STOR608 Sprint 1: Optimisation Under Uncertainty

Group 2:
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- Due to the stochastic nature of the demands, they are treated as random variables.
- In order to carry out an optimisation, we treat the problem as a two-stage stochastic programme.

Network Planning with Random Demand

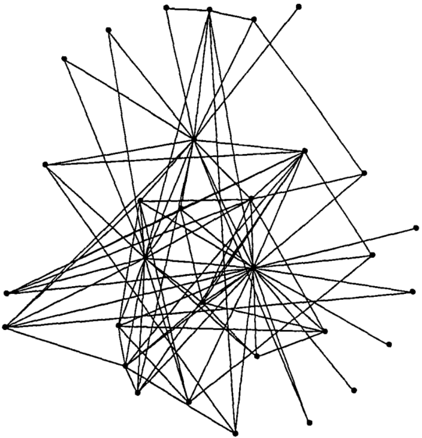

Figure: Graphical representation of the network planning problem Sen et al. (1994).

Notation

Symbol	Explanation
n	Number of links/edges in the graph
$x \in \mathbb{R}^n$	A vector denoting the amount of additional capacity for each link j
m	The number of point-to-point pairs served by the network
$\tilde{d} \in \mathbb{R}^m$	A r.v. that represents the demands for the m point-to-point pairs
b	Total capacity in the system

More Notation

Symbol	Explanation
$R(i)$	denote a set of routes (r) that can be used for connections associated with point-to-point pair i
$A_{ir} \in \mathbb{R}^n$	An indicator vector where $(A_{ir})_j$ is 1 if link j i is in route r (0 otherwise)
$e \in \mathbb{R}^n$	The current link capacities for each link j
f_{ir}	Number of connections associated with pair i using route $r \in R(i)$
s_i	Number of unserved requests for each pair i

Model Summary (Level 1)

Using this notation, we formulate the initial problem in the following way:

$$\min_{x} \mathbb{E}[h(x, \tilde{d})],$$

subject to

$$\sum_{j=1}^{n} x_j \leqslant b,$$

$$x \geqslant 0.$$

Model Summary (Level 2)

Within the first level of the model we must optimise the following linear programme:

$$h(x, d) = \min_{s,f} \sum_{i=1}^{m} s_i,$$

subject to

$$\sum_{i} \sum_{r \in R(i)} A_{ir} f_{ir} \leqslant x + e,$$

$$\sum_{r \in R(i)} f_{ir} + s_i = d_i,$$

$$f_{ir}, s_i \geqslant 0,$$

for all $i \in \{1, \ldots m\}$ and $r \in R(i)$.

Deterministic Equivalent Form

We can solve the previous problem as a single linear programme by converting the problem into Deterministic Equivalent form over T scenarios, each with probability $p^{(t)}$:

$$\min_{x,s,f} \left\{ \sum_{t=1}^{T} p^{(t)} \left(\sum_{i=1}^{m} s_i^{(t)} \right) \right\}$$

subject to

$$\sum_{i=1}^{m} \sum_{r \in R(i)} A_{ir} f_{ir}^{(t)} \leqslant x + e, \qquad \sum_{j=1}^{n} x_j \leqslant b,$$

$$\sum_{r \in R(i)} f_{ir}^{(t)} + s_i^{(t)} = d_i^{(t)}, \qquad x, f, s \geqslant 0,$$

for all $i \in \{1, \ldots, m\}$, $t \in \{1, \ldots, T\}$.

Python Implementation

Using the previous formulation allows the problem to be minimised using a single optimisation over the three variables x, s, f.

We chose $p^{(t)} = 1/T$ as we have no prior information on the likelihood of individual scenarios.

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Conditional on the number of scenarios T, the number of constraints is $T(n+m)+1$ (excluding non-negativity).

Increasing the number of scenarios reduces the variability within the model, at the cost of increased time complexity. The increase in time taken will limit our computational capabilities going forward.

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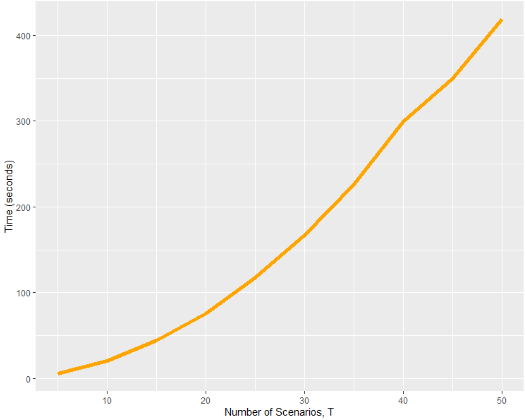

Figure: Optimisation times for varying T

Optimality Gap

We find some feasible value of x using a large value of T',

$$\tilde{x} = \left\{ x \in \mathcal{X} : \min_{x,s,f} \frac{1}{T'} \sum_{t=1}^{T'} h(x, \tilde{\xi}) \right\}.$$

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Then optimality gap is defined as

$$G_T^i = \frac{1}{T} \sum_{t=1}^{T} h(\tilde{x}, \xi_{it}) - \min_{x,s,f} \frac{1}{T} \sum_{t=1}^{T} h(x, \xi_{it}).$$

Optimality Gap Results

For sufficiently large T, we know that over k macroreplications

$$\frac{\bar{G}_T - \mathbb{E}(G_T)}{\frac{\sigma_T}{\sqrt{k}}} \sim t_{k-1},$$

where

$$\bar{G}_T = \frac{1}{k}\sum_{i=1}^{k} G_T^k,$$

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This means we can calculate a one-sided confidence interval for $\mathbb{E}\left(G_T\right)$:

$$\left(0, \bar{G}_T + t_{k-1,\alpha}\frac{\sigma_T}{\sqrt{k}}\right).$$

Optimality Gap Results

Using $k = 50$ macroreplications and $T = 20$ scenarios we obtain estimates $\bar{G}_T = 5.010$ and $\sigma_T = 0.908$.

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This means we have a 95% confidence interval for for $\mathbb{E}(G_T)$ of $(0, 5.225)$.

Optimality Gap Results

Plotting a histogram of the standardised outputs, we expect the results to follow the distribution of a t-distribution on $k - 1 = 49$ degrees of freedom.

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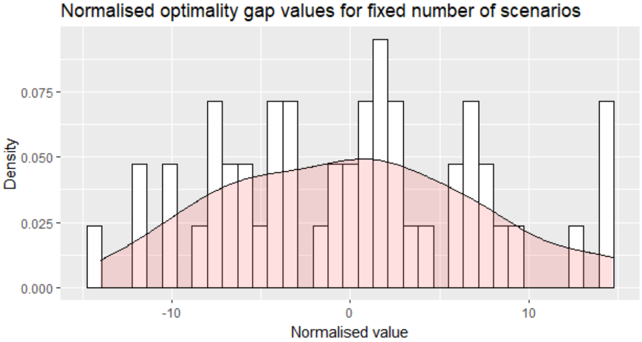

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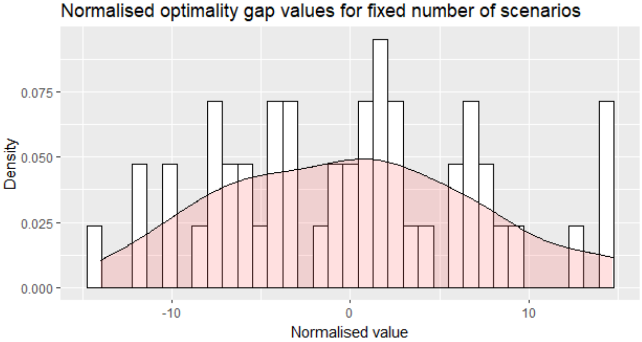

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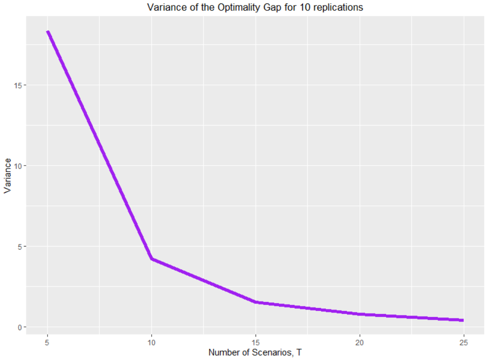

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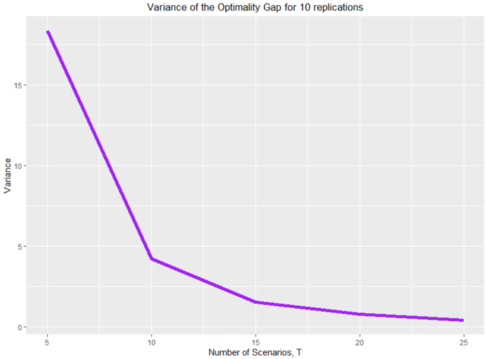

Therefore, as T increases, the estimate of the optimality gap tends to the true optimality gap.

Risk reduction methods

Possible methods to make the formulation more risk averse:

1. VaR.
2. CVaR
3. Minimise Variance.
4. Minimise Semivariance.
5. Minimise the squared losses.
6. More recent proposals e.g. most probable maximum size of risk events (MPMR) [Chen and Cheng (2022)]

β-CVaR

β-CVaR is defined as

$$\beta\text{-CVaR}(\theta) = \int_{\beta}^{1} F^{-1}(u)du$$

which can be thought of as the expected loss if the worst case threshold is every passed.

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From Acerbi and Tasche (2002) we know that β-CVaR is a coherent risk measure, whereas VaR is not.

Minimising β-CVaR

We previously estimated $\min \mathbb{E}(h(x,\xi))$ by

$$\min_{x,s,f} \left\{ \sum_{t=1}^{T} \frac{1}{T} h(x,\xi)^{(t)} \right\}.$$

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By a theorem from Rockafellar and Uryasev (2002) we have

$$\min \beta\text{-CVaR} = \min \left(\alpha + \frac{1}{1-\beta} \mathbb{E}\left(h(x,\xi) - \alpha \right)_+ \right),$$

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and so we can similarly estimate $\min \beta$-CVaR by

$$\min_{\alpha,x,s,f} \left\{ \alpha + \frac{1}{1-\beta} \sum_{t=1}^{T} \frac{1}{T} \left(h(x,\xi)^{(t)} - \alpha \right)_+ \right\}.$$

Auxiliary Variables

Python cannot compute the minimum of a maximum, so we introduce a new variable z, a vector of length T. We then implement

$$\min_{\alpha,x,s,f,z} \left\{ \alpha + \frac{1}{1-\beta} \sum_{t=1}^{T} \frac{1}{T} z^{(t)} \right\},$$

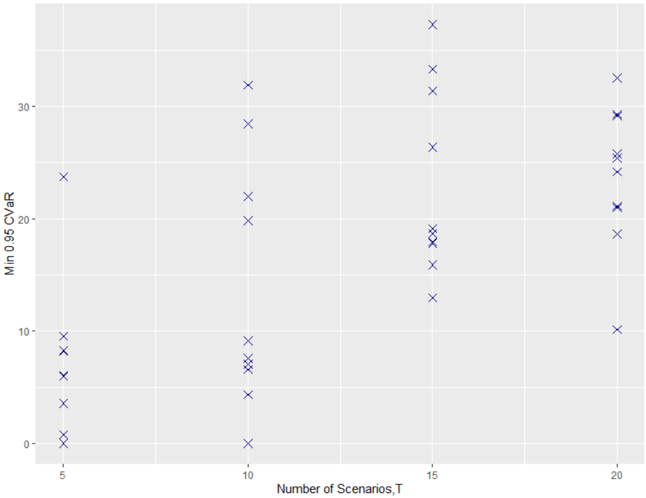
with additional constraints

$$z^{(t)} \geqslant h(x,\xi)^{(t)} - \alpha,$$
$$z^{(t)} \geqslant 0,$$

for all $t \in \{1, \ldots, T\}$.

Optimality Gap Replications

This shows the minimum 0.95-CVaR as the value of T changes.



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Thank You

Thank you for listening. Any questions?