LECTURE 8: SPECIAL PRODUCTION FUNCTIONS, PART II

ANSWERS AND SOLUTIONS

True/False Questions

False_ The elasticity of scale of a fixed proportions production function is not defined because the fixed proportions production function is not differentiable.

True_ The *MRTS* between two inputs for a fixed proportions production function is either zero or infinity or not defined depending on the input mix. No other values are possible.

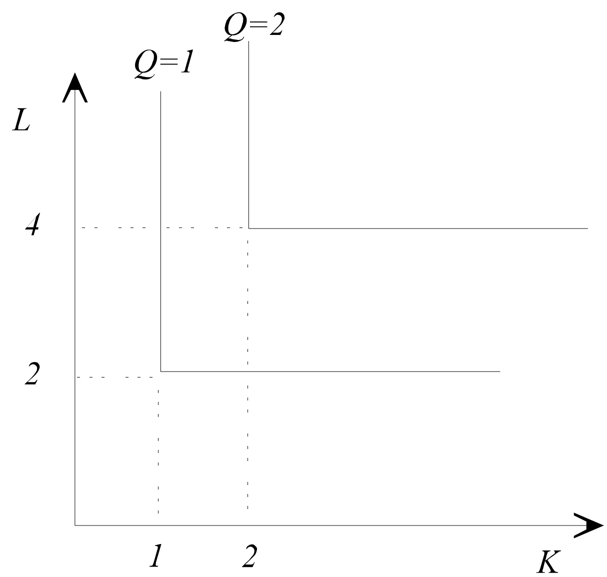
False_ If a firm's production function is linear, then the marginal product of each input is constant and independent of the level of the other inputs.

False_ The $MRTS_{x,y}$ of a linear production function that has two inputs, x and y, is constant if the production function is of constant returns to scale, increasing if the production function is of increasing returns to scale, and decreasing if the production function is of decreasing returns to scale.

False_ The $MRTS_{x,y}$ of a linear production function that has two inputs, x and y, is constant if the production function is of constant returns to scale, decreasing if the production function is of increasing returns to scale, and increasing if the production function is of decreasing returns to scale.

Short Questions

1. Consider a firm whose production function is characterized by the following isoquants.



A. Is the production function of this firm Cobb-Douglas, Leontieff, or Linear?

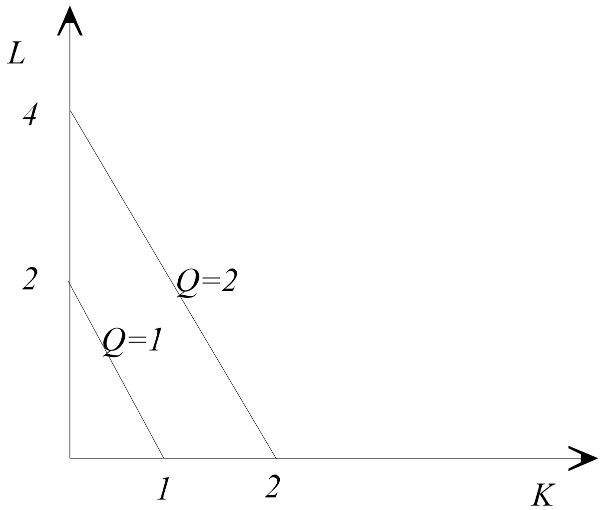
This is a Leontieff (fixed proportions) production function.

B. Write the mathematical expression for this production function.

It is

$$Q = \min\left\{K, \frac{1}{2}L\right\}$$

2. Consider a firm whose production function is characterized by the following isoquants:



A. Is the production function of this firm Cobb-Douglas, Leontieff, or Linear?

This production function is the Linear production function, because the isoquants are straight lines that are downward sloping.

B. Write the mathematical expression for this production function.

Notice that this production function has constant returns to scale because doubling the inputs doubles the output. Also notice that 2 units of labor (L) are equivalent to 1 unit of capital (K), and either of them is sufficient to produce one unit of output.

Therefore, the production function is

$$q = \frac{1}{2} L + K$$

3. Consider the production function

$$q = K^{\gamma} E^{0.2} L^{\frac{1}{\theta}}$$

Answer the following questions. No need to show any calculations. Just provide the answer, using what we have learned from the lectures.

a. Is this production function Cobb-Douglas, Leontieff, Linear, or of some other type?

It is Cobb-Douglas.

b. Does this production exhibit decreasing or increasing *MRTS*?

This exhibits decreasing *MRTS* (all Cobb-Douglas production functions have this property).

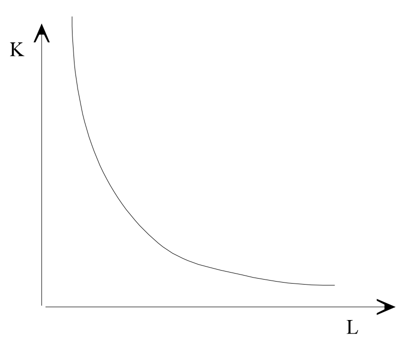
c. What is the elasticity of scale of this production function?

It is equal to the sum of the exponents, i.e., $\gamma + 0.2 + \frac{1}{\theta}$.

d. Which of the three inputs to this production function are essential for production?

All of the them (you cannot produce any output if any of the inputs is zero).

e. Sketch an isoquant of this production function with *K* on the vertical axis and *L* on the horizontal axis, for *E*=1. No need to label any points; just make sure the overall shape is consistent with the above production function.



The isoquant does not touch either of the two axis, and its slope is decreasing (in absolute value) as we move to the right.

4. Consider the production function

$$q = \min\{2K, \frac{1}{3}L\}^{1/\theta}$$

where θ is an unknown positive parameter. Answer the following questions using what we learned from the lectures (no need to provide explanations).

a. Is this production function Cobb-Douglas, Leontieff, Linear, or of another type?

   It is a Leontieff production function.

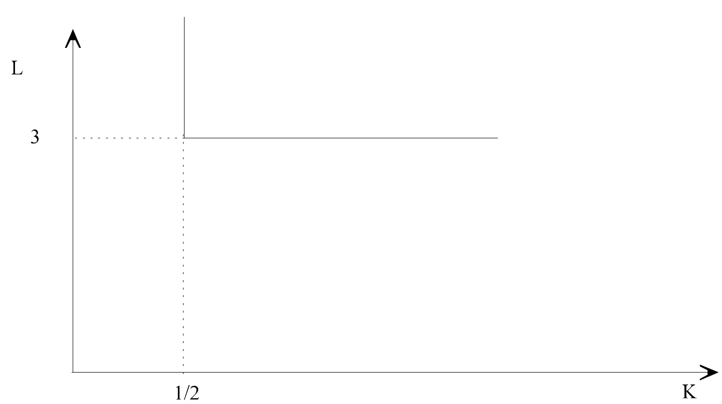
b. What is the elasticity of scale of this production function?

   It is $1/\theta$.

c. Are any of the two inputs essential for producing output? If so, which one(s)?

   Yes, both of them. This is always true for Leontieff (fixed proportions) production technologies.

d. Draw below the isoquant for this production function that corresponds to $q = 1$, by putting labor (L) in the vertical axis and capital (K) on the horizontal axis.



5. Consider the production function

$$f(K, L) = (K^2 + \beta\ L^2)^{\epsilon}$$

where K is a measure of capital inputs, L is a measure of labor inputs, the parameter β reflects productivity specific to labor and the parameter ϵ is related to the returns to scale.

a. What is the $MRTS_{K,L}$?

$$MRTS_{K,L} = \frac{MP_K}{MP_L}$$

$$= \frac{\epsilon\ (K^2 + \beta\, L^2)^{\epsilon-1}\ 2\ K}{\epsilon\ (K^2 + \beta\, L^2)^{\epsilon-1}\ 2\ \beta\ L}$$

$$= \frac{1}{\beta}\ \frac{K}{L}$$

b. What is the elasticity of scale for this productions function?

$$\epsilon_{scale} = \frac{\partial \log[f(m\mathbf{y})]}{\partial \log(m)}\bigg|_{m=1}$$

$$= \frac{\partial \log[((m\,K)^2 + \beta\ (m\,L)^2)^{\epsilon}]}{\partial \log(m)}\bigg|_{m=1}$$

$$= \frac{\partial \log[m^{2\epsilon}\ (K^2 + \beta\ L^2)^{\epsilon}]}{\partial \log(m)}\bigg|_{m=1}$$

$$= \frac{\partial [2\,\epsilon \log(m) + \epsilon\ \log(K^2 + \beta\ L^2)]}{\partial \log(m)}\bigg|_{m=1}$$

$$= 2\,\epsilon$$

Problems

1. A power plant can produce electricity using natural gas or fuel oil, or a combination of the two. In particular, its production function for electricity is given by

$$E = (\alpha \; G + \beta \; F)^{\epsilon}$$

where is E is the output of electricity, G in the input of natural gas, F is the input of fuel oil, and α, β, and ϵ are parameters.

a. Show that the elasticity of scale is equal to ϵ.

The definition of the elasticity of scale is

$$\epsilon_{scale} = \frac{\partial \log[f(m\,\mathbf{x})]}{\partial \log(m)}\bigg|_{m=1}$$

Substituting the above production function, we obtain

$$\epsilon_{scale} = \frac{\partial \log[(\alpha\, m\, G + \beta\, m\, F)^{\epsilon}]}{\partial \log(m)}\bigg|_{m=1} \quad \Rightarrow$$

$$\epsilon_{scale} = \frac{\partial \log[m^{\epsilon}\;(\alpha\, G + \beta\, F)^{\epsilon}]}{\partial \log(m)}\bigg|_{m=1} \quad \Rightarrow$$

$$\epsilon_{scale} = \frac{\partial\left[\log(m^{2}) \; + \; \log\!\left((\alpha\, G + \beta\, F)^{\epsilon}\right)\right]}{\partial \log(m)}\bigg|_{m=1} \quad \Rightarrow$$

$$\epsilon_{scale} = \frac{\partial\,[\epsilon \log(m) \; + \; \epsilon \; \log(\alpha\, G + \beta\, F)]}{\partial \log(m)}\bigg|_{m=1} \quad \Rightarrow$$

$$\epsilon_{scale} = \epsilon$$

b. Draw the isoquant for $E = 9$, if $\alpha = 2$, $\beta = 3$ and $\epsilon = 2$.

Substituting the above information into the production function yields the relationship

$$9 = (2\ G + 3\ F)^2 \Rightarrow$$

$$3 = 2\ G + 3\ F$$

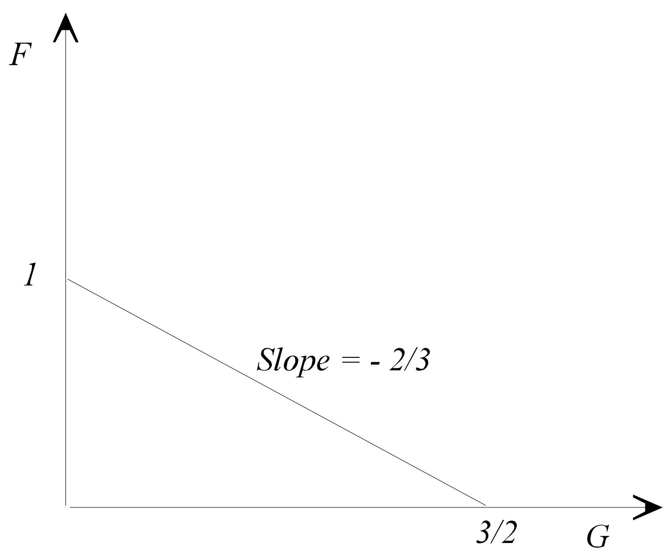
One can plot the isoquant either with *G* on the vertical axis, or *F* on the vertical axis. If the isoquant is plotted with *G* on the vertical axis and *F* on the horizontal axis, then one would have to solve the above equation for *G*. If instead the isoquant is plotted with *F* on the vertical axis and *G* on the horizontal axis, one would have to solve the above equation for *F*. It does not matter which of two plots is given. We choose the latter of the two.

Solving for *F*, the above equation yields

$$F = 1 - \frac{2}{3} G$$

This is the equation of the isoquant. It is a straight line with *F*-axis intercept of 1 and a slope of -2/3. The G-axis intercept is equal to 3/2.

The isoquant is plotted below.



2. Fixing a bug in the computer code of Windows requires 100 hours of an experienced programmer or 300 hours of an inexperienced programmer, or a linear combination of experienced and inexperienced programmers.

a. How many Windows computer code bugs could Microsoft fix if it had at its disposal 1,000 hours worth of experienced programers and 1,500 hours worth of inexperienced programmers?

1,000 hours of experienced programmers can fix 1000/100 = 10 computer bugs, and 1,500 hours of inexperienced programmers can fix 1500/300 = 5 computer bugs. So in total, Microsoft will be able to fix 10 + 5 = 15 computer bugs for Windows.

b. Write down the expression for the production function that relates the number of available hours of experienced programmers and the number of available hours of inexperienced programmers to the number of bugs of Windows computer code that can be fixed.

In general, the number of bugs can be fixed if there are *E* hours of experienced programmers is E/100. The number of bugs that can be fixed if there are I hours of inexperienced programmers is *I*/300. There is perfect substitutability between the two types of programmers, since they can be used in a linear combination to yield an equivalent level of output. Moreover, from the description of the problem, it appears that the production function has constant returns to scale. Therefore, the production function is the Linear constant returns to scale production function with coefficients of 1/100 for *E* and 1/300 for *I*, or

$$q = \frac{1}{100} E + \frac{1}{300} I$$

3. Producing sweetening each unit of diet *FineSoda* requires either θ units of NutraSweet or λ units of Splenda, or a linear combination of the two. The sweetening process is constant returns to scale, so increasing the output by any given percentage would require increasing the inputs by that same percentage.

a. Write down the production function for sweetening *FineSoda*.

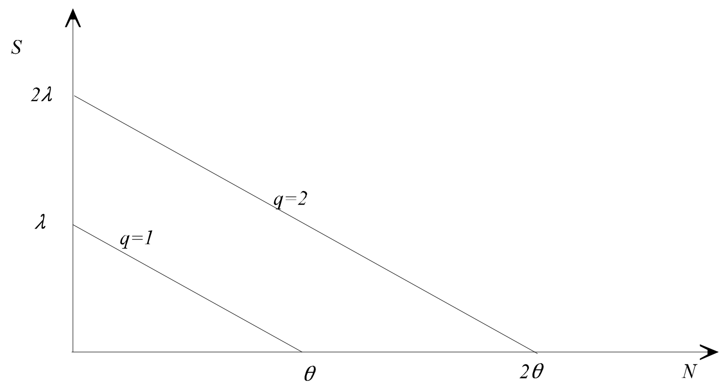
The production function is

$$q = \frac{1}{\lambda} S + \frac{1}{\theta} N$$

where *S* is the amount of Splenda used and *N* the amount of NutraSweet used.

b. Graph the isoquant that corresponds to sweetening one unit of *FineSoda*. In the same figure, graph the isoquant that corresponds to sweetening two units of *FineSoda*. Make sure to draw the two isoquants so as to reflect the fact that this is a constant returns to scale production function.

The two isoquants are drawn below. The S intercept of the q=1 isoquant indicates the amount of Splenda that can sweeten one unit of *FineSoda*. The N intercept of that isoquant indicates the amount of NutraSweet that can sweeten one unit of *FineSoda*. The isoquant is a straight line that connects these two intercepts because this is a linear production function.



The intercepts for the q=2 isoquant are twice as far from the origin as the intercepts of the q=1 isoquant to reflect that this is a constant returns to scale production function.

4. The number of heart transplants that can be done at Humana hospital depends on the number of surgeon-hours and nurse-hours available. There is no substitutability between surgeons and nurses: For each surgery one needs 4 surgeon-hours and 60 nurse-hours.

a. Write down the production function for heart transplants at Human hospital (assume for simplicity that one can have fractional transplants).

To get one transplant, one needs at least 4 surgeon-hours and at least 60 nurse-hours. To get two transplants, one needs at least 8 surgeon-hours and at least 120 nurse-hours. In general, to get q transplants, one needs at least 4 q surgeon-hours and at least 60 q nurse hours. Notice that regardless of the number of transplants performed, the ratio of nurses to surgeons is 60/4 = 15. Therefore, if S is the number of surgeon hours and N the number of nurse-hours, regardless of the number of transplants, q, the number of hours of each type of labor used will be given by the equation

$$N = 15\ S$$

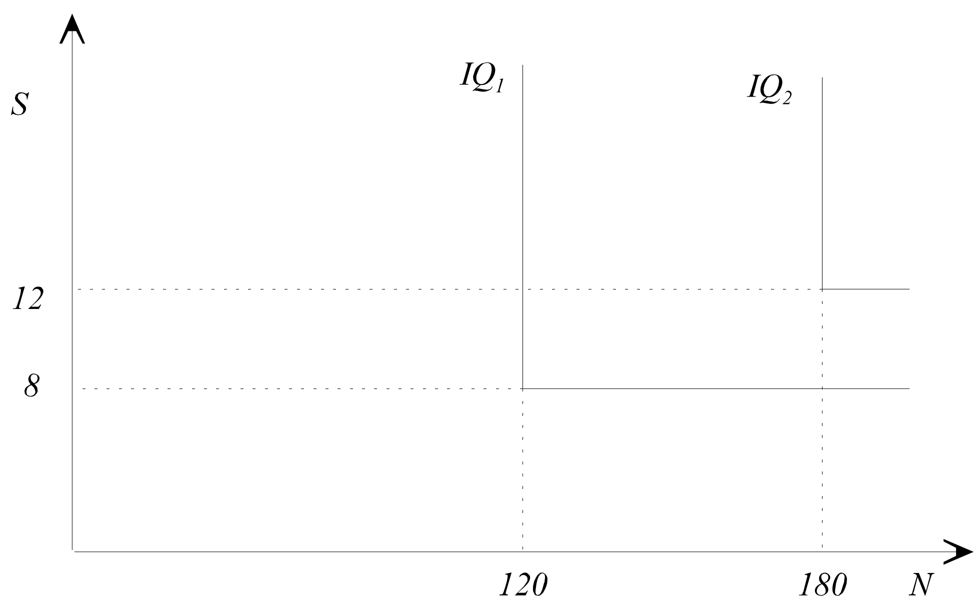
If there are more nurses available (relative to the number of surgeons), they will not be used, and the number of transplants will not be any higher. If there are more surgeons available (relative to the number of nurses), they will not be used, and the number of transplants will also not be any higher. Therefore, the number of transplants performed is given by the function

$$q = \min\left\{\frac{1}{4}\ S,\ \frac{1}{60}\ N\right\}$$

This is the production function of heart transplants in Humana hospital.

b. Graph the isoquant that corresponds to 2 transplants. In the same figure, graph the isoquant that corresponds to 3 transplants.

To do 2 transplants, one needs 120 nurse-hours and 8 surgeon hours. An increase of either alone does not increase the number of transplants feasible. A decrease of either will make performing 2 transplants impossible (not matter how many units of the other are used). This relationship between inputs and outputs is graphed below by IQ_1 (the figure is not in scale).



Similarly, IQ_2 is the isoquant that corresponds to 3 transplants.