Modelling Discontinuous Behaviour in Ecology

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Catastrophe Points

Modelling and predicting abrupt changes in ecosystems is an important problem for environmental preservation. Many ecological processes can be modelled as non-linear dynamical systems aiming to minimise a potential $V(y;c)$:

$$\frac{\partial y}{\partial t} = -\frac{\partial V(y;c)}{\partial y}, \quad c \in \mathcal{R}^n, y \in \mathcal{R}^m, \tag{1}$$

where y is the response variable(s) being modelled and c is the control variable(s). Catastrophe points are degenerate equilibria which occur when the Hessian matrix $\partial^2 V(y,c)/\partial y_i y_j$ has zero eigenvalues. At these points, small changes in the control variables can lead to sudden discontinuous behaviour.

Example: Modelling Hysteresis

When it is not possible to predict the state of a system without knowledge of its past evolution, this is called hysteresis. Consider a response variable y with a control variable c with the following governing equation:

$$V(y;c) = -cy - \frac{3}{2}y^2 + \frac{1}{4}y^4,$$
$$\frac{\partial y}{\partial t} = -\frac{\partial V}{\partial y} = c + 3y - y^3. \tag{2}$$

For any fixed c, y will evolve to a state where $\partial y/\partial t = 0$. These solutions are plotted in Figure 1; the stable equilibria of $V(y;c)$ with respect to y are black and the unstable equilibria are red. Starting in the top right, decreasing c causes a gradual decline in y until we reach catastrophe point A. Then, if c is decreased more, y will decline abruptly to the lower stable equilibria. After this, increasing c will not cause y to recover until catastrophe point B.

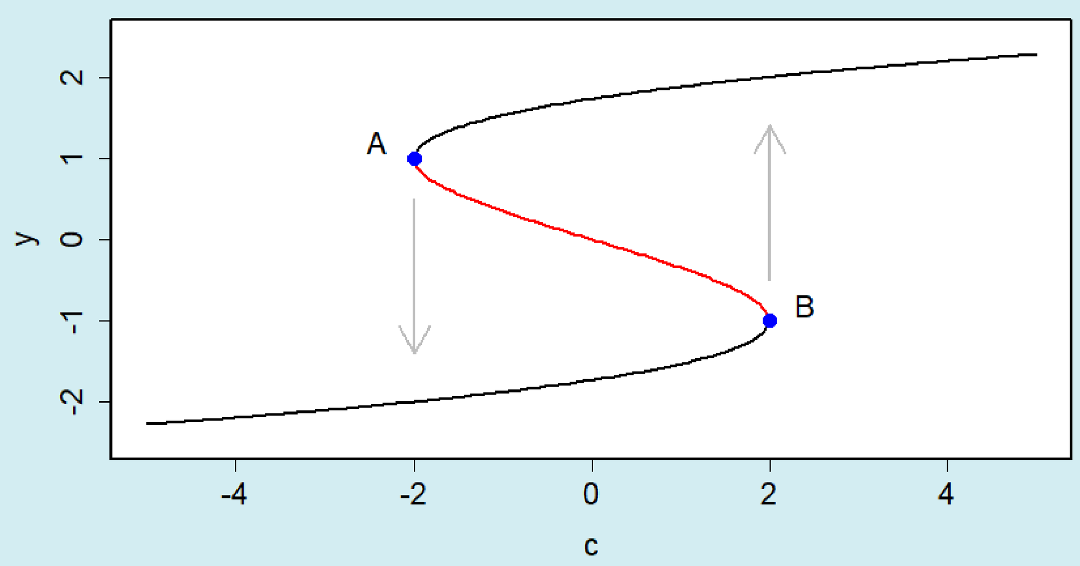


Figure 1. The equilibria of Equation 2 with stable states in black and unstable states in red, demonstrating hysteresis for $-2 \le c \le 2$. A and B are catastrophe points.

The Cusp Catastrophe Model

The simplest catastrophe model with discontinuities is the cusp, which has two control variables and one response variable. It is governed by:

$$\frac{\partial y}{\partial t} = \alpha + \beta y - y^3. \tag{3}$$

The equilibria surface is plotted in Figure 2. The values of α and β where there are two stable states is shaded; here discontinuous effects may occur.

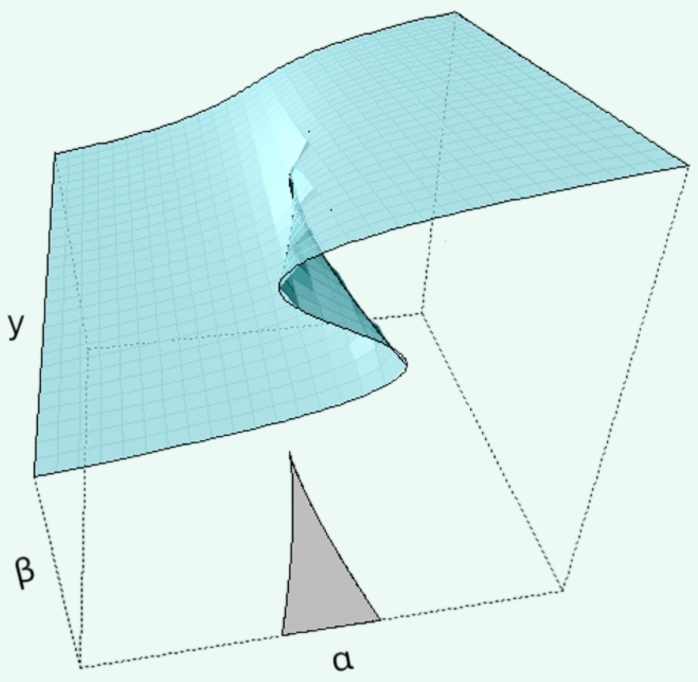


Figure 2. The cusp catastrophe surface

Parameter estimation is difficult, as most techniques rely on minimising the differences between model predictions and the data, but catastrophe models predict multiple possible values when there are multiple stable states.

Stochastic Catastrophe Theory

Ecological systems are inherently random, but Equation 1 describes a deterministic system. To fix this, Cobb and Watson (1980) developed stochastic catastrophe theory. Under this, the system is governed by the stochastic differential equation:

$$dy = -\frac{\partial V(y;c)}{\partial y}dt + \sigma(y;c)dW_t, \tag{4}$$

where $\sigma^2(y;c)$ is the infinitesimal variance and W_t is the Weiner process. The probability density function $f(y,t;c)$ will evolve to a stationary distribution $f^*(y;c)$ such that $\partial f^*/\partial t = 0$. If $\sigma^2(y;c)$ is constant, then

$$\log f^*(y;c) = \log A - \frac{2V(y;c)}{\epsilon}, \tag{5}$$

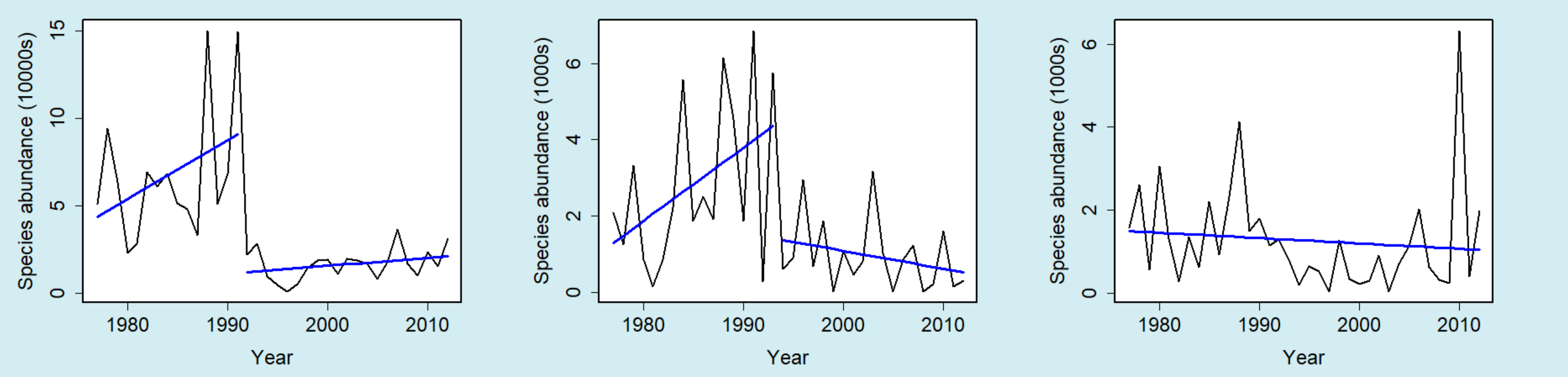
where A is a constant. Thus local maxima of f^* and local minima of V occur at the same values of y: the highest probability densities occur at the steady states which deterministic catastrophe theory predicts the system is in. Once probability densities are defined, maximum likelihood methods are available to estimate parameters and judge model fit.

If $\sigma^2(y;c)$ is nonconstant, then the location and number of local maxima of f^* may not correspond to local minima of V. Wagenmakers et al. (2005) propose performing inference on an invariant function instead.

Example: Wintering Waterfowl and Changepoints Links

If a stochastic catastrophe model is fitted to time series data, then catastrophe points will look like changepoints. Almaraz and Green (2024) show that the eruption of Mt. Pinatubo led to a decline in the abundance of wintering waterfowl in 1992, by fitting a stochastic catastrophe model to abundance data of 10 wintering waterfowl species from 1978 to 2013.

To compare the effectiveness of a changepoint analysis to their approach, I used the Pruned Exact Linear Time (PELT) algorithm (Killick et al., 2012) to look for changes in the regression structure of their data for different species.



(a) Common Teal (b) Gadwell (c) Shelduck

Figure 3. Fitting piecewise linear regression to the abundance data of wintering waterfowl.

- **Common Teal**: PELT found a single changepoint in 1992, agreeing with the stochastic cusp catastrophe model.
- **Gadwell**: PELT found a changepoint in the 1990s but it was after 1992.
- **Shelduck**: PELT was ineffective due to significant recovery after 1992.

This suggests changepoint analysis may be useful to use alongside catastrophe theory to find catastrophe points. However, changepoint analysis lacks predictive power and causal insight, as it only uses a statistical model of the data, whereas catastrophe theory introduces a mechanistic model.

Open Research and References

- Extend stochastic catastrophe theory for settings with autocorrelated noise by replacing dW_t with a different process.
- Further compare changepoint analysis and stochastic catastrophe models.

[Almaraz and Green, 2024] Almaraz, P. and Green, A (2024). Catastrophic bifurcation in the dynamics of a threatened bird community triggered by planetary-scale environmental perturbation. *Biological Conservation*, 291: Article 110466

[Cobb and Watson, 1980] Cobb, L. and Watson, B. (1980). Statistical catastrophe theory: An overview. *Mathematical Modelling*, 1(4):311-317

[Killick et al., 2012] Killick, R., Fearnhead, P., and Eckley, I. A. (2012) Optimal Detection of Changepoints With a Linear Computational Cost. *Journal of the American Statistical Association*, 107(500):1590-1598.

[Wagenmakers et al., 2005] Wagenmakers, E. J. et al. (2005) Transformation invariant stochastic catastrophe theory. *Physica D: Nonlinear Phenomena*, 211(3):263-276

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