Irrationality and Transcendence with Rational Approximations

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Proving Irrationality with Rational Approximations

One method for showing a number, α, is irrational is by assuming it is rational a/b and defining a series of rational approximations to α. We then define a series of integers c_n which violate the "Fundamental Principle of Number Theory": an impossible integer, bounded between 0 and 1.

For instance, consider rational approximations to e by truncating the power series of e:

$$\frac{p_n}{n!} = \sum_{i=1}^{n} \frac{1}{i!} = 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}.$$

Suppose e= a/b. We create an integer for $n \geq b$ by defining:

$$c_n = n! \left| \frac{a}{b} - \frac{p_n}{n!} \right| = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots = \frac{1}{n}.$$

Our integer c_n will never be 0, and is bounded between 0 and $1/n$, which will clearly be less than 1 for large n. e is therefore irrational!

Beukers' Method for Irrationality of $\zeta(3)$

Beukers developed a more sophisticated method for proving the irrationality of real numbers.

1. Define a non-zero function f(x) such that the series of integrals:
$$I_n = \int_0^1 P_n(x) f(x) dx = A_n + B_n \alpha,$$
where A_n, B_n are rational numbers, and P_n are the Legendre polynomials.
2. After performing integration by parts n times on I_n, we get the equation
$$I_n = \frac{(-1)^n}{n!} \int_0^1 x^n (1-x)^n \frac{d^n f}{dx^n} dx.$$
We use this to find an upper bound for I_n in the form CM^n, such that M is less than 1.
3. Now we follow a similar method to above. Assume $\alpha = a/b$, so α is rational. Then
$$0 < \left| A_n + B_n \frac{a}{b} \right| \leq CM^n$$
Multiplying through by any denominators, we get an integer between 0 and some upper bound, which is less than 1 for large n.

$\zeta(s)$ is defined as $\sum_{n=1}^{\infty} n^{-s}$. For even s, $\zeta(s)$ is known to be rational functions of π and thus transcendental, but less is known about $\zeta(s)$ for odd s, though $\zeta(3)$ is known to be irrational. Beukers' refined Apery's proof of this. He used

$$f(x) = \int_0^1 \frac{P_n(y)}{1-xy} \log(xy)\, dx dy,$$

and showed that

$$I_n = \frac{A_n}{d_n^3} + B_n \zeta(3),$$

where d_n is the lowest common multiple of the first n natural numbers, and A_n, B_n are **integers**. Assuming $\zeta(3) = a/b$ is rational, this leads us to the inequality

$$0 < |A_n d_n^3 a + B_n b| < d_n^3 a M^n < a * 0.6^n.$$

where M is calculated to be around 0.03. This thus proves $\zeta(3)$ is irrational, as for large enough n, we have an integer bounded between 0 and 1. However, it is an open question as to whether $\zeta(3)$ is transcendental.

The method above fails to prove $\zeta(5)$ is irrational, because although we can find an integral $I_0 = \zeta(5)$, the family I_n are not combinations of rational numbers and $\zeta(5)$. However, Wadim Zudilin showed that one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ or $\zeta(11)$ must be irrational.

Approximation

How closely can we approximate a real number, α by rational numbers, p/q (where p, q are always coprime integers)? The rationals are infinitely dense on the line so if we don't limit the size of q, we can let p/q get arbitrarily close to α. But what if the distance between α and p/q also depends on q? Let us consider the equation:

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

1. **Case 1:** $\alpha = a/b$ **is rational**
There are only finitely many solutions. If $p/q \neq \alpha$,
$$\left| \alpha - \frac{p}{q} \right| = \left| \frac{a}{b} - \frac{p}{q} \right| = \left| \frac{aq - pb}{bq} \right| \geq \left| \frac{1}{bq} \right|,$$
since we have that $aq - bp$ is a non-zero integer. Therefore, we require $q < b$, but for each q, there will only be finite options for p. This realisation is why it is possible to prove α is irrational using rational approximations.
2. **Case 2:** α **is irrational**
Dirichet proved that in this case, there are infinite solutions. He used the pigeonhole principal, giving a name to an idea which had been around for millennia.

The Approximation Exponent

For any $\alpha \in \mathbb{R}$, define the approximation exponent $\mu = \mu(\alpha)$ as the smallest number so that for any $\delta > 0$

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{q^{\mu+\delta}}$$

has only finitely many solutions.

Dirichet's Theorem is therefore equivalent to saying $\mu(\alpha) \geq 2$ for any irrational α.

Number α	$\mu(\alpha)$
Rational a/b	1
Irrational algebraic number	2
π	$2 \leq \mu(\pi) \leq 7.10320\ldots$
e	2
$\log 2$	$2 \leq \mu(\log 2) \leq 3.57455\ldots$
$\zeta(3)$	$2 \leq \mu(\zeta(3)) \leq 5.51389\ldots$
Champernowne constant $= 0.1234567891011\ldots$	10
Louiville Numbers	∞

Liouville's Theorem (1844)

If α is a real algebraic number with degree d, then for any rational $\frac{p}{q}$,

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c(\alpha)}{q^d}$$

where $c(\alpha)$ is some real constant that depends only on α.

Discovering Transcendental Numbers

Liouville's Theorem can be proved using the minimal polynomial for α, F(x), and considering $F(p/q)$ and using the mean value theorem. A simple corollary to it is that $\mu(\alpha) \leq d$ for any algebraic α of degree d. Combined with Dirichet's Theorem, this demonstrates that quadratic irrationals have approximation exponent 2.

Over the next 100 years, mathematicians such as Thue, Siegel and Roth worked to reduce the bound on $\mu(\alpha)$ for algebraic numbers, leading to Roth being awarded the Fields Medal in 1958 for showing all algebraic numbers have approximation exponent 2. The proof involves constructing multivariate auxiliary polynomials depending on δ, and finding a contradiction using the large number of rational approximations of α.

Liouville has also given us a way to construct transcendental numbers! We want numbers that can be approximated by rationals extremely well, so that $\mu(\alpha) = \infty$. Then by Liouville theorem, it is impossible for α to be the root of any polynomial of degree $d < \infty$. For example:

$$\alpha = \sum_{n=1}^{\infty} 10^{-n^n}.$$

The approximations found by taking the first N terms in the series are extremely good, because many 0s follow each 1 in the decimal expansion of α, which allows us to prove $\mu(\alpha) = \infty$. Numbers of this type are called **Liouville numbers**. However, most transcendental numbers are not Liouville numbers. For example, e and π have finite approximation exponent.

Transcendence of e

We can also use rational approximation to prove the transcendence of e in the following way:

1. Approximate e^t for $t \geq 1$ by rationals by writing:
$$e^t = \frac{M_t + \epsilon_t}{M},$$
where M, M_t are integers and the error term, ϵ_t, is small. For p a large prime, we choose:
$$M = \int_0^\infty x^{p-1} e^{-x} \frac{((x-1)(x-2)\ldots(x-n))^p}{(p-1)!} dx,$$
$$M_t = e^t \int_t^\infty x^{p-1} e^{-x} \frac{((x-1)(x-2)\ldots(x-n))^p}{(p-1)!} dx,$$
$$\epsilon_t = e^t \int_0^t x^{p-1} e^{-x} \frac{((x-1)(x-2)\ldots(x-n))^p}{(p-1)!} dx.$$
2. Assume e is algebraic, so it is the root of a polynomial P(x) with integer coefficients, a_t. $\sum_{t=0}^{n} a_t e^t = 0$, with $n \geq 1$, $a_0, a_n \neq 0$. Substituting our approximation in,
$$0 = M a_0 + \sum_{t=1}^{n} a_t M_t + \sum_{t=1}^{n} a_t \epsilon_t$$
3. We show that $M a_0 + \sum_{t=1}^{n} a_t M_t$ is a non-zero integer.
4. Then we show ϵ_t is small enough that $|\sum_{t=1}^{n} a_t \epsilon_t| < 1$.
This proves e is transcendental since the sum of a non-zero integer and a number with absolute value less than 1 cannot be zero. $M a_0 + \sum_{t=1}^{n} a_t M_t$ is an impossible integer. A similar method proves π is transcendental.

References

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