a single unit of capital.

a. What is the marginal product of labor the US and in Chinese truck plants described above?

The marginal product of labor in a truck plant is

$$MP_L \equiv \frac{\partial Q}{\partial L} = \alpha\ 0.6\ K^{0.3}\ L^{-0.4}$$

Using the information above, we can write the marginal product of labor of the US and Chinese plants as

$$MP_L^{US} = 2 \cdot 0.6 \cdot 5^{0.3} \cdot 2^{-0.4} \approx 1.47$$

and

$$MP_L^{Ch} = 0.6 \cdot 2^{-0.4} = 0.45$$

Labor in the Chinese plant is about 30% as productive as labor in the US plant.

b. Suppose that the plant efficiency in the Chinese plant increases (through technological diffusion) to the level of the US plant. What is the new marginal product of labor in the Chinese plant?

Increasing the value of α to 2 yields a marginal product of labor in the Chinese plant of

$$MP_L^{Ch} = 2 \cdot 0.6 \cdot 2^{-0.4} = 0.91$$

c. Suppose instead of the increase in plant efficiency, investment in the Chinese plant increases (through capital inflows from abroad) so that the capital in the Chinese plant reaches the level of its US counterpart. What is the new marginal produce of labor in the Chinese plant?

Increasing the capital in the plant to 5 while leaving α to 1 results in a marginal product of labor in the Chinese plant of

$$MP_L^{Ch} = 0.6 \cdot 5^{0.3} \cdot 2^{-0.4} \approx 0.74$$

d. Which of the two changes described above is the most beneficial in terms of raising labor productivity?

Bringing the state of the technology in the Chinese plant to the US level yields a bigger increase in labor productivity than bringing the level of capital up to the US level. This, of course, is not a general result; it depends on the nature of the production function, and the levels of capital and state of technology in the US and China.

2. Consider the production function of a manufacturing plant given by

$$Q = (K^{0.5} + 4\ L^{0.5} + E^{0.5})^2$$

The plant is currently having a capital stock equal to 1, and employees 4 units of labor. Its energy use is 2 units.

a. What is the marginal product of labor for this plant, and for this level of input use?

The expression for the marginal product of labor is obtained by differentiating the production function with respect to *L*. Once this differentiation is done, then we can plug in the numerical values of input usage by the firm to obtain the numerical value for the marginal product of labor.

The expression for the marginal product of labor is

$$\frac{\partial Q}{\partial L} = 2\ (K^{0.5} + 4\,L^{0.5} + E^{0.5})\ 0.5\ 4\ L^{-0.5}$$

Given that K=1, L=4, and E=2, the above expression becomes

$$\frac{\partial Q}{\partial L} = 2\ (1^{0.5} + 4\ 4^{0.5} + 2^{0.5})\ 0.5\ 4\ 4^{-0.5}$$

$$= 2\ (1\ +\ 4\ \sqrt{4}\ +\ \sqrt{2})\ \frac{2}{\sqrt{4}}$$

$$= 2\ (1\ +\ 8\ +\ \sqrt{2})$$

$$= 2\ 10.41$$

$$= 20.82$$

b. Suppose an energy crisis forces the plant to cut back its energy usage to only 1 unit. What is the effect of this reduction in energy inputs on the marginal product of labor? Has labor become more or less productive?

Given that E is now equal to 1, we can plug in K=1, L=4, and E=1 in the expression for the marginal product of labor obtained in (a). This gives:

$$\frac{\partial Q}{\partial L} = 2\ (1^{0.5} + 4\ 4^{0.5} + 1^{0.5})\ 0.5\ 4\ 4^{-0.5}$$

$$= 2\ (1\ +\ 4\ \sqrt{4}\ +\ 1)\ \frac{2}{\sqrt{4}}$$

$$= 2\ (1\ +\ 8\ +\ 1)$$

$$= 2\ 10$$

$$= 20$$

The marginal product of labor has been reduced by 0.82 from 20.82, or in percentage terms by approximately 4%.

c. Suppose to compensate for this loss of energy inputs the plant decides to increase its capital stock. What would the increase in the capital stock have to be for the marginal product of labor to go up to what it was in part (a)?

The firm wants to increase the marginal product of labor back up to 20.82 by increasing capital. One easy way to find out the answer to this particular problem is to observe that if K=2, and E=1, this is equivalent from the point of view of the production function and the marginal product of labor to K=1 and E=2. Therefore, an increase in the capital stock by one unit from 1 to 2 should compensate for the reduction of energy inputs from 2 units to 1 unit.

Since it will often not be able to solve this problem by inspection, we will now find the answer mathematically. The firm needs to set the capital stock to K_{new} so that the expression for the marginal product of labor is equal to 20.82 when L=4 and E=1. Therefore, we have

$$\frac{\partial Q}{\partial L} = 2\ (K_{new}^{0.5} + 4\ 4^{0.5} + 1^{0.5})\ 0.5\ 4\ 4^{-0.5} = 20.82 \quad \Rightarrow$$

$$2\ (K_{new}^{0.5} + 4\ \sqrt{4} + 1)\ \frac{2}{\sqrt{4}} = 20.82 \quad \Rightarrow$$

$$2\ (K_{new}^{0.5} + 9) = 20.82 \quad \Rightarrow$$

$$K_{new}^{0.5} + 9 = 10.41 \quad \Rightarrow$$

$$K_{new}^{0.5} = 1.41 \quad \Rightarrow$$

$$K_{new} = 2$$

Therefore, the firm needs to add one unit of capital to the one unit of capital it already has to reach a total of 2 units of capital stock.