Proving The Irrationality Of Real Numbers: 3 Methods from Hermite, Niven and Beukers

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Method for Proving Irrationality with Rational Approximations

- Assume a number, α is rational a/b.
- Define a sequence of rational approximations to α
- Define a sequence of integers c_n, using the absolute difference between α and the rational approximations to α.
- Show that for large enough n, these "integers" are between 0 and 1.

Fractions

Throughout this presentation, I will write fractions a/b, and it is assumed that a, b are coprime integers, and $b \neq 0$.

Example: Irrationality of e

- Consider rational approximations to e:

$$\frac{p_n}{n!} = \sum_{i=1}^{n} \frac{1}{i!} = 1 + \frac{1}{2!} + \dots + \frac{1}{n!}.$$

- Assume $e = a/b$. We create an integer for $n \geq b$ by defining:

$$c_n = n! \left| \frac{a}{b} - \frac{p_n}{n!} \right| = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \dots$$
$$< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots = \frac{1}{n}.$$

Our integer c_n will never be 0, and is bounded between 0 and $1/n$. e is therefore irrational!

Proof for the Irrationality of π: Defining Polynomials

Assume $\pi = a/b$, and define the polynomial sequences:

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$
$$G(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - ... + (-1)^n f^{(2n)}(x).$$

Properties of $f(x)$

1. $f(x) = f(a/b - x)$ by substitution.
2. $f(0)$ and $f^{(i)}(0)$ are integers since the lowest degree of x in f(x) is n, so differentiating n times will produce a multiple of $n!$
3. Therefore, $f(a/b) = f(\pi)$, $f^{(i)}(\pi)$ are also integers.
4. $f(x) = G(x) + G''(x)$ since $f^{(2n+2)}(x) = 0$.

Proof for the Irrationality of π: An Integer

Using the product rule:

$$\frac{d}{dx}[G'(x)\sin x - G(x)\cos(x)] = G''(x)\sin x + G(x)\sin x$$
$$= f(x)\sin x$$

by property 4. Thus, we find

$$\int_0^\pi f(x)\sin x dx = [G'(x)\sin x - G(x)\cos(x)]_0^\pi$$
$$= G(\pi) + G(0).$$

This is an integer, as $f^{(i)}(\pi) = f^{(i)}(0) = 0$.

Proof for the Irrationality of π: Bounding the Integer

We can bound the integral:

$$0 < \int_0^{\pi} f(x)\sin x dx = \int_0^{\pi} \frac{x^n(a-bx)^n}{n!}\sin x dx < \frac{\pi^n a^n}{n!}.$$

For large n, n! grows faster than any x^n, so the upper bound becomes arbitrarily small. Because we showed that this integral was an integer, π must be irrational!

The Zeta Function

Definition

The zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

for $Re(s) > 1$ and its analytic continuation.

We are only interested in the positive integers values of s.

- For even s, we know that $\zeta(s)$ is rational functions of π. For example, $\zeta(2) = \pi^2/6$.
- We don't know exact values of $\zeta(s)$ for odd s, and we only know that at least one of $\zeta(5), \zeta(7), \zeta(9)$, and $\zeta(11)$ is irrational.
- Apery showed that $\zeta(3)$ is irrational in 1978, and Beukers' simplified the proof a year later. It's still not known whether $\zeta(3)$ is transcendental.

Beukers' Method for the Irrationality of $\zeta(3)$

- To show α is irrational, define a non-zero function f(x) such that the sequence of integrals:

$$I_n = \int_0^1 P_n(x) f(x) dx = A_n + B_n \alpha,$$

where A_n, B_n are rational numbers, P_n are the Legendre polynomials, and $I_n \neq 0$.

Legendre Polynomials

Legendre polynomials $P_n(x)$ are a sequence of orthogonal polynomials of degree n, and are used here because we can easily perform integration by parts with them. The first few Legendre polynomials are:

$$P_0(x) = 1, P_1(x) = x$$
$$P_2(x) = 1/2(3x^2 + 1)$$
$$P_3(x) = 1/2(5x^3 - 3x).$$

Beukers' Method for the Irrationality of $\zeta(3)$

- After performing integration by parts n times on I_n, we get

$$I_n = \frac{(-1)^n}{n!}\int_0^1 x^n(1-x)^n \frac{d^n f}{dx^n} dx.$$

  We use this to find an upper bound for I_n in the form CM^n, such that M is between 0 and 1.

- Now we follow a similar method to before. Assume $\alpha = a/b$, so α is rational. Then

$$0 < \left| A_n + B_n \frac{a}{b} \right| \leq CM^n$$

  Multiplying through by any denominators, we get an integer between 0 and some upper bound, which is less than 1 for large n.

Irrationality of $\zeta(3)$

For this, we choose

$$f(x) = \int_0^1 \frac{P_n(y)}{1-xy}\log(xy)dy,$$

and define

$$J_{r,s} := \int_0^1 \int_0^1 \frac{x^r y^s \log xy}{1-xy} dxdy.$$

for $0 \leq r, s \leq n$.

The 3 Cases

- $r = s = 0$
- $r = s \neq 0$
- $r \neq s$

Example Case: r=s=0

$$J_{0,0} = \int_0^1 \int_0^1 \frac{\log(xy)}{1-xy} dxdy = 2\int_0^1 \int_0^1 \log(x) \left(\sum_{i=0}^{\infty} x^i y^i \right) dxdy$$
$$= 2\sum_{i=0}^{\infty} \left(\int_0^1 \log(x) x^i dx \int_0^1 y^i dy \right)$$

Integrating $\log(x)x^i$ by parts, gets

$$\int_0^1 \log(x) x^i dx = \left[\frac{\log(x) x^{i+1}}{i+1} \right]_0^1 - \frac{1}{i+1} \int_0^1 x^i dx = \frac{1}{(i+1)^2},$$

$$\text{so } J_{0,0} = 2\sum_{i=0}^{\infty} \frac{1}{(i+1)^3} = 2\zeta(3).$$

- Looking at the other cases we find that

$$J_{r,s} = \frac{A_n}{d_n^3} + B_n\zeta(3),$$

where A_n, B_n are **integers**, and d_n is the lowest common multiple of the first n natural numbers.

- Due to the linearity of integrals

$$I_n = \int_0^1 \int_0^1 \frac{P_n(y)P_n(x)}{1-xy} \log(xy)dydx,$$

is in the same form.

Bounding I_n

We evaluate I_n and after algebraic manipulation, can find

$$|I_n| = \int_0^1 \int_0^1 \int_0^1 \frac{[(x-x^2)(y-y^2)(z-z^2)]^n}{[(1-(1-z)x)(1-yz)]^{n+1}} dxdydz$$
$$\leq M^n \int_0^1 \frac{1}{((1-(1-z)x)(1-yz)} dxdydz = CM^n.$$

where M is the maximum of $\frac{(x-x^2)(y-y^2)(z-z^2)}{(1-(1-z)x)(1-yz)}$ for x, y, z all in the range [0,1].
M can be evaluated to be $17 - 12\sqrt{2} < 0.03$.

The Final Contradiction

We now have can form the inequality:

$$0 < |I_n| = \left|\frac{A_n}{d_n^3} + B_n\zeta(3)\right| < C * 0.03^n$$

Assuming that $\zeta(3) = a/b$ and multiplying through by denominators:

$$0 < \left|A_n b + B_n * a * d_n^3\right| < C * b * d_n^3 * 0.03^n.$$

Lemma (A Bound on d_n)

For large n, we have that $d_n < e^n$.

Using the lemma, we achieve that

$$0 < \left|A_n b + B_n * a * d_n^3\right| < C * b * (0.03e^3)^n < C * b * 0.6^n,$$

and for large enough n, we have a contradiction. $\zeta(3)$ is irrational!

References

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Frits Beukers (1979)
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