On Matroid Parity and Matching Polytopes*

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October 2017

Abstract

The matroid parity problem is a natural extension of the matching problem to the matroid setting. It can be formulated as a $0-1$ linear program using the so-called rank and line constraints. One can obtain a more compact formulation using what we call projected rank inequalities. Our main result is that, for either formulation, the elementary closure of the continuous relaxation is equal to its $\{0, \frac{1}{2}\}$-closure. To prove this, we first derive some auxiliary results concerned with matchings, gammoids and laminar matroids.

1 Introduction

Edmonds’ seminal works on polyhedra associated with matchings and matroids [3, 4] essentially marked the birth of what is now known as polyhedral combinatorics (see, e.g., [2]). Moreover, the matroid parity problem, an elegant generalisation of the matching and matroid intersection problems, has been studied in depth (e.g., [8, 10, 11]). The polyhedra associated with the matroid parity problem have received far less attention. These polyhedra, and their connections with matching polyhedra, are the subject of this paper.

Let us recall some definitions. A $c$-capacitated $b$-matching in an undirected graph $G(V,E)$ ($b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$) is a vector $x \in \mathbb{Z}_+^E$ such that

$$x(\delta(i)) \leq b_i \quad \forall \ i \in V,$$

$$0 \leq x_e \leq c_e \quad \forall \ e \in E,$$

*The research of the first author was partially funded by the Greek National Strategic Reference Framework, through the program “Education and Lifelong Learning”.

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where, as usual, $\delta(i)$ denotes the set of edges incident on node $i$ and, for any $E' \subset E$, let $x(E')$ denote $\sum_{e \in E'} x_e$. The associated $c$-capacitated $b$-matching polytope is defined as

$$\mathcal{PB}_{b,c} = \text{conv} \left\{ x \in \mathbb{Z}_+^E : x \text{ satisfies (1) and (2)} \right\}.$$ 

One can optimise linear functions over $\mathcal{PB}_{b,c}$ in polynomial time. Edmonds [3] and Pulleyblank [13] showed that a linear description of $\mathcal{PB}_{b,c}$ is given by (1), (2) and the blossom inequalities

$$x(E(H)) + x(T) \leq \left\lfloor \frac{b(H) + c(T)}{2} \right\rfloor, \forall H \subset V, T \subseteq \delta(H) : b(H) + c(T) \text{ odd},$$

where $E(H)$ denotes the set of edges with both end-nodes in $H$, and $b(H)$ and $c(T)$ denote $\sum_{i \in H} b_i$ and $\sum_{e \in T} c_e$, respectively. A $c$-capacitated $b$-matching is perfect if it satisfies the constraints (1) at equality.

Given a matroid $M(F, \mathcal{I})$, the matroid polytope is the convex hull of the incidence vectors $x \in \{0, 1\}^F$ of subsets in $\mathcal{I}$; its linear description is given [4] by the non-negativity inequalities and the rank inequalities

$$x(S) \leq r(M)(S) \quad \forall S \subseteq F,$$

where $r_M(S) = \max_{A \subseteq S} \{|A| : A \in \mathcal{I}\}$ is the rank function. One can optimise linear functions over the matroid polytope in polynomial time [4].

The matroid parity (MP) problem assumes a matroid $M$ with a ground set $F$ of even cardinality, along with a partition of $F$ into two-element subsets called lines and indexed by $\mathcal{L}$, where each line is assumed to be independent. The problem calls for a set of lines of maximum cardinality (or weight) such that their union is independent in $M$. To define the associated polytope, the incidence vectors of independent sets $x \in \{0, 1\}^F$ and lines $y \in \{0, 1\}^\mathcal{L}$, where $(x, y)$ must satisfy the rank inequalities (4) plus the line constraints

$$x_i = x_j = y_{ij} \quad \forall \{i, j\} \in \mathcal{L}. \quad (5)$$

Using (5) to project out the $x$ variables, we derive the projected rank inequalities, apparently first discovered by Vande Vate [15]:

$$\sum_{\ell \in \mathcal{L}} |S \cap \ell| y_{\ell} \leq r_M(S) \quad \forall S \subseteq F.$$ 

The matroid parity (MP) polytope is then:

$$\mathcal{PM}_{M, \mathcal{L}} = \text{conv} \left\{ y \in \{0, 1\}^\mathcal{L} : (6) \text{ hold} \right\}.$$ 

Optimising linear functions over $\mathcal{PM}_M$ is difficult in general. It is $NP$-hard when the matroid $M$ has a compact description [10] and, if $M$ is given only implicitly via an independence oracle, it can take exponential time [9].
It can however be done efficiently under certain conditions, e.g., if \( M \) is given by a linear representation \([6, 8]\), or if \( M \) is a \textit{gammoid} \([14]\). Moreover, Lee \textit{et al.} \([11]\) have provided some nice approximation results.

In this paper, then, we investigate MP polytopes in more depth. Our main result is that, for any instance of the matroid parity problem, the elementary closure of the continuous relaxation of either the “rank-and-line” formulation (4)–(5) or the “projected rank” formulation (6) is equal to its \( \{0, \frac{1}{2}\} \)-closure. Along the way, we derive several results concerned with MP polytopes, matchings, gammoids and laminar matroids. Specifically, we prove the following: (i) when \( M \) is a \textit{gammoid}, the MP polytope is a projection of a perfect matching polytope into a suitable subspace; (ii) when \( M \) is \textit{laminar}, the MP polytope is affinely congruent to a perfect matching polytope; and (iii) even if \( M \) is laminar, the MP polytope can have facets that are defined by inequalities with non-ternary left-hand side coefficients (all terms in italics will be defined later on).

The rest of the paper has a simple structure. After a section presenting some additional background and notation, four sections present the results (i)-(iii) mentioned above plus the main result.

2 Background

Given a set system \((F, \mathcal{A})\) where \( \mathcal{A} = \{A_j : j \in J\} \), a \textit{transversal} of \( \mathcal{A} \) is a set \( S \subset F \) such that there exists a bijection \( \pi : J \to S \) with \( \pi(j) \in A_j \) for all \( j \in J \). A transversal defined over a subset of \( \mathcal{A} \) is called \textit{partial} and the collection of partial transversals of \( \mathcal{A} \) forms the independent set \( I \) of a matroid with ground set \( F \); such a matroid is called \textit{transversal matroid} and hereafter denoted as \( M[\mathcal{A}] \). Transversals can be associated to matchings in a bipartite graph \( G(V_1, V_2, E) \) such that \( V_1 \) and \( V_2 \) have one node per member of \( F \) and \( \mathcal{A} \) respectively, while \( E = \{e_{ij} : i \in A_j, i \in V_1, j \in V_2\} \).

Let us extend the relation of transversals to matchings to include also \textit{linkings}. For a digraph \( G(V, A) \) and two subsets \( S, T \) of \( V \) (not necessarily distinct) with \(|S| = |T|\), we say that \( S \) can be linked \textit{onto} \( T \) if there are \(|S|\) node-disjoint (directed) paths in \( G \) with sources in \( S \) and sinks in \( T \); \( S \) is linked \textit{into} \( T \) if it is linked onto some subset of \( T \). Define the node sets \( N_v = \{v\} \cup \{u \in V : (v, u) \in A\} \) for all \( v \in V \) and the set family \( N_S = \{N_v : v \in S\} \) for any \( S \subseteq V \). The relationship between linkings and transversals arises from the \textit{linkage lemma} of Ingleton and Piff \([7]\): for any two node-subsets \( S \) and \( T \) of a digraph \( G(V, A) \), \( S \) is linked into \( T \) if and only if \( V \setminus S \) contains a transversal of \( N_{V \setminus T} \).

2.1 Gammoids

Perfect \([12]\) introduced a broad class of matroids called \textit{gammoids}, defined over linkings in a digraph \( G(V, A) \). For a fixed node-subset \( T \) of \( G(V, A) \),
let $L(G,T)$ denote the set family of nodes that can be linked into $T$, i.e., $S \subseteq V$ is in $L(G,T)$ if and only if it is linked onto some $X \subseteq T$. Then, a strict gammoid is a matroid $M(F,I)$ such that $F = V$ and $I = L(G,T)$.

**Definition 1** A gammoid is a matroid that is isomorphic to a restriction of a strict gammoid.

We denote as $M[L(G,T)]$ a strict gammoid and following standard notation we write $M[L(G,T)]|F$ to denote a gammoid (i.e. a restriction of the matroid $M[L(G,T)]$ by the set $V \setminus F$). Gammoids are related to matchings via the linkage lemma. Moreover, for a strict gammoid $M[L(G,T)]$, one can derive a representation of its dual $(M[L(G,T)])^* = M[N_{V\setminus T}]$; take a transversal matroid $M[A]$ on a ground set $F$ plus some transversal $T = \{v_1,\ldots,v_q\}$ of $A$. For each $i = \{1,\ldots,q\}$ draw a directed edge $(v_i,j)$ for $j \in A_i \setminus v_i$. In this way we yield a digraph $G(V,A)$ such that $V^G = F$ and $(M[A])^* = M[L(G,V \setminus T)]$. In what follows we occasionally write $V^G$ instead of $V$ to denote the node-set of a graph (or digraph) $G$ and use $E^G$ and $A^G$ in a similar fashion.

Tong et al. [14] represent a gammoid using a bipartite graph as follows. Assume a strict gammoid $M[L(G,T)]$ and consider its dual, that is the transversal matroid $M[N_{V\setminus T}]$. Construct the bipartite graph $G'(V_1,V_2,E)$ that corresponds to $M[N_{V\setminus T}]$ by setting $V_1^{G'} = V^G$, $V_2^{G'} = V^G \setminus T$ and $E^{G'} = \{(u,v) : u \in N_v, v \in V^G \setminus T\}$. Based on our discussion, $S \in L(G,T)$ if and only if there is some matching in $G'$ that matches each node in $V_1^{G'}$ with exactly one node in $V_2^{G'} \setminus S$. In their construction, Tong et al. add another copy of $V^G$ to represent the ground set of the strict gammoid. Specifically, the bipartite graph takes the form $\bar{G}(V_1,V_2 \cup V_3,E)$, where $V_1^{\bar{G}}$ and $V_3^{\bar{G}}$ are both copies of $V^G$, and $V_2^{\bar{G}} = V^G \setminus T$. Let us use $v_i$ to denote a node $v \in V^G$ that is also a member of the node set $V_i^{\bar{G}}$, $i \in \{1,2,3\}$. The edge set of $\bar{G}$ can be written as $E^{\bar{G}} = E^{G'} \cup \{(v_1,v_3) : v_1 \in V_1^{\bar{G}}, v_3 \in V_3^{\bar{G}}, v \in F\}$. Then, $S \in L(G,T)$ if and only if there is some matching $\bar{G}$ that covers every node in $V_2^{\bar{G}}$, as well as every node in $V_3^{\bar{G}}$.

### 2.2 Laminar matroids

Given a ground set $F$, a family $\mathcal{F} = \{F_i \subseteq F : i = 1,\ldots,k\}$ is called laminar if, for all $F_i,F_j \in \mathcal{F}$ and $i \neq j$ exactly one of the following holds: $F_i \subset F_j$, $F_j \subset F_i$ or $F_i \cap F_j = \emptyset$. We say that two sets $F_i,F_j \subset F$, $i \neq j$ cross, if and only if none of the three laminarity conditions hold. On the other hand, a chain is any laminar family $\mathcal{F}$ such that $F_i \cap F_j \neq \emptyset$ for all $F_i,F_j \in \mathcal{F}$, $i \neq j$. Note that a set family is laminar if and only if it can be represented graphically as an arborescence (anti-arborescence) [9].
Definition 2 A matroid $M(F, I)$ is called laminar if there is a laminar family $F$ and positive integers $U = \{u_1, \ldots, u_k\}$, such that $S \in I$ if and only if $|S \cap F_j| \leq u_j$ for $j = 1, \ldots, k$.

Let us denote by $M[F, F, U]$ a laminar matroid. We assume that $u_j \leq |F_j|$ for $j = 1, \ldots, k$, and that all sets $F_j$ are non-redundant, in terms of sustaining the independent sets of $M[F, F, U]$.

Laminar matroids are gammoids and to show this one can construct a suitable digraph [5]. Let us first though introduce some notation to be also used in later sections. For a laminar matroid $M[F, F, U]$ and $F_i, F_j \in F$ such that $F_j \subset F_i$ and there is no $F_k \in F$ satisfying $F_j \subset F_k \subset F_i$, we call $j$ a ‘child’ of $i$. If $\chi(i)$ denotes the set of children of $i$, the set $F(i) = F_i \setminus \bigcup_{j \in \chi(i)} F_j$ contains any elements in $F_i$ that are not elements of $i$’s children.

We construct the digraph $G'(V, A)$ by introducing one node $V^{G'}(i)$ for each member $i$ of the ground set $F$ and one node $V^{G'}(F^i)$ for each member $F_i$ of $F$. The arc set is $A^{G'} = A^1 \cup A^2$, where

$$A^1 = \{(V^{G'}(i), V^{G'}(F^j)) : i \in F, i \in F(j)\}$$

and

$$A^2 = \{(V^{G'}(F^i), V^{G'}(F^j)) : F_i, F_j \in F, i \in \chi(j)\}.$$ 

The capacity of each arc in $A^1$ is one, while for arcs $(V(F^i), V(F^j)) \in A^2$ the capacity is $u_i$. Note that the underlying undirected graph of $G'$ is a forest with as many components as the maximal members of $F$. We can easily convert this forest to a tree by introducing the pseudonode $V(F^{k+1})$ that represents the ground set $F$. If we take into account the direction of the edges, $G'$ is an anti-arborescence and thus there is always a unique path linking every node in $V^{G'}(F)$ with the root node $V^{G'}(F^{k+1})$.

Now, every collection of paths in $G'$ that respects the capacities $u_1, \ldots, u_k$ and has source nodes in $V^{G'}(S) \subseteq V^{G'}(F)$ and destination the node $V^{G'}(F^{k+1})$ can be mapped into a collection of node-disjoint paths in a digraph $G(V, A)$ with source nodes $V^{G}(S)$ and sink nodes in $V^{G}(T)$. Hence, we set $V(G) = V(F) \cup V(f)$, where

$$f = \{f^j_i : \text{ for } i \text{ such that } F_i \in F, j \in \{1, \ldots, u_i\}\},$$

and draw, for each $F_i \in F$ and $j \in F$ such that $j \in F(i)$, arcs from $V(j)$ to each $V(f^j_i)$, $p \in \{1, \ldots, u_i\}$. It is straightforward to check that $S \subseteq F$ is independent in $M$ if and only if there are $|S|$ node-disjoint paths with source nodes in $V^{G}(F)$, and sink nodes in $V^{G}(T)$. It follows that any laminar matroid $M[F, F, U]$ has such a representation and thus is a gammoid $M[L(G, T)]|F$. 

5
3 The MP polytope for gammoids

Recall the representation of a gammoid $M[\langle L(G, T) \rangle | F]$ as the bipartite graph $G(V_1, V_2 \cup V_3, E)$. A set $S \subseteq F$ is independent in $M$ if and only if there is a matching in $\bar{G}$ that covers all the nodes in $V_2^G$ and nodes in $V_3^G(S)$. To obtain a representation of the parity problem over a gammoid construct the graph $\tilde{G}$ such that $V_{\tilde{G}} = V_G$ and $E_{\tilde{G}} = E_G \cup \{(v, u) : \{v, u\} \in \ell, \ell \in L\}$. That is, $\tilde{G}$ is a copy of $G$ plus the edges corresponding to lines $\ell \in L$.

We observe that $\tilde{G}$ is not bipartite and that any matching $\tilde{G}$ defines an independent set of $M[\langle L(G, T) \rangle | F]$. Moreover, an edge $(v, u) \in \tilde{E}$ that corresponds to the line $\{v, u\} \in \mathcal{L}$ is in a matching of $\tilde{G}$ if and only if that line is not selected. The authors in [14] impose large weights to all edges adjacent to nodes in $V_{\tilde{G}}^2$ to satisfy the condition of covering that node set. Instead, one can drop these weights and formulate the problem as a perfect matching problem.

**Proposition 1** The parity problem over the gammoid $M[\langle L(G, T) \rangle | F]$ and the set of lines $\mathcal{L}$ can be formulated as a perfect matching problem on an undirected graph with $|F| + |V \setminus T| + |A| + |\mathcal{L}|$ edges and $|V| + |F| + |V \setminus T|$ nodes.

**Proof.** We provide an integer programming (IP) formulation of the parity problem. For each $v \in F$, define the binary variable $x_v$ that takes the value 1 if and only if $v$ is to be included in the linking $S$. Define also the binary variable $\bar{y}_\ell$ taking the value 1 if and only if $\ell$ is not selected. For each $v = 1, \ldots, |V \setminus T|$ and $u \in F_v$, let the binary variable $a_{vu}$ be 1 if and only if the arc $(v, u) \in A_G$ is included in the node-disjoint paths that link $S$ and $T$. Finally, for each $v \in V_G$, define the node-set $C_v = \{u \in V \setminus T : v \in N_u\}$ representing the subset of nodes in $V^G$ that are adjacent at the tails of arcs in $A^G$ that have their head adjacent to node $v$.

Assuming $S$ is independent in $M[\langle L(G, T) \rangle | F]$ and by the definition of vectors $x, \bar{y}$ and $a$, it is easy to check that the incidence vector of $S$ corresponds to a feasible solution to the following IP.

$$\begin{align*}
\text{max} & \quad \frac{1}{2}x(S) \\
\text{s.t.} & \quad \sum_{u \in N_v} a_{vu} = 1 \quad v \in V \setminus T \quad (7) \\
& \quad \sum_{u \in C_v} a_{vu} + x_v = 1 \quad v \in F \quad (8) \\
& \quad \sum_{u \in C_v} a_{vu} = 1 \quad v \in V \setminus F \quad (9) \\
& \quad x_v + \bar{y}_\ell = 1 \quad \forall \ell \in \mathcal{L}, v \in \ell \quad (10) \\
& \quad x_v, \bar{y}_\ell \in \mathbb{Z}_+ \quad \forall \ell \in \mathcal{L}, v \in \ell \quad (11) \\
& \quad a_{vu} \in \mathbb{Z}_+ \quad \forall v \in V \setminus T, u \in V. \quad (12)
\end{align*}$$

Constraints (7), (8)-(9) and (10) are the degree equations for the nodes in $V_2^G$, $V_1^G$ and $V_3^G$, respectively. \hfill \square
Proposition 1 has the following interesting corollary:

**Corollary 1** Every MP polytope for a gammoid is a projection of a perfect matching polytope.

**Proof.** The convex hull of solutions to (7)–(12) is a perfect matching polytope. To obtain the corresponding MP polytope, it suffices to project the matching polytope into the space of the \( \bar{y} \) variables and then complement each \( \bar{y} \) variable. \( \square \)

### 4 The MP polytope for laminar matroids

Since laminar matroids are gammoids, Proposition 1 implies that the laminar matroid parity problem can be formulated as a perfect 1-matching problem. Laminarity allows for stronger results, i.e., the derivation of a \( b \)-matching formulation in a lower dimension and the proof that the MP polytope (i.e., the polytope defined only on the \( y \)-space) is affinely congruent to a perfect matching polytope.

Assume a laminar matroid \( M[F, \mathcal{F}, U] \) and the anti-arborescence representation \( G'(V, A) \) described in Section 2.2. By definition, the laminar matroid \( M[F, \mathcal{F}, U] \) is equivalent to a gammoid represented by \( G' \) and thus it has a bipartite representation \( \bar{G}(V_1, V_2 \cup V_3, E) \). We now exploit the connection between graphs \( G' \) and \( \bar{G} \). Members of \( V_1^G \) and \( V_2^G \), represent the sink and source nodes respectively of each arc in the anti-arborescence \( G' \). Members of \( V_3^G \) on the other hand represent the potential source nodes of each collection of paths from a subset of \( V_{G'}(F) \) to the root node \( V_{G'}(F_k+1) \).

For any \( i \in F \) then, nodes in \( V_{G'}(i) \), concern the leaves of \( G' \). Moreover, the special structure of \( G' \) implies that for any \( i \in F \):

- \( V_1^G(i) \) is adjacent to exactly two nodes, namely \( V_3^G(i) \) and \( V_2^G(i) \),
- \( V_2^G(i) \) is also adjacent to exactly two nodes; that is \( V_1^G(i) \) and \( V_1^G(F_j) \) for some \( F_j \in \mathcal{F} \cup F_{k+1} \) such that \( i \in F(j) \), and finally
- \( V_3^G(i) \) is adjacent to \( V_1^G(i) \) only.

For a given linking \( S \in L(G', F^{k+1})|F \) and the corresponding matching in \( \bar{G} \), we see that edge \((V_2^G(i), V_1^G(F_j))\) belongs to that matching for some \( F_j \in \mathcal{F} \cup F_{k+1} \) such that \( i \in F(j) \) if and only if \( V_1^G(i) \in S \). Then, because of laminarity and for any \( i \in F \), we can identify the pairs of nodes \( (V_2^G(i), V_3^G(i)) \) and \( (V_1^G(i), V_1^G(F_j)) \) for \( i \in F(j) \). We implement this transformation by simply deleting node sets \( V_1^G(F) \) and \( V_2^G(F) \).

The node \( V_1^G(F^{k+1}) \) is a pseudo-node that corresponds to set \( F \), thus the degree equation that corresponds to \( V_1^G(F^{k+1}) \) is implied by the degree equations that correspond to the maximal members of \( \mathcal{F} \) (i.e. the nodes
Without loss of generality then, we do not represent $V^G(F^{k+1})$ explicitly in $\bar{G}$, and the multi-edges $(V^G_2(F^i), \cdot)$ for all $F_i \in \bar{F}$ have only one end point.

As in Section 3, we define the laminar matroid parity problem by introducing a line set $L$ and construct the graph $\bar{G}(V, E)$ such that $V^G = V^G$ and $E^G = E^G \cup \{(i, j) : \{i, j\} \in \ell, \ell \in L\}$.

**Lemma 1** The parity problem on a laminar matroid $M = [F, F, U]$ and a line set $L$ is reducible to a perfect $b$-matching problem on a graph with $|F| + |L|$ edges, $2|F|$ multi-edges and $|F| + 2|F|$ nodes.

**Proof.** As in the proof of Proposition 1, for each $i \in F$, define the binary variable $x_i$, taking the value 1 if and only if $i$ is to be included in the set $S \in I$, and the binary variable $\bar{y}_\ell$, being 1 if and only if line $\ell$ is not selected. For $i = 1, \ldots, k$, define the variable $z_i \in \{0, \ldots, u_i\}$, representing the quantities $|S \cap F_i|$. Finally, for $i = 1, \ldots, k$, define the variable $\bar{z}_j \in \{0, \ldots, u_j\}$, representing the quantities $u_j - |S \cap F_j|$. Then, the incidence vector $(x, y, z, \bar{z})$ of an $S \in I$ is a feasible solution to the following IP.

$$\begin{align*}
\text{max} & \quad \frac{1}{2} x(F) \\
\text{s.t.} & \quad x(F(i)) + \sum_{j \in \chi(i)} z_j + \bar{z}_i = u_i & i = 1, \ldots, k \\
& \quad z_i + \bar{z}_i = u_i & i = 1, \ldots, k \\
& \quad x_e + \bar{y}_\ell = 1 & \forall \ell \in L, e \in \ell \\
& \quad x_e, \bar{y}_\ell, \in \mathbb{Z}_+ & \forall \ell \in L, e \in \ell \\
& \quad z_i, \bar{z}_j \in \mathbb{Z}_+ & i = 1, \ldots, k.
\end{align*}$$

Constraints (13) and (14) are the degree equations of nodes in $V^G_1$ and $V^G_2$ respectively, while (15) are the degree equations of nodes in $V^G_3$. □

**Lemma 2** The laminar MP polytope is affinely congruent to a perfect matching polytope.

**Proof.** Let a laminar matroid parity instance be given by a matroid $M[F, F, U]$ and a line set $L$ and let $\mathcal{P}_M \subset \mathbb{R}^{|L|}$ be the associated polytope. We prove that there is a perfect $b$-matching polytope in $\mathbb{R}^{|F| + |L| + 2|F|}$ that is affinely congruent to $\mathcal{P}_M$.

Define the polytope

$$\mathcal{P}^*_M = \text{conv}\{ (x, \bar{y}, z, \bar{z}) \in \mathbb{Z}_+^{|F| + |L| + 2|F|} : (13) - (15) \text{ hold} \}.$$
Lemma 1 implies that a vector $y^*$ lies in $\mathcal{P}_M$ if and only if the corresponding vector $(x^*, \bar{y}^*, z^*, \bar{z}^*)$ lies in $\mathcal{P}^+_M$, where:

$$x_e^* = x_f^* = y_{ef}^* \quad (\{e, f\} \in \mathcal{L})$$

$$\bar{y}_\ell^* = 1 - y_{\ell}^* \quad (\ell \in \mathcal{L})$$

$$z_i^* = \sum_{\ell \in \mathcal{L}} |\ell \cap F_i| y_\ell^* \quad (i = 1, \ldots, k)$$

$$\bar{z}^*_i = u_i - \sum_{\ell \in \mathcal{L}} |\ell \cap F_i| y_\ell^* \quad (i = 1, \ldots, k).$$

Since this mapping is affine and invertible, $\mathcal{P}_M$ is affinely congruent to $\mathcal{P}^+_M$. □

5 Facet-defining inequalities

The perfect matching polytope for a graph $G = (V, E)$ and vector $b \in \mathbb{Z}_+^V$ is completely described by the degree equations $x(\delta(i)) = b_i$ for all $i \in V$, the non-negativity inequalities $x_e \geq 0$ for all $e \in E$, and the simple blossom inequalities

$$x(E(H)) \leq \left\lfloor \frac{b(H)}{2} \right\rfloor,$$

where $H \subset V$ is such that $b(H)$ is odd. From this it follows that the only inequalities that can define facets of $\mathcal{P}^+_M$ are the non-negativity inequalities for the $x$, $\bar{y}$, $z$ and $\bar{z}$ variables, together with the simple blossom inequalities, which can now involve combinations of those variables. Using this fact together with Lemma 2, one can derive a complete linear description of the laminar matroid parity polytope $\mathcal{P}_M$.

The non-negativity inequalities are the easiest to handle:

- For each $\{e, f\} \in \mathcal{L}$, both inequalities $x_e \geq 0$ and $x_f \geq 0$ for $\mathcal{P}^+_M$ map to the inequality $y_{ef} \geq 0$ for $\mathcal{P}_M$.

- For each $\ell \in \mathcal{L}$, the inequality $\bar{y}_\ell \geq 0$ for $\mathcal{P}^+_M$ maps to the upper bound inequality $y_\ell \leq 1$ for $\mathcal{P}_M$.

- For each $i = 1, \ldots, k$, the inequality $z_i \geq 0$ is redundant, in light of equations (14), which imply $z_i = x(F_i)$ for all $i = 1, \ldots, k$.

- For each $i = 1, \ldots, k$, the inequality $\bar{z}^*_i \geq 0$ for $\mathcal{P}^+_M$ is equivalent to $x(F_i) \leq u_i$, due to equations (14). This latter inequality in turn maps to the projected rank inequality

$$\sum_{\ell \in \mathcal{L}} |F_i \cap \ell| y_\ell \leq u_i.$$
The simple blossom inequalities for $P_M$ map to a new and non-trivial family of valid inequalities for $P_M$, which we call projected blossom inequalities. It is known that blossom inequalities (20) can be derived by summing together the degree inequalities for all nodes in $H$ and the upper bounds for all edges in $T$, dividing the resulting inequality by two, and rounding down; i.e., they are “$\{0, \frac{1}{2}\}$-cuts” in the sense of Caprara & Fischetti [1]. Now, recall the perfect $b$-matching polytope $P_M^+$, defined in Section 4 and let us introduce the corresponding undirected graph $G^+ = (V^+, E^+)$ that has one edge for each variable $x, \bar{y}, z, \bar{z}$ and one node for each degree equation (13)-(15). We also define the sets $T = \{1, \ldots, k\}$, $U = \{k + 1, \ldots, 2k\}$ and $S = \{2k + 1, \ldots, 2k + |F|\}$, which index the equations (13), (14) and (15), respectively. (By construction, $T$, $U$ and $S$ form a partition of $V^+$.)

Note that any set $F_i \in F$ is associated with two equations in our IP formulation: one of the form (13), indexed by $i \in T$, and the other of the form (14), indexed by $(i + k) \in U$. Furthermore, any element $f \in F$ is associated with one degree equation of the form (15), while any line $\ell \in L$ is associated with two of them.

Now, for a given simple blossom inequality, let $\bar{T} \subseteq T$, $\bar{U} \subseteq U$ and $\bar{S} \subseteq S$ denote the index sets of the equations that are used in their derivation as a $\{0, \frac{1}{2}\}$-cut. (By construction, $T$, $U$ and $S$ form a partition of $V^+$.) We can now state the following lemma.

**Lemma 3** If a simple blossom inequality defines a facet of $P_M^+$, then the corresponding sets $\bar{T}$, $\bar{U}$ and $\bar{S}$ satisfy the following conditions:

1. $\sum_{i \in \bar{T}} u_i + \sum_{i \in \bar{U}} u_{i-k} + |\bar{S}|$ is odd.
2. $S = \{i + 2k : \exists \{i, j\} \in L \text{ such that } i, j \in \bigcup_{n \in \bar{T}} F(n)\}$
3. If $(i + k) \in \bar{U}$, then $j \in \bar{T}$ where $i \in \chi(j)$.

**Proof.**

1. If condition 1 does not hold, no rounding down occurs on the right-hand side.

2. Suppose condition 2 does not hold. Then there is some element $i \in F$ and some line $\{i, j\} \in L$ for which we are using the equation $x_i + \bar{y}_{ij} = 1$ in the derivation of the blossom inequality, yet for which the variable $x_i$ does not appear in any other equation that we are using. Now consider two cases:
   (i) $j+2k$ does not lie in $\bar{S}$. Then both $x_i$ and $\bar{y}_{ij}$ will receive a coefficient of zero in the blossom inequality. Then, the blossom inequality will be either unchanged or strengthened if we remove $i + 2k$ from $\bar{S}$.
   (ii) $j+2k$ does lie in $\bar{S}$. Then the net contribution of the two equations, before dividing by two and rounding down, is $x_i + x_j + 2\bar{y}_{ij} \leq 2$. After
dividing by two and rounding down, the left-hand side coefficient of $x_i$ will be zero. So the best possible scenario is that we have added $x_j + \overline{y}_{ij} \leq 1$ to the blossom inequality. There is no point doing this, since $x_j + \overline{y}_{ij} = 1$.

3. Suppose condition 3 does not hold. Then there is some degree equation $i + k$ in $\overline{U}$ for which the degree equation $j \in T$, corresponding to the unique parent of $i$, is not included in the derivation of the blossom inequality. We observe that the variable $\overline{z}_i$ appears in the degree equations $i + k$ and $i \in T$, while the variable $z_i$ appears in $i + k$ and its parent $j \in T$. Again, we consider two cases:

(i) the degree equation $i$ is not in $\overline{T}$. Then, we are using the equation $z_i + \overline{z}_i = u_i$ in the derivation of the blossom inequality, even though neither $z_i$ nor $\overline{z}_i$ appear in any other equation that is used. After dividing by two and rounding down, the left-hand side coefficients of both $z_i$ and $\overline{z}_i$ will be zero. Then, we could get a stronger inequality by removing $i$ from $\overline{T}$.

(ii) $i$ does lie in $\overline{T}$. Then, the net contribution of the equations $i + k$ and $i$, before dividing by two and rounding down, is

$$x(F(i)) + \sum_{j \in \chi(i)} z_j + 2\overline{z}_i + z_i \leq 2u_i.$$  

Thus, after dividing by two and rounding down, the contribution to the left-hand side of the blossom inequality is at most $x(F(i)) + \sum_{j \in \chi(i)} z_j + \overline{z}_i$, while the contribution to the right-hand side is exactly $u_i$. Since $x(F(i)) + \sum_{j \in \chi(i)} z_j + \overline{z}_i = u_i$, we could get a stronger blossom inequality by removing $i + k$ from $\overline{U}$ and $i$ from $\overline{T}$. □

For given sets $\overline{S}$, $\overline{T}$ and $\overline{U}$ that respect the conditions of Lemma 3, we can derive a simple blossom inequality for $P^+_M$. Before we present the general form of such an inequality, it is helpful to introduce some further index sets. We let $\overline{S}'$ denote a subset of $\overline{S}$ such that for any $\{i, j\} \in L$, the degree equation $i + k$ and $i \in T$ have either $(i + k) \in S$ or $(j + k) \in S'$, but not both. In addition, we define the following two index sets associated with condition 3 of Lemma 3:

$$Z = \{i \in \{1, \ldots, k\} : i \not\in \overline{T}, (i + k) \in \overline{U}, i \in \chi(j) \text{ for some } j \in \overline{T}\}$$

$$\tilde{Z} = \{i \in \{1, \ldots, k\} : i \in \overline{T}, (i + k) \in \overline{U}, i \in \chi(j) \text{ for some } j \in \overline{T}\}$$

Note that the sets $Z$ and $\tilde{Z}$ correspond to the possible scenarios for membership in $\overline{T}$ and $\overline{U}$. Using this notation we obtain the following general
form of the simple blossom inequalities for $P_M$:

$$x(\delta^+(\tilde{T})) + y(\delta^+(\tilde{S}')) + z(\delta^+(Z \cup \tilde{Z})) + \bar{z}(\delta^+(\tilde{Z})) \leq \left[ \sum_{i \in \tilde{T}} u_i + \sum_{i \in \tilde{U}} u_{i-k} + |\tilde{S}| \right].$$  \hspace{1cm} (21)

We can now state the main result of this section.

**Theorem 1** The laminar matroid parity polytope $P_M$ is completely described by the bound constraints $0 \leq y_\ell \leq 1$ for all $\ell \in L$, the projected rank inequalities (6), and the set of \{0, $\frac{1}{2}$\}-cuts obtained from them.

**Proof.** It suffices to show that every non-dominated simple blossom inequality for $P_M$ becomes a \{0, $\frac{1}{2}$\}-cut for $P_M$ when projected into $\mathbb{R}^{|L|}$. So, consider a simple blossom inequality of the form (21). First, we use (16) and (17) to project out the $\bar{y}$ variables. Note that $2(\delta^+(\tilde{S}')) = 2|\tilde{S}'| = |\tilde{S}|$, and therefore subtracting $|\tilde{S}|$ from the right hand side of (21) does not change its parity. Thus, the inequality (21) is equivalent to:

$$x(\delta^+(\tilde{T})) - x(\delta^+(\tilde{S}')) + z(\delta^+(Z \cup \tilde{Z})) \leq \left[ \sum_{i \in \tilde{T}} u_i + \sum_{i \in \tilde{U}} u_{i-k} \right].$$

Now, condition 2 of Lemma 3 implies that $x(\delta^+(\tilde{T})) = x(\delta^+(\tilde{S}'))$, and therefore the inequality reduces to:

$$z(\delta^+(Z \cup \tilde{Z})) + \bar{z}(\delta^+(\tilde{Z})) \leq \left[ \sum_{i \in \tilde{T}} u_i + \sum_{i \in \tilde{U}} u_{i-k} \right].$$

Next, we eliminate the $\bar{z}$ variables, using (14), to obtain:

$$z(\delta^+(Z)) \leq \left[ \sum_{i \in \tilde{T}} u_i + \sum_{i \in \tilde{U}} u_{i-k} \right] - \sum_{i \in Z} u_i. \hspace{1cm} (22)$$

Now we simplify the right-hand side. Note that if $i \in \tilde{Z}$, then $i \in \tilde{T}$ and $i + k \in \tilde{U}$. Thus

$$\sum_{i \in \tilde{T}} u_i + \sum_{i \in \tilde{U}} u_{i-k} - 2\sum_{i \in Z} u_i = \sum_{i \in \tilde{T} \setminus \tilde{Z}} u_i + \sum_{i \in \tilde{U} \setminus \tilde{Z}} u_{i-k} = \sum_{i \in \tilde{Z}} u_i.$$

We can therefore re-write the inequality (22) in the following simplified form:

$$z(\delta^+(Z)) \leq \left[ \sum_{i \in \tilde{Z}} u_i \right]. \hspace{1cm} (23)$$
Finally, we will project out the $z$ variables. To this end, we define the set family $Q = \{ F_i \in \mathcal{F} : i \in \mathbb{Z} \}$ and let
\[
\alpha_\ell = \frac{1}{2} \left( \sum_{i:F_i \in Q} |\ell \cap F_i| \right), \quad \ell \in \mathcal{L},
\]
\[
\beta = \left[ \frac{\sum_{i:F_i \in Q} r_M(F_i)}{2} \right].
\]

Using equation (18), we project the inequality (23) into $\mathbb{R}^L$ to yield:
\[
\sum_{\ell \in \mathcal{L}} \alpha_\ell y_\ell \leq \beta.
\]

Inequality (24) is a $\{0, \frac{1}{2}\}$-cut for $\mathcal{P}_M$, derived from the projected rank inequalities (6) for the members of $Q$. \hfill \square

This yields the following corollary.

**Corollary 2** The laminar matroid parity polytope has Chvátal rank 1.

Recall once more that the $c$-capacitated $b$-matching polytope is completely described by (1)–(3). Thus, all its facet-defining inequalities have binary left-hand-side coefficients. On the other hand, the following statements imply a more elaborate structure for the laminar matroid parity polytope.

**Proposition 2** A projected rank inequality that defines a facet of the laminar matroid parity polytope may have ternary coefficients.

**Proposition 3** A projected blossom inequality that defines a facet of the laminar matroid parity polytope may have non-ternary coefficients.

**Proposition 4** The coefficient of a variable $y_\ell$, $\ell \in \mathcal{L}$, in a non-dominated projected blossom inequality for the laminar matroid parity polytope is at most $\left[ \frac{\sum_{i=1}^{k} |\ell \cap F_i|}{2} \right]$.

**Proof.** Any non-dominated blossom inequality takes the form (24), i.e., a $\{0, \frac{1}{2}\}$-cut, hence the result. \hfill \square

An $O(|F|)$ bound on the coefficients of a projected blossom inequality follows easily, while Example 1 shows that the bound of Proposition 4 is attainable.

**Example 1** Assume a laminar matroid defined over the ground set $F = \{1, \ldots, 20\}$, the set family $\mathcal{F} = \{F_1, \ldots, F_5\}$ and the line set
\[
\mathcal{L} = \{ \{1,11\}, \{2,12\}, \{3,13\}, \{4,14\}, \{5,15\}, \{6,16\}, \{7,17\}, \{8,18\}, \{9,19\}, \{10,20\} \}.
\]
The members of $F$ along with their upper bounds are given in Table 1. Observe that each of the three columns of this table corresponds to a chain in $F$. The matroid parity polytope is described by five projected rank inequalities.

The facet-defining inequality (25) has non-binary left-hand side coefficients. Taking the $\{0, 1/2\}$-cut of (25)-(29) yields the facet-defining inequality:

$$y_{1,11} + y_{2,12} + y_{3,13} + y_{4,14} + y_{5,15} + y_{6,16} + 2y_{7,17} + 2y_{8,18} + 3y_{9,19} + y_{10,20} \leq 7,$$

that is a projected blossom having non-ternary left-hand side coefficients.

6 Elementary closure

Now we return to the matroid parity problem for general (i.e., not necessarily laminar) matroids. Let $P_1$ be the elementary closure of the feasible region of the LP relaxation of the matroid parity problem. That is, let $P_1$ be the polytope defined by the intersection of bound and projected rank inequalities, together with the set of Chvátal-Gomory (C-G) cuts derived from them. $P_1$ is a natural polyhedral outer-approximation of $P_M$. An apparently weaker such approximation is derived when the multipliers applied for the calculation of the C-G cut can only take the value 0 or 1/2. We follow [1] in calling the corresponding polytope $P_{1/2}$. We establish that $P_1 = P_{1/2}$ using the fact that the laminar matroid parity polytope is fully described by its $\{0, 1/2\}$-cuts. For conciseness, let us call a set of inequalities (6) laminar if their supports form a laminar set.
Theorem 2 A C-G cut is a facet of $P_1$ only if it is obtainable by a laminar set of rank inequalities.

Proof. Let $R$ be the set of inequalities defining $P_M$. A facet of $P_1$ is defined by a C-G cut, hence let $\lambda \in [0, 1)^{|R|}$ be the corresponding multipliers and $R' = \{ i \in R : \lambda_i > 0 \}$. Assuming to the contrary that the set of rank inequalities indexed by $R'$ is not laminar, there is a pair of projected rank inequalities $i, j \in R'$ whose supports $S_i, S_j$ cross, i.e., $S_i \setminus S_j \neq \emptyset \neq S_j \setminus S_i$. The contribution of these two inequalities in the C-G cut, before rounding down, is the sum of

$$\lambda_i (\sum_{\ell \in L} |S_i \cap \ell| y_{\ell}) \leq \lambda_i r_M(S_i) \quad \text{and} \quad \lambda_j (\sum_{\ell \in L} |S_j \cap \ell| y_{\ell}) \leq \lambda_j r_M(S_j).$$

(31)

Assume without loss of generality that $\lambda_i \geq \lambda_j > 0$ and observe that (31) can alternatively be written as the sum of

$$(\lambda_i - \lambda_j) (\sum_{\ell \in L} |S_i \cap \ell| y_{\ell}) \leq (\lambda_i - \lambda_j) r_M(S_i) \quad \text{and} \quad \lambda_j (\sum_{\ell \in L} |S_i \cap \ell| y_{\ell}) \leq \lambda_j r_M(S_i).$$

(33)

(34)

Consider now the two rank inequalities derived by ‘uncrossing’ the sets $S_i$ and $S_j$, i.e., the inequalities

$$\sum_{\ell \in L} |(S_i \cup S_j) \cap \ell| y_{\ell} \leq r_M(S_i \cup S_j) \quad \text{and} \quad \sum_{\ell \in L} |(S_i \cap S_j) \cap \ell| y_{\ell} \leq r_M(S_i \cap S_j).$$

(35)

(36)

It becomes easy to show that, for each $\ell \in L$,

$$|(S_i \cup S_j) \cap \ell| + |(S_i \cap S_j) \cap \ell| = |S_i \cap \ell| + |S_j \cap \ell|,$$

(37)

by noticing the following partitions of $S_i \cap \ell$ and $(S_i \cap S_j) \cap \ell$:

$$S_i \cap \ell = ((S_i \setminus S_j) \cap \ell) \cup ((S_i \cap S_j) \cap \ell),$$

$$S_i \cup S_j \cap \ell = ((S_i \setminus S_j) \cap \ell) \cup ((S_i \cap S_j) \cap \ell) \cup ((S_j \setminus S_i) \cap \ell).$$

Also, the submodularity of the rank function $r_M$ implies

$$r_M(S_i \cup S_j) + r_M(S_i \cap S_j) \leq r_M(S_i) + r_M(S_j).$$

(38)

But then, the sum (35)-(36), each multiplied by $\lambda_j$, plus (33) provides an inequality with the same left-hand side with the sum of (31)-(32) (because of (37)) and a no-larger right-hand side (because of (38)). This suggests a substitution strategy for strengthening the facet-defining C-G cut, i.e., the substitution in $R'$ of (31)-(32) with (35)-(36), each multiplied by $\lambda_j$, plus (33). By repeating this uncrossing argument for any pair of crossing inequalities in $R'$, one can substitute every non-laminar subset of rank inequalities in $R$ with a laminar one and derive a C-G cut that is at least as strong. □
Corollary 3 For any matroid M and any set of lines \( \mathcal{L} \), \( \mathcal{P}_{1/2} = \mathcal{P}_1 \).

Proof. Theorem 2 implies that an inequality \( \alpha y \leq \beta \) that is facet-defining for \( \mathcal{P}_1 \) can be derived as a C-G cut from a laminar set of projected rank inequalities. Let \( R' \subset R \) be this laminar set and \( A'y \leq b' \) be the system of linear inequalities that it defines. Then, \( \{ y \in \{0,1\}^{|E|} : A'y \leq b' \} \) is a laminar matroid parity polytope. By Theorem 1, this polytope is described by the bounds, projected rank inequalities and \( \{0, \frac{1}{2}\} \)-cuts. Hence, \( \alpha y \leq \beta \) is also a \( \{0, \frac{1}{2}\} \)-cut. □

Since matroid parity is equivalent to matroid matching, recall that the matroid matching polytope is not equal to the elementary closure of the linear relaxation examined in [15], despite the fact that this relaxation admits some structural properties of the matching polytope like half-integral vertices.

References


