On Matroid Parity and Matching Polytopes

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Abstract

The matroid parity (MP) problem is a powerful (and $\mathcal{NP}$-hard) extension of the matching problem. Whereas matching polytopes are well understood, little is known about MP polytopes. We prove that, when the matroid is laminar, the MP polytope is affinely congruent to a perfect b-matching polytope. From this we deduce that, even when the matroid is not laminar, every Chvátal-Gomory cut for the MP polytope can be derived as a $\{0, \frac{1}{2}\}$-cut from a laminar family of rank constraints. We also prove a negative result concerned with the integrality gap of two linear relaxations of the MP problem.

Keywords: matroid parity; matroid matching; polyhedral combinatorics

1 Introduction

In the late 1960s, Jack Edmonds proved that two important combinatorial optimisation problems, the *matching* and *matroid intersection* problems, can be solved in polynomial time [7, 9]. Since then, researchers have defined various problems that generalise both. Examples include the *matchoid* problem of Jenkyns [19], the *matroid matching* problem of Lovász [23, 24], and the *matroid parity* problem of Lawler [21]. It was quickly shown that those three problems are equivalent [25]. In this paper, we focus on the third problem, which we call the "MP" problem.

The MP problem has been studied in depth (e.g., [4, 13, 15, 18, 20, 22–24, 26, 27, 31]). It is strongly $\mathcal{NP}$-hard if the underlying matroid M has a compact description [23], and can take exponential time if M is given only via an independence oracle [20, 23]. On the other hand, it can be solved efficiently if M is given by a linear representation [4, 15, 23, 26, 27].

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A highly successful approach to $\mathcal{NP}$-hard combinatorial optimisation problems is the *polyhedral* approach, in which strong valid inequalities are derived for the convex hull of feasible solutions (e.g., [5]). However, whereas matching polytopes are well understood [11, 30], very little is known about MP polytopes [17, 32].

We believe that an improved understanding of MP polytopes would ultimately lead to improved algorithms for the MP problem. To this end, we do the following. First, we show that, when M is *laminar*, the MP polytope is affinely congruent to a perfect b-matching polytope. Second, we derive a complete linear description of laminar MP polytopes. Third, we show that, regardless of whether M is laminar, every non-dominated Chvátal-Gomory (CG) cut for the MP polytope can be derived as a $\{0, \frac{1}{2}\}$-cut from a laminar family of so-called "projected rank" constraints. Finally, we show that, even for very simple (but non-laminar) matroids, the projected rank constraints and CG-cuts can yield a poor upper bound for the MP problem.

The rest of the paper has a simple structure: after a brief literature review, four sections present the aforementioned results.

Throughout the paper, we assume that the reader is familiar with elementary graph theory and the basics of integer programming. We also use the following standard notation. Given an undirected graph $G = (V, E)$ and any node set $S \subset V$, $\delta(S)$ denotes the set of edges with exactly one end-node in S, and $E(S)$ denotes the set of edges with both end-nodes in S. Given a vector $x \in \mathbb{R}^q$ and a set $S \subseteq \{1, \ldots, q\}$, $x(S)$ denotes $\sum_{i \in S} x_i$. When G is a digraph, we write A (for "arcs") instead of E (for "edges"). We occasionally write V^G instead of V to denote the node-set of a graph (or digraph) G, and use E^G or A^G similarly.

2 Literature Review

2.1 Matchings

Given an undirected graph $G(V, E)$ and a vector $b \in \mathbb{Z}_+^V$, a *b-matching* is a set E' of edges, possibly with repetitions, such that each vertex i is incident on at most b_i edges in E'. The b-matching is called *perfect* if each vertex i is incident on exactly b_i edges. (Note that $b(V)$ must be even for a perfect b-matching to exist.) Given a weight vector $w \in \mathbb{Z}_+^E$, one can find a maximum-weight (perfect) b-matching in polynomial time [8]. When all b_i are equal to one, the b-matching is called simply a *matching*.

2.2 Matroids

A *matroid* consists of a *ground set* F and a family $\mathcal{I} \subseteq 2^F$ of *independent sets*. It satisfies two properties: (i) every subset of an independent set is independent, (ii) if I_1 and I_2 are independent and $|I_2| > |I_1|$, then for some

$x \in I_2 \setminus I_1$ the set $I_1 \cup \{x\}$ is independent. One can find a maximum-weight independent set in polynomial time by the greedy algorithm [10]. In fact, the more general *matroid intersection* problem is also solvable in polynomial time. That problem assumes two matroids with the same ground set, and asks for a common independent set of maximum weight [9].

Many different kinds of matroids have been defined, such as partition matroids, graphic matroids, series-parallel matroids, linear matroids and gammoids [28]. Of particular interest to us will be *laminar* matroids [14]. A set family $\mathcal{F} = \{F_i \subseteq F : i = 1, \ldots, k\}$ is called laminar if, for any $i \neq j$, either $F_i \subset F_j$ or $F_j \subset F_i$ or $F_i \cap F_j = \emptyset$ (e.g., [20]). A matroid $M(F, \mathcal{I})$ is called laminar if there is a laminar family $\mathcal{F} = \{F_1, \ldots, F_k\}$ and a set of positive integers $U = \{u_1, \ldots, u_k\}$ such that $S \in \mathcal{I}$ if and only if $|S \cap F_j| \leq u_j$ for $j = 1, \ldots, k$. We assume w.l.o.g. that all sets F_j are non-redundant (that is, we cannot increase any of the u_j without changing the matroid).

It is known that (a) partition matroids are laminar [14], (b) laminar matroids are gammoids [12], (c) series-parallel matroids are both gammoids and graphic matroids (e.g., [1]), and (d) gammoids and graphic matroids are linear (e.g. [28]).

2.3 Matroid parity

In the MP problem, we are given a matroid M with ground set F of even cardinality, and a partition of F into two-element subsets called *lines*. We seek a maximum-cardinality set of lines whose union is independent in M [23, 24]. As already mentioned, the problem is strongly $\mathcal{NP}$-hard, but solvable in polynomial time for linear matroids. A polynomial-time 2/3-approximation algorithm is known for the general case [13].

Let $\mathcal{L}$ denote the set of lines. In the *weighted* MP problem, we are given a non-negative weight for each $\ell \in \mathcal{L}$, and we seek a solution of maximum total weight. The problem is solvable in polynomial time for linear matroids [18], but no approximation algorithm is known for general matroids.

The weighted MP problem on a partition matroid can be easily reduced to weighted b-matching [21]. Less obviously, the weighted MP problem on gammoids can be reduced to weighted matching [31].

2.4 Polytopes

A vector $x \in \mathbb{Z}^E$ is the incidence vector of a b-matching if and only if it satisfies:

$$x\big(\delta(i)\big) \leq b_i \qquad \forall\, i \in V, \tag{1}$$

$$x_e \geq 0 \qquad \forall\, e \in E. \tag{2}$$

The *b-matching polytope* is

$$\mathscr{P}_b = \operatorname{conv}\Big\{x \in \mathbb{Z}^E : \ x \text{ satisfies (1) and (2)}\Big\}.$$

It is known [11] that $\mathscr{P}_b$ is described by (1), (2), and the following *blossom* inequalities:

$$x\big(E(H)\big) \leq \left\lfloor \tfrac{b(H)}{2} \right\rfloor, \forall H \subset V : b(H) \text{ odd}. \tag{3}$$

The *fractional* b-matching polytope, which we denote by $\tilde{\mathscr{P}}_b$, is the relaxation of $\mathscr{P}_b$ obtained by permitting x to take fractional values. It is known that $\tilde{\mathscr{P}}_b$ has half-integral extreme points [2].

The *matroid polytope* of a matroid $M(F, \mathcal{I})$ is the convex hull of the incidence vectors $x \in \{0,1\}^F$ of subsets in $\mathcal{I}$. It is known [10] that its linear description is given by non-negativity and the following *rank* inequalities

$$x(S) \leq r_M(S) \qquad \forall S \subseteq F, \tag{4}$$

where $r_M(S) = \max_{A \subseteq S} \{|A| : A \in \mathcal{I}\}$ is the rank function.

As for the MP problem, the incidence vectors of independent sets $x \in \{0,1\}^F$ and lines $y \in \{0,1\}^{\mathcal{L}}$ must satisfy the rank constraints (4) plus the *line* constraints

$$x_i = x_j = y_{ij} \qquad \forall \{i,j\} \in \mathcal{L}. \tag{5}$$

Using (5) to project out the x variables yields the *projected rank* inequalities

$$\sum_{\ell \in \mathcal{L}} |S \cap \ell| \, y_\ell \leq r_M(S) \qquad \forall \, S \subseteq F. \tag{6}$$

Thus, the MP polytope can be defined in y-space [32], as

$$\mathscr{P}_{M,\mathcal{L}} \quad = \quad \operatorname{conv}\left\{ y \in \{0,1\}^{\mathcal{L}} : y \text{ satisfies (6)} \right\}.$$

Note that b-matching polytopes are MP polytopes.

By allowing y to be fractional, one obtains the *fractional* MP polytope, denoted here by $\tilde{\mathscr{P}}_{M,\mathcal{L}}$, whose extreme points are half-integral [32]. More recently [17], it was shown that the LP formulation of the fractional MP problem has half-integral dual solutions.

2.5 Chvátal-Gomory cuts

Let $P = \left\{ x \in \mathbb{R}^p : \, Ax \leq b \right\}$ be a polyhedron, where $b \in \mathbb{Z}^q$ and $A \in \mathbb{Z}^{qp}$, and let P_I be the convex hull of the integral points in P. A *Chvátal-Gomory cut* (CG-cut for short) is a valid inequality for P_I of the form $(\lambda^T A)x \leq \lfloor \lambda^T b \rfloor$, where $\lambda \in \mathbb{Q}^q_+$ [6, 16]. The set of points in P that satisfy all CG-cuts is called the *elementary closure* of P [6].

A CG-cut is called a "$\left\{0, \frac{1}{2}\right\}$-cut" if $\lambda \in \left\{0, \frac{1}{2}\right\}^q$ [3]. We will follow [3] in letting P_1 denote the elementary closure of P, and letting $P_{1/2}$ denote the set of points in P that satisfy all $\left\{0, \frac{1}{2}\right\}$-cuts. Then, by definition, $P_I \subseteq P_1 \subseteq P_{1/2} \subseteq P$.

It is well-known [3, 6] that the blossom inequalities (3) can be derived as $\left\{0, \frac{1}{2}\right\}$-cuts from the system (1), (2). In other words, if P is the fractional

b-matching polytope $\tilde{\mathscr{P}}_b$, then $P_I = P_1 = P_{1/2} = \mathscr{P}_b$. This is not true for fractional MP polytopes in general. Indeed, even when the matroid M is series-parallel, it is possible for $\mathscr{P}_{M,\mathcal{L}}$ to be strictly contained in the elementary closure of $\tilde{\mathscr{P}}_{M,\mathcal{L}}$ [32, Example 5.1].

3 From Laminar MP to Perfect b-Matching

As mentioned in Subsection 2.3, MP over gammoids can be reduced to weighted matching. Given that laminar matroids are gammoids, this holds also for laminar MP. In this section, we give a much simpler reduction, from laminar MP to *perfect b-matching*. We then use the reduction to derive a polyhedral result.

In order to proceed, we need further definitions and notation. Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a laminar family defined over a ground set F, and let F_i and F_j be any two distinct members of $\mathcal{F}$. If $F_j \subset F_i$ and there does not exist a third member F_ℓ such that $F_j \subset F_\ell \subset F_i$, we say that j is a 'child' of i. The set of children of i will be denoted by $\chi(i)$. Also, for $i = 1, \ldots, k$, we define the set $F(i) = F_i \setminus \bigcup_{j \in \chi(i)} F_j$. That is, $F(i)$ contains the elements in F_i, if any, that are not members of F_i's children.

We are now ready to prove a key theorem.

Theorem 1 *Given a laminar matroid $M = (F, \mathcal{I})$ and a line set $\mathcal{L}$, the laminar MP problem can be formulated as an integer linear program (ILP) with $|F| + |\mathcal{L}|$ binary variables, $2|\mathcal{F}|$ general integer variables and $|F| + 2|\mathcal{F}|$ constraints.*

Proof. Let $\{F_1, \ldots, F_k\} \subset F$ be the sets in the laminar family $\mathcal{F}$ associated with M, and let $u_1, \ldots, u_k$ be the associated upper bounds on $|S \cap F_i|$, for any $S \in \mathcal{I}$, $i = 1, \ldots, k$.

For each $i \in F$, define the binary variable x_e, taking the value 1 if and only if e is to be included in the set $S \in \mathcal{I}$. For each $\ell \in \mathcal{L}$, consider the binary variable $\bar{y}_\ell$, taking the value 1 if and only line ℓ is *not* to be selected. Finally, for $i = 1, \ldots, k$, define the general positive integer variables z_i, $\bar{z}_i \in \{0, \ldots, u_i\}$, representing the quantities $|S \cap F_i|$ and $u_i - |S \cap F_i|$, respectively.

Assuming $S \in \mathcal{I}$ to be a union of lines, it is easy to check that its incidence vector $(x, \bar{y}, z, \bar{z})$ corresponds to a feasible solution to the following

ILP:

$$
\begin{array}{lrll}
\max & \frac{1}{2}x(F) & & \\
\text{s.t.} & x(F(i)) + \sum_{j\in\chi(i)} z_j + \bar{z}_i = u_i & i = 1,\dots,k & (7)\\
& z_i + \bar{z}_i = u_i & i = 1,\dots,k & (8)\\
& x_e + \bar{y}_\ell = 1 & \forall \ell \in \mathcal{L},\ e \in \ell & (9)\\
& x_e,\ \bar{y}_\ell,\ \in \mathbb{Z}_+ & \forall \ell \in \mathcal{L},\ e \in \ell & \\
& z_i,\ \bar{z}_i \in \mathbb{Z}_+ & i = 1,\dots,k. &
\end{array}
$$

To complete the proof, we need to show that, if $(x', y', z', \bar{z}')$ is a feasible solution to the ILP, then it represents a feasible solution to the laminar MP instance.

The easy part is to show that the set S corresponding to x' is a union of lines. Indeed, the equations (9) imply that, for any line $\{e, f\} \in \mathcal{L}$, the variables x_e and x_f have the same value.

The hard part is to show that S is independent in M, i.e., that $x'(F_i) \le u_i$ for all $i \in 1\dots,k$. From the equations (8), this is equivalent to showing

$$z_i' = x'(F_i) \qquad (i \in 1,\dots,k). \tag{10}$$

Note first that constraints (7) and (8) imply

$$z_i' = x'(F(i)) + \sum_{j\in\chi(i)} z_j' \qquad (i = 1,\dots,k). \tag{11}$$

We will establish (10) by induction. First, consider any set $F_i \in \mathcal{F}$ that has no children. For such a set, the equation (11) immediately reduces to (10). Now take any set F_i that has children and assume by induction that condition (10) is true for all $j \in \chi(i)$. Then:

$$z_i' \;=\; x'(F(i)) + \sum_{j\in\chi(i)} z_j' \;=\; x'(F(i)) + \sum_{j\in\chi(i)} x'(F_j) \;=\; x'(F_i).$$

We conclude that any feasible solution to the ILP corresponds to a solution to the laminar MP instance and vice-versa. Moreover, the cardinality of the set S is equal to the objective value of the ILP solution. $\square$

Corollary 1 *The laminar MP problem can be polynomially reduced to the perfect b-matching problem.*

Proof. Observe that the constraints (7)–(9) are equations with binary left-hand side coefficients and integral right hand sides. It was shown in [11] that any ILP with this property is equivalent to an instance of the perfect b-matching problem defined over a graph $G(V, E)$. Each variable of the ILP corresponds to an edge of G. Each equation of the ILP corresponds to a node $i \in V$, whose degree bound b_i is set to the right-hand side of the equation. From the construction in Theorem 1, the ILP corresponding to a given laminar MP instance has $|F| + |\mathcal{L}| + 2|\mathcal{F}|$ variables and $2(|F| + |\mathcal{F}|)$ constraints. So the resulting perfect b-matching instance is defined on a graph of polynomial size. □

Let us recall that two polytopes are called *affinely congruent* or *combinatorially equivalent* if there is an affine transformation from each polytope to the other.

Corollary 2 *When M is laminar, $\mathscr{P}_{M,\mathcal{L}}$ is affinely congruent to a perfect b-matching polytope.*

Proof. Let a laminar MP instance be given by a matroid $M[F, \mathcal{F}, U]$ and a line set $\mathcal{L}$. We prove that there is a perfect b-matching polytope in $\mathbb{R}^{|F|+|\mathcal{L}|+2|\mathcal{F}|}$ that is affinely congruent to $\mathscr{P}_{M,\mathcal{L}}$.

Define the polytope

$$\mathscr{P}^+ = \text{conv}\left\{(x, \bar{y}, z, \bar{z}) \in \mathbb{Z}_+^{|F|+|\mathcal{L}|+2|\mathcal{F}|} : (7) - (9) \text{ hold}\right\}.$$

Theorem 1 implies that a vector y^* lies in $\mathscr{P}_{M,\mathcal{L}}$ if and only if the corresponding vector $(x^*, \bar{y}^*, z^*, \bar{z}^*)$ lies in $\mathscr{P}^+$, where:

$$x_e^* = x_f^* = y_{ef}^* \qquad (\{e, f\} \in \mathcal{L}) \tag{12}$$

$$\bar{y}_\ell^* = 1 - y_\ell^* \qquad (\ell \in \mathcal{L}) \tag{13}$$

$$z_i^* = \sum_{\ell \in \mathcal{L}} |\ell \cap F_i| y_\ell^* \qquad (i = 1, \ldots, k) \tag{14}$$

$$\bar{z}_i^* = u_i - \sum_{\ell \in \mathcal{L}} |\ell \cap F_i| y_\ell^* \qquad (i = 1, \ldots, k). \tag{15}$$

This mapping is affine and invertible. □

4 Linear Description of the Laminar MP Polytope

In this section, we use the reduction in the previous section to derive a complete linear description of the $\mathscr{P}_{M,\mathcal{L}}$ in the laminar case.

Recall the definition of the perfect b-matching polytope $\mathscr{P}^+$, defined in the previous section. The only inequalities that can define facets of $\mathscr{P}^+$ are the non-negativity inequalities for the x, $\bar{y}$, z and $\bar{z}$ variables, together with the blossom inequalities, which can now involve combinations of those variables.

The non-negativity inequalities are the easiest to handle:

- For each $\ell = \{e, f\} \in \mathcal{L}$, both inequalities $x_e \geq 0$ and $x_f \geq 0$ for $\mathscr{P}^+$ map to the inequality $y_\ell \geq 0$ for $\mathscr{P}_{M,\mathcal{L}}$.

- For each $\ell \in \mathcal{L}$, the inequality $\bar{y}_\ell \geq 0$ for $\mathscr{P}^+$ maps to the upper bound inequality $y_\ell \leq 1$ for $\mathscr{P}_{M,\mathcal{L}}$.
- For each $i = 1, \ldots, k$, the inequality $z_i \geq 0$ is redundant, in light of equations (8), which imply $z_i = x(F_i)$ for all $i \in 1, \ldots, k$.
- For each $i = 1, \ldots, k$, the inequality $\bar{z}_i \geq 0$ for $\mathscr{P}^+$ is equivalent to $x(F_i) \leq u_i$, due to equations (8). This latter inequality in turn maps to the following projected rank inequality for $\mathscr{P}_{M,\mathcal{L}}$:

$$\sum_{\ell \in \mathcal{L}} |F_i \cap \ell| y_\ell \leq u_i.$$

We will show that the blossom inequalities for $\mathscr{P}^+$ map to a new and non-trivial family of valid inequalities for $\mathscr{P}_{M,\mathcal{L}}$, which we call *projected* blossom inequalities. To that end, let us introduce the corresponding undirected graph $G^+ = (V^+, E^+)$ that has one edge for each variable $x, \bar{y}, z, \bar{z}$ and one node for each degree equation (7)-(9). We also define the sets $T = \{1, \ldots, k\}$, $U = \{k+1, \ldots, 2k\}$ and $S = \{2k+1, \ldots, 2k+|F|\}$, which index the equations (7), (8) and (9), respectively. (By construction, T, U and S form a partition of V^+.)

Note that any set $F_i \in \mathcal{F}$ is associated with two equations in our ILP formulation: one of the form (7), indexed by $i \in T$, and the other of the form (8), indexed by $(i + k) \in U$. Furthermore, any element $f \in F$ is associated with one degree equation of the form (9), while any line $\ell \in \mathcal{L}$ is associated with two of them.

For a given blossom inequality, let $\bar{T} \subseteq T$, $\bar{U} \subseteq U$ and $\bar{S} \subseteq S$ denote the index sets of the equations that are used in their derivation as a $\left\{0, \frac{1}{2}\right\}$-cut. (By construction, $\bar{T}$, $\bar{U}$ and $\bar{S}$ form a partition of H.) We can now state the following lemma.

Lemma 1 *If a blossom inequality defines a facet of $\mathscr{P}^+$, then the corresponding sets $\bar{T}$, $\bar{U}$ and $\bar{S}$ satisfy the following conditions:*

1. $\sum_{i \in \bar{T}} u_i + \sum_{i \in \bar{U}} u_{i-k} + |\bar{S}|$ *is odd.*
2. $\bar{S} = \{i + 2k : \exists \{i, j\} \in \mathcal{L}$ *such that* $i, j \in \bigcup_{n \in \bar{T}} F(n)\}$
3. *If* $(i + k) \in \bar{U}$*, then* $j \in \bar{T}$ *where* $i \in \chi(j)$*.*

Proof.

1. If condition 1 does not hold, no rounding down occurs on the right-hand side.

2. Suppose condition 2 does not hold. Then there is some element $i \in F$ and some line $\{i, j\} \in \mathcal{L}$ for which we are using the equation $x_i + \bar{y}_{ij} = 1$ in the derivation of the blossom inequality, yet for which the variable x_i does not appear in any other equation that we are using. Now consider two cases:

   (i) $j+2k$ does not lie in $\bar{S}$. Then both x_i and $\bar{y}_{ij}$ will receive a coefficient of zero in the blossom inequality. Then, the blossom inequality will be either unchanged or strengthened if we remove $i + 2k$ from $\bar{S}$.

   (ii) $j+2k$ does lie in $\bar{S}$. Then the net contribution of the two equations, before dividing by two and rounding down, is $x_i + x_j + 2\bar{y}_{ij} \leq 2$. After dividing by two and rounding down, the left-hand side coefficient of x_i will be zero. So the best possible scenario is that we have added $x_j + \bar{y}_{ij} \leq 1$ to the blossom inequality. There is no point doing this, since $x_j + \bar{y}_{ij} = 1$.

3. Suppose condition 3 does not hold. Then there is some degree equation $i + k$ in $\bar{U}$ for which the degree equation $j \in T$, corresponding to the unique parent of i, is not included in the derivation of the blossom inequality. We observe that the variable $\bar{z}_i$ appears in the degree equations $i + k$ and $i \in T$, while the variable z_i appears in $i + k$ and its parent $j \in T$. Again, we consider two cases:

   (i) the degree equation i is not in $\bar{T}$. Then, we are using the equation $z_i + \bar{z}_i = u_i$ in the derivation of the blossom inequality, even though neither z_i nor $\bar{z}_i$ appear in any other equation that is used. After dividing by two and rounding down, the left-hand side coefficients of both z_i and $\bar{z}_i$ will be zero. Then, we could get a stronger inequality by removing i from $\bar{T}$.

   (ii) i does lie in $\bar{T}$. Then, the net contribution of the equations $i + k$ and i, before dividing by two and rounding down, is

$$x(F(i)) + \sum_{j \in \chi(i)} z_j + 2\bar{z}_i + z_i \leq 2u_i.$$

   Thus, after dividing by two and rounding down, the contribution to the left-hand side of the blossom inequality is at most $x(F(i)) + \sum_{j \in \chi(i)} z_j + \bar{z}_i$, while the contribution to the right-hand side is exactly u_i. Since $x(F(i)) + \sum_{j \in \chi(i)} z_j + \bar{z}_i = u_i$, we could get a stronger blossom inequality by removing $i + k$ from $\bar{U}$ and i from $\bar{T}$.

□

For given sets $\bar{S}$, $\bar{T}$ and $\bar{U}$ that respect the conditions of Lemma 1, we can derive a blossom inequality for $\mathscr{P}^+$. Before we present the general form of such an inequality, it is helpful to introduce some further index sets. We let $\bar{S}'$ denote a subset of $\bar{S}$ such that for any $\{i,j\} \in \mathcal{L}$ for which $(i+k) \in \bar{S}$ and $(j+k) \in \bar{S}$ we have either $(i+k) \in \bar{S}'$ or $(j+k) \in \bar{S}'$, but not both. In addition, we define the following two index sets associated with condition 3 of Lemma 1:

$$\begin{aligned} Z &= \big\{ i \in \{1,\ldots,k\} : i \notin \bar{T}, (i+k) \in \bar{U}, i \in \chi(j) \text{ for some } j \in \bar{T} \big\} \\ \tilde{Z} &= \big\{ i \in \{1,\ldots,k\} : i \in \bar{T}, (i+k) \in \bar{U}, i \in \chi(j) \text{ for some } j \in \bar{T} \big\} \end{aligned}$$

Note that the sets Z and $\tilde{Z}$ correspond to the possible scenarios for membership in $\bar{T}$ and $\bar{U}$. Using this notation we obtain the following general form of the blossom inequalities for $\mathscr{P}^+$:

$$\begin{aligned} & x(\delta^+(\bar{T})) + \bar{y}(\delta^+(\bar{S}')) + z(\delta^+(Z \cup \tilde{Z})) + \bar{z}(\delta^+(\tilde{Z})) \\ & \qquad \leq \left\lfloor \frac{\sum_{i \in \bar{T}} u_i + \sum_{i \in \bar{U}} u_{i-k} + |\bar{S}|}{2} \right\rfloor . \end{aligned} \tag{16}$$

We can now state the main result of this section.

Theorem 2 *Every non-dominated projected blossom inequality for $\mathscr{P}_{M,\mathcal{L}}$ can be derived as a $\left\{0, \frac{1}{2}\right\}$-cut from the projected rank inequalities (6) and the bound constraints $0 \leq y_\ell \leq 1$ for all $\ell \in \mathcal{L}$.*

Proof. Consider a blossom inequality of the form (16). First, we use (12) and (13) to project out the $\bar{y}$ variables. Note that $2(\delta^+(\bar{S}')) = 2|\bar{S}'| = |\bar{S}|$, and therefore subtracting $|\bar{S}|$ from the right hand side of (16) does not change its parity. Thus, the inequality (16) is equivalent to:

$$\begin{aligned} & x(\delta^+(\bar{T})) - x(\delta^+(\bar{S}')) + z(\delta^+(Z \cup \tilde{Z})) + \bar{z}(\delta^+(\tilde{Z})) \\ & \qquad \leq \left\lfloor \frac{\sum_{i \in \bar{T}} u_i + \sum_{i \in \bar{U}} u_{i-k}}{2} \right\rfloor . \end{aligned}$$

Now, condition 2 of Lemma 1 implies that $x(\delta^+(\bar{T})) = x(\delta^+(\bar{S}')$, and therefore the inequality reduces to:

$$z(\delta^+(Z \cup \tilde{Z})) + \bar{z}(\delta^+(\tilde{Z}) \leq \left\lfloor \frac{\sum_{i \in \bar{T}} u_i + \sum_{i \in \bar{U}} u_{i-k}}{2} \right\rfloor .$$

Next, we eliminate the $\bar{z}$ variables, using (8), to obtain:

$$z(\delta^+(Z)) \leq \left\lfloor \sum_{i \in \bar{T}} u_i + \sum_{i \in \bar{U}} u_{i-k} \right\rfloor - \sum_{i \in \tilde{Z}} u_i. \tag{17}$$

Now we simplify the right-hand side. Note that if $i \in \tilde{Z}$, then $i \in \bar{T}$ and $i + k \in \bar{U}$. Thus

$$\sum_{i \in \bar{T}} u_i + \sum_{i \in \bar{U}} u_{i-k} - 2 \sum_{i \in \tilde{Z}} u_i = \sum_{i \in \bar{T} \setminus \tilde{Z}} u_i + \sum_{i \in \bar{U} \setminus \tilde{Z}} u_{i-k} = \sum_{i \in Z} u_i.$$

We can therefore re-write the inequality (17) in the following simplified form:

$$z(\delta^+(Z)) \le \left\lfloor \sum_{i \in Z} u_i \right\rfloor . \tag{18}$$

Finally, we will project out the z variables. To this end, we define the set family $\mathcal{Q} = \{F_i \in \mathcal{F} : i \in Z\}$ and let

$$\begin{aligned} \alpha_\ell &= \frac{1}{2} \left(\sum_{\{i : F_i \in \mathcal{Q}\}} |\ell \cap F_i| \right), \quad \ell \in \mathcal{L}, \\ \beta &= \left\lfloor \frac{\sum_{\{i : F_i \in \mathcal{Q}\}} r_M(F_i)}{2} \right\rfloor . \end{aligned}$$

Using equation (14), we project the inequality (18) into $\mathbb{R}^{\mathcal{L}}$ to yield:

$$\sum_{\ell \in \mathcal{L}} \alpha_\ell y_\ell \le \beta. \tag{19}$$

Inequality (19) is a $\left\{0, \frac{1}{2}\right\}$-cut for $\mathscr{P}_{M,\mathcal{L}}$, derived from the projected rank inequalities (6) for the members of Q. $\square$

This yields the following corollary.

Corollary 3 *If P is a fractional MP polytope on a laminar matroid, then $P_I = P_1 = P_{1/2}$.*

This generalises the classical result on fractional b-matching polytopes mentioned in Subsection 2.5 (since fractional b-matching polytopes are equivalent to fractional MP polytopes for partition matroids).

There is however a sense in which laminar MP polytopes are "more complicated" than b-matching polytopes. Recall once more that b-matching polytopes are completely described by (1)–(3). Thus, all of their facet-defining inequalities have binary left-hand-side coefficients. This is not the case for laminar MP polytopes. Indeed, facet-defining projected rank inequalities can have ternary coefficients, and facet-defining projected blossom inequalities can have non-ternary coefficients. This is shown in the following example.

Example 1 Let M be the laminar matroid defined over the ground set $F = \{1, \dots, 20\}$, with set family $\mathcal{F} = \{F_1, \dots, F_5\}$, where

$$\begin{aligned} F_1 &= \{1, \dots, 9, 19\} & u_1 &= 5 \\ F_2 &= \{5, \dots, 9, 19\} & u_2 &= 4 \\ F_3 &= \{7, \dots, 9, 19\} & u_3 &= 3 \\ F_4 &= \{11, 13, 17, 10\} & u_4 &= 2 \\ F_5 &= \{12, 14, 18, 20\} & u_5 &= 1. \end{aligned}$$

Let the line set $\mathcal{L}$ be as follows:

$$\big\{\{1,11\},\{2,12\},\{3,13\},\{4,14\},\{5,15\},\{6,16\},\\ \{7,17\},\{8,18\},\{9,19\},\{10,20\}\big\}.$$

One can check (either by hand or with the help of a computer) that the following five projected rank inequalities define facets of $\mathscr{P}_{M,\mathcal{L}}$:

$$\begin{aligned} \sum_{i=1,\ldots,9} y_{i,10+i} + 2y_{9,19} &\leq 5 && (20)\\ \sum_{i=5,\ldots,9} y_{i,10+i} + 2y_{9,19} &\leq 4 && (21)\\ \sum_{i=7,8,9} y_{i,10+i} + 2y_{9,19} &\leq 3 && (22)\\ y_{1,11} + y_{3,13} + y_{7,17} + y_{10,20} &\leq 2 && (23)\\ y_{2,12} + y_{4,14} + y_{8,18} + y_{10,20} &\leq 1. && (24) \end{aligned}$$

Three of these have non-binary left-hand side coefficients. Now, taking the $\{0, \frac{1}{2}\}$-cut of (20)-(24) yields the following projected blossom inequality:

$$\begin{aligned} &y_{1,11} + y_{2,12} + y_{3,13} + y_{4,14} + y_{5,15} +\\ &y_{6,16} + 2y_{7,17} + 2y_{8,18} + 3y_{9,19} + y_{10,20} \leq 7. \end{aligned}$$

One can check that this inequality is also facet-defining. Moreover, it has non-ternary left-hand side coefficients. □

On the positive side, the coefficients in a facet-defining projected blossom inequality cannot be *very* large:

Proposition 1 *For any given $\ell \in L$, the coefficient of the variable y_ℓ in a facet-defining projected blossom inequality is $O\big(|F|\big)$.*

Proof. Any facet-defining blossom inequality takes the form (19). Thus, the coefficient of y_ℓ cannot exceed $\Big\lfloor \frac{\sum_{i=1,\ldots,k} |\ell \cap F_i|}{2} \Big\rfloor$. From this the $O\big(|F|\big)$ bound follows easily. □

5 The MP Polytope for Arbitrary Matroids

Next, we consider the MP polytope in the case of an arbitrary matroid. Throughout this section, for notational simplicity, we write $\tilde{\mathscr{P}}$ and $\mathscr{P}$ for $\tilde{\mathscr{P}}_{M,\mathcal{L}}$ and $\mathscr{P}_{M,\mathcal{L}}$, respectively. Then, the elementary closure $\tilde{\mathscr{P}}_1$ is a natural polyhedral outer-approximation of $\mathscr{P}$.

For conciseness, let us call a set of projected rank inequalities (6) *laminar* if their supports form a laminar set. We have the following result.

Theorem 3 *If a CG-cut defines a facet of $\tilde{\mathscr{P}}_1$, it can be derived using a laminar set of projected rank inequalities.*

Proof. Let R be the set of non-dominated projected rank inequalities. Consider a CG-cut that defines a facet of $\tilde{\mathscr{P}}_1$. Let $\lambda \in [0,1)^R$ be the corresponding multiplier vector and let $R' = \{i \in R : \lambda_i > 0\}$. Assuming to the contrary that the set of projected rank inequalities indexed by R' is not laminar, there is a pair of inequalities $i, j \in R'$ whose supports S_i, S_j cross, i.e., $S_i \setminus S_j \neq \emptyset \neq S_j \setminus S_i$. The contribution of these two inequalities in the CG-cut, before rounding down, is the sum of

$$\lambda_i \big(\textstyle\sum_{\ell \in \mathcal{L}} |S_i \cap \ell| \, y_\ell \big) \leq \lambda_i r_M(S_i) \quad \text{and} \tag{25}$$
$$\lambda_j \big(\textstyle\sum_{\ell \in \mathcal{L}} |S_j \cap \ell| \, y_\ell \big) \leq \lambda_j r_M(S_j). \tag{26}$$

Assume without loss of generality that $\lambda_i \geq \lambda_j > 0$ and observe that (25) can alternatively be written as the sum of

$$(\lambda_i - \lambda_j) \big(\textstyle\sum_{\ell \in \mathcal{L}} |S_i \cap \ell| \, y_\ell \big) \leq (\lambda_i - \lambda_j) r_M(S_i) \quad \text{and} \tag{27}$$
$$\lambda_j \big(\textstyle\sum_{\ell \in \mathcal{L}} |S_i \cap \ell| \, y_\ell \big) \leq \lambda_j r_M(S_i). \tag{28}$$

Consider now the two projected rank inequalities derived by 'uncrossing' the sets S_i and S_j, i.e., the inequalities

$$\textstyle\sum_{\ell \in \mathcal{L}} |(S_i \cup S_j) \cap \ell| \, y_\ell \leq r_M(S_i \cup S_j) \quad \text{and} \tag{29}$$
$$\textstyle\sum_{\ell \in \mathcal{L}} |(S_i \cap S_j) \cap \ell| \, y_\ell \leq r_M(S_i \cap S_j). \tag{30}$$

It becomes easy to show that, for each $\ell \in \mathcal{L}$,

$$|(S_i \cup S_j) \cap \ell| + |(S_i \cap S_j) \cap \ell| = |S_i \cap \ell| + |S_j \cap \ell|, \tag{31}$$

by noticing the following partitions of $S_i \cap \ell$ and $(S_i \cup S_j) \cap \ell$:

$$S_i \cap \ell = \big((S_i \setminus S_j) \cap \ell\big) \cup \big((S_i \cap S_j) \cap \ell\big),$$
$$(S_i \cup S_j) \cap \ell = \big((S_i \setminus S_j) \cap \ell\big) \cup \big((S_i \cap S_j) \cap \ell\big) \cup \big((S_j \setminus S_i) \cap \ell\big).$$

Also, the submodularity of the rank function r_M implies

$$r_M(S_i \cup S_j) + r_M(S_i \cap S_j) \leq r_M(S_i) + r_M(S_j). \tag{32}$$

But then, the sum (29)-(30), each multiplied by λ_j, plus (27) provides an inequality with the same left-hand side as the sum of (25)-(26) (because of (31)) and a no-larger right-hand side (because of (32)). This suggests a substitution strategy for strengthening the CG-cut, i.e., the substitution in R' of (25)-(26) with (29)-(30), each multiplied by λ_j, plus (27). By repeating this uncrossing argument for any pair of crossing inequalities in R', one can substitute every non-laminar subset of projected rank inequalities in R with a laminar one and derive a CG-cut that is at least as strong. $\square$

We can now establish that $\tilde{\mathscr{P}}_1 = \tilde{\mathscr{P}}_{1/2}$ using the fact that the laminar MP polytope is fully described by its $\{0, \frac{1}{2}\}$-cuts.

Corollary 4 *For any matroid M and set of lines $\mathcal{L}$, $\tilde{\mathscr{P}}_{1/2} = \tilde{\mathscr{P}}_1$.*

Proof. Theorem 3 implies that an inequality $\alpha y \leq \beta$ that is facet-defining for $\tilde{\mathscr{P}}_1$ can be derived as a CG-cut from a laminar set of projected rank inequalities. Let $R' \subset R$ be this laminar set and $A'y \leq b'$ be the system of linear inequalities that it defines. Then, $\{y \in \{0,1\}^{|E|} : A'y \leq b'\}$ is a laminar MP polytope. By Theorem 2, this polytope is described by the bounds, projected rank inequalities and $\{0, \frac{1}{2}\}$-cuts. Hence, $\alpha y \leq \beta$ is also a $\{0, \frac{1}{2}\}$-cut. □

We remark that Corollary 4 can be proved in a different way, using the fact [17] that the system of inequalities defining $\tilde{\mathscr{P}}$ is totally dual half-integral. Our proof, however, highlights the strong connection between MP polytopes and laminarity.

6 Integrality Gaps

As mentioned in Subsection 2.5, Vande Vate [32] showed that $\mathscr{P}_{M,\mathcal{L}}$ can be strictly contained in the elementary closure of $\tilde{\mathscr{P}}_{M,\mathcal{L}}$, even when M is a series-parallel matroid. We now prove a stronger result.

Proposition 2 *Even when M is a series-parallel matroid, the integrality gap of the relaxation defined by all projected rank and non-negativity inequalities can be as large as $\big(|\mathcal{L}|+1\big)/2$. Moreover, even when all CG-cuts are added, the integrality gap can be as large as $|\mathcal{L}|/2$.*

Proof. Let $k \geq 3$ be an odd integer. Let $G = (V, E)$ be a graph defined as follows. The vertex set is $\{0, \ldots, k+1\}$. For $i = 1, \ldots, k$, the edges $\{0, i\}$ and $\{i, k+1\}$ are present in E. Note that G is series-parallel. Let M be the graphic matroid of G. Given that M is graphic, the independent sets are forests. One can check that the non-dominated rank inequalities for the forest polytope are:

$$\begin{array}{rl} x_e \leq 1 & (e \in E) \\ \sum_{i \in S} \big(x_{0,i} + x_{i,k+1}\big) \leq |S| + 1 & \big(S \subseteq \{1, \ldots, k\} : |S| \geq 2\big). \end{array}$$

We now define an MP instance on M as follows. There are k lines, and the ith line consists of the edges $\{0, i\}$ and $\{i, k+1\}$. Each line has unit profit. The optimal solution to this MP instance is trivial: we can select at most one line. The projected rank inequalities are:

$$\begin{array}{rl} y_i \leq 1 & (i = 1, \ldots, k) \\ 2\sum_{i \in S} y_i \leq |S| + 1 & \big(S \subseteq \{1, \ldots, k\} : |S| \geq 2\big). \end{array}$$

Thus, an optimal solution to the fractional MP problem is obtained by setting all y variables to $(k+1)/2k$. This yields the upper bound $(k+1)/2$ as stated. One can check that the non-dominated $\{0, \frac{1}{2}\}$-cuts take the form:

$$\sum_{i \in S} y_i \leq \left\lfloor \frac{|S|+1}{2} \right\rfloor \qquad \big(S \subseteq \{1, \ldots, k\} : |S| \geq 3 \text{ and odd}\big).$$

Thus, one can satisfy all $\{0, \frac{1}{2}\}$-cuts (and therefore all CG-cuts) by setting the y variables to $1/2$. This yields the upper bound $k/2$ as stated. □

We find the above integrality gap result surprising, given that the MP problem for gammoids (and therefore also series-parallel matroids) can be solved in polynomial time.

We end the paper with a few suggestions for further research. On the theoretical side, one could attempt to find a complete linear description of the MP polytope for series-parallel matroids or, more ambitiously, gammoids or graphic matroids. One could also search for new valid inequalities for the MP polytope of a general matroid. On the algorithmic side, one could design and test an exact algorithm for the MP problem, perhaps based on branch-and-cut [29]. To this end, fast heuristics for identifying violated projected rank inequalities and $\{0, \frac{1}{2}\}$-cuts would be desirable.

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