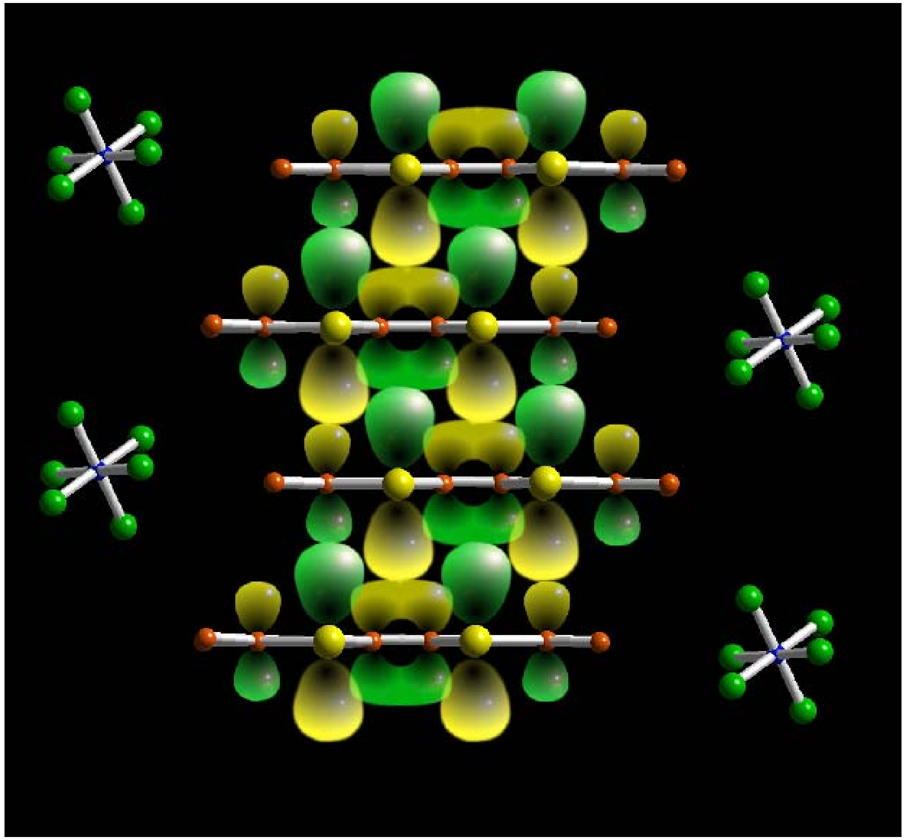
Bosonization: a primer

T. Giamarchi

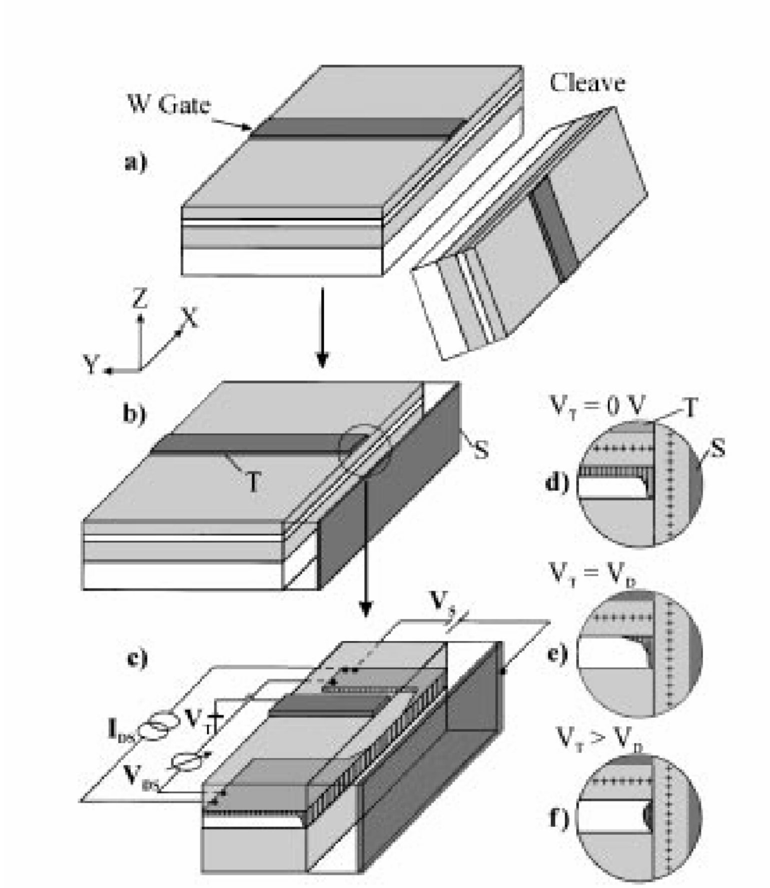
References on 1D Physics

- Emery, V. J. (1979). Highly conducting one dimensional solids, pp. 247. Plenum.
- Solyom, J. (1979). Adv. Phys., 28, 209.
- Schulz, H. J. (1995). *Les Houches LXI* pp. 533. Elsevier.
- Voit, J. (1995). Rep. Prog. Phys., 58, 977.
- von Delft, J. and Schoeller, H. (1998). Ann. Phys., 7, 225.
- Schonhammer, K. (2002). J. Phys. C, 14, 12783.
- Senechal, D. (2003). CRM Series in Mathematical Physics, Springer. cond-mat/9908262.
- Gogolin, A. O., Nersesyan, A. A., and Tsvelik, A. M. (1999). Bosonization and Strongly Correlated Systems. Cambridge University Press, Cambridge.
- Giamarchi, T. (2004) Quantum Physics in one dimension. Oxford University Press, Oxford.

1d Systems

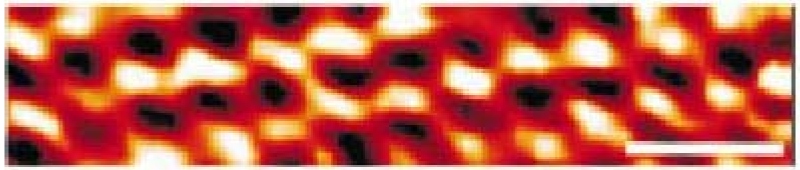


Organic conductors

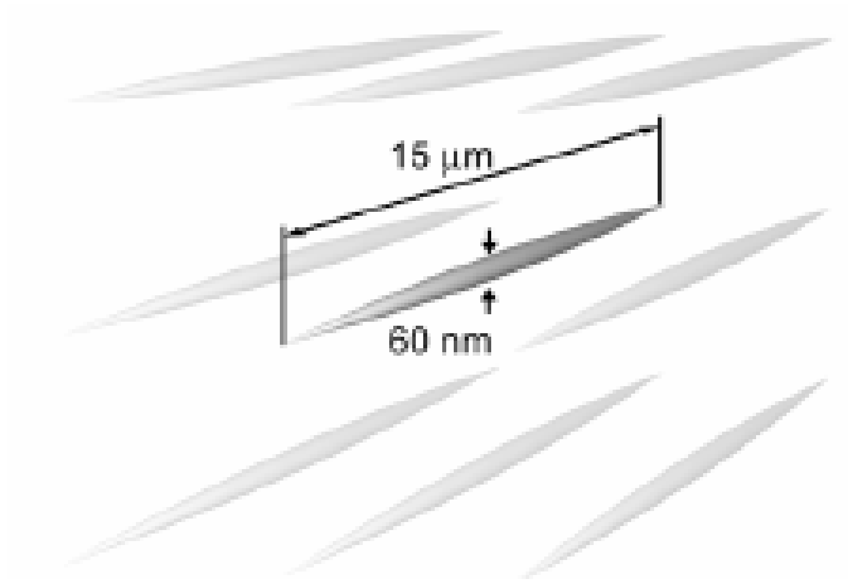


Quantum wires

Cold Bosons



Nanotubes



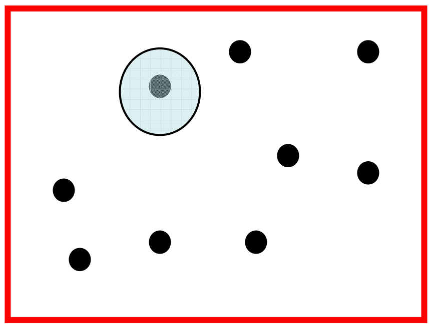
But also:
josephson junctions
ladders
Edge states …..

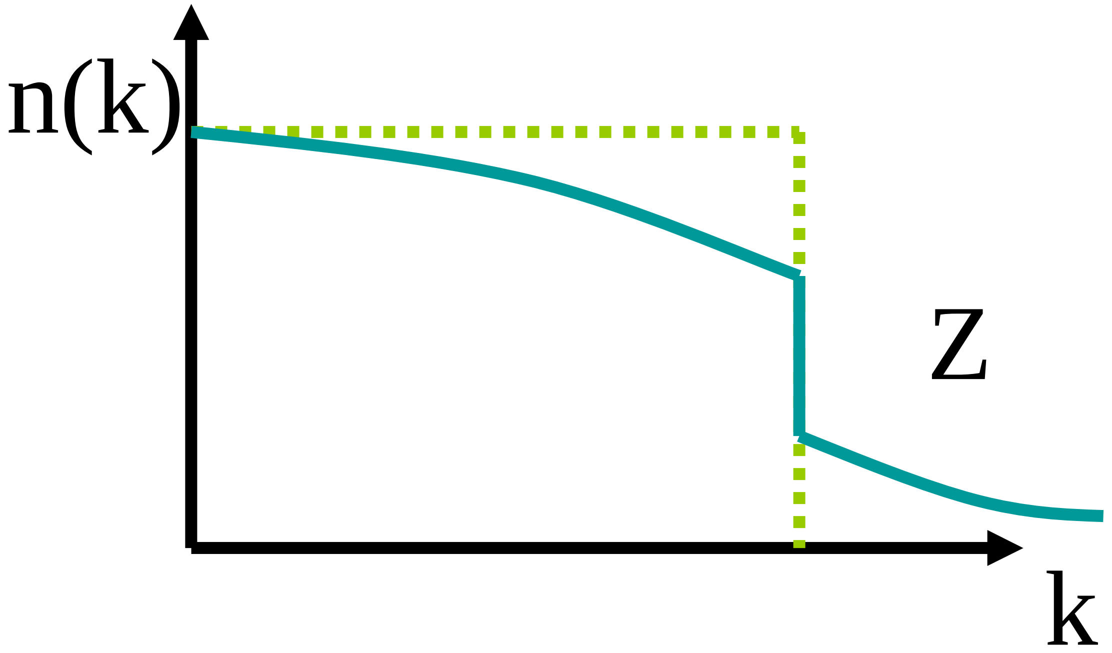
N_0 ~ 10 to 10^3 atoms

T. Stoferle *et al.* PRL **92** 130403 (2004)

Fermi liquid : crash course

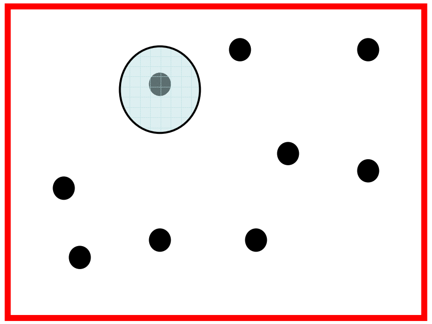
- Individual fermionic excitations exist (as for free electrons)

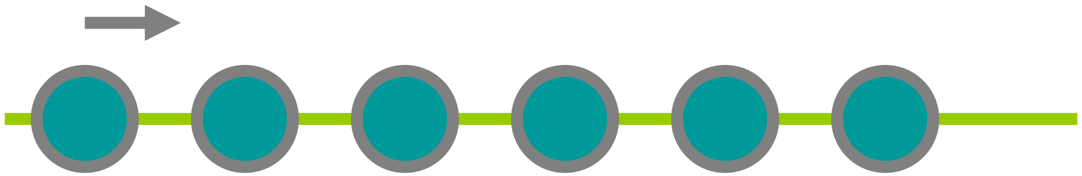




D=1

• No individual excitation can exist (only collective ones)





• Strong quantum fluctuations (``no'' ordered state or mean field possible)

How to study

- Exact methods (Bethe Ansatz)

Exact

spectrum; limited to very special models

- Numerics

``Exact''

special models, size limitations,

quantities specific to models

- Low energy methods methods

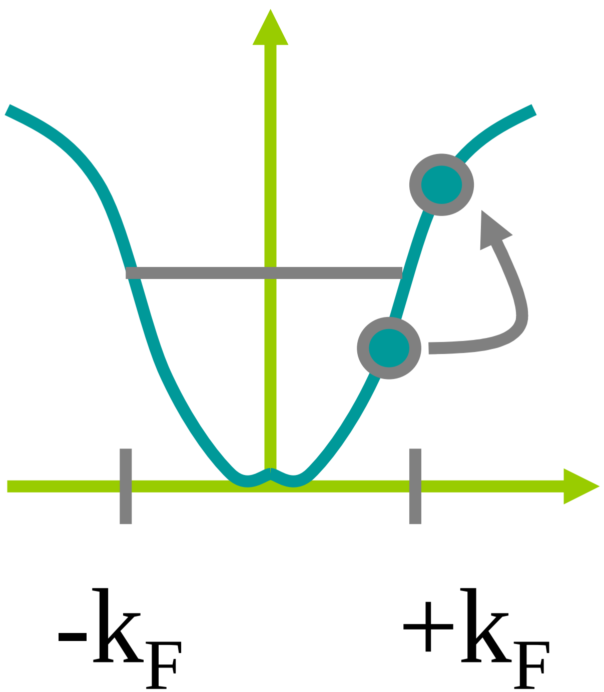
How to solve

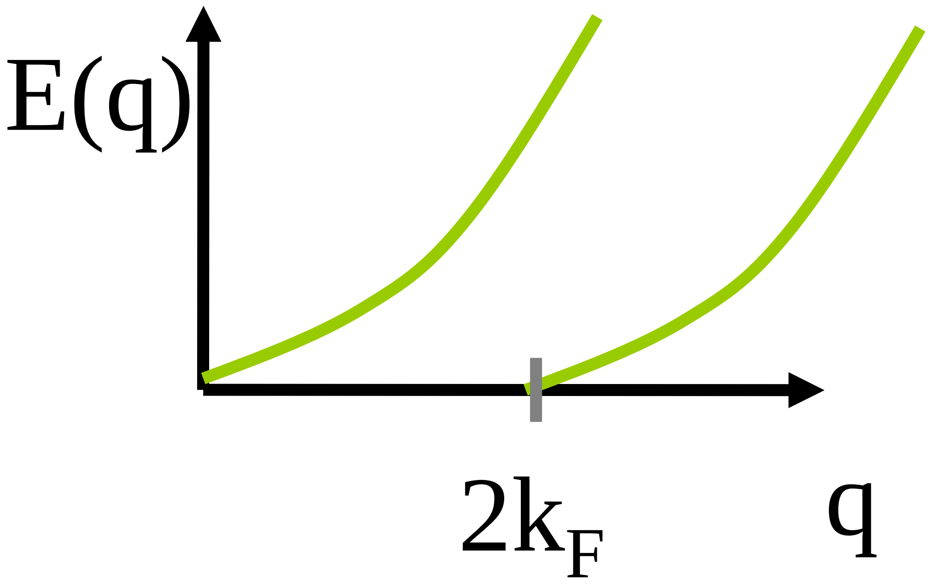
- Identify the low energy excitations

- Bosonization method

Particle hole excitations

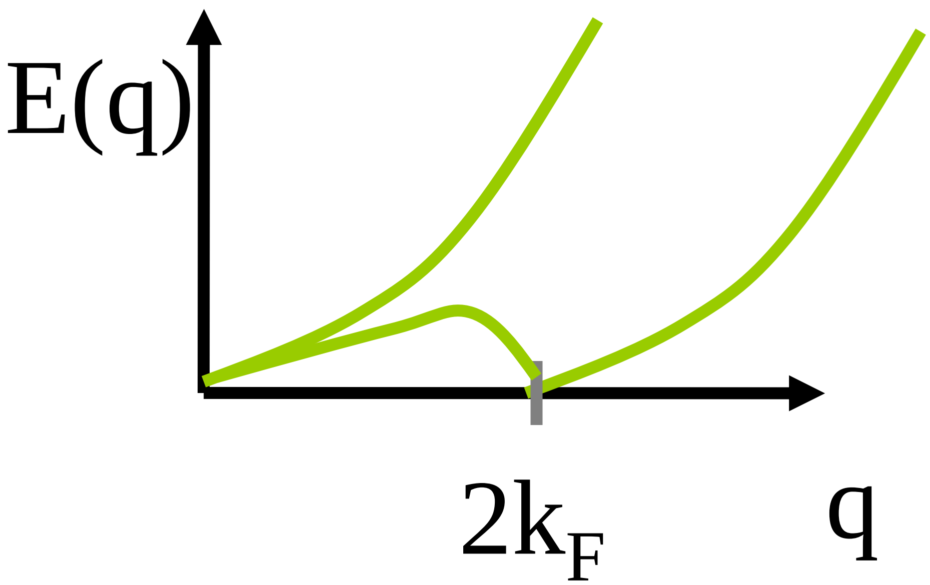
$$E(k,q) = \varepsilon(k+q) - \varepsilon(k)$$





D>1:

continuum



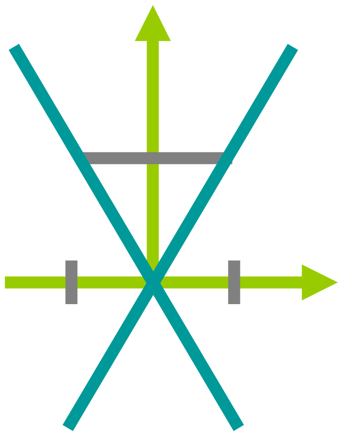
D=1:

Well defined excitations

$$E(q) = v_F q$$

Gory details

$$H = \sum_{|k|<\Lambda} v_F k (c^\dagger_{k,R} c_{k,R} - c^\dagger_{k,L} c_{k,L})$$



$$\psi(x) = e^{\imath k_F x}\psi_R(x) + e^{-\imath k_F x}\psi_L(x)$$

$$[\rho_L(q), \rho_R(q')] = 0$$

$$[\rho_L(q), \rho_L(q')] = \frac{Lq}{2\pi}\delta_{q,q'}$$

$$[\rho_R(q), \rho_R(q')] = -\frac{Lq}{2\pi}\delta_{q,q'}$$

$$[H, \rho_R(q)] = v_F q \rho_R(q)$$

$$[H, \rho_L(q)] = v_F q \rho_L(q)$$

$$H = \sum_{q \neq 0} \frac{\pi v_F}{L} [\rho_R(q)\rho_R(-q) + \rho_L(q)\rho_L(-q)]$$

$$H = \sum_{p \neq 0} v_F |p| b_p^* b_p$$

More convenient to use

$$\nabla\Phi(x) = -\pi[\rho_R(x) + \rho_L(x)]$$

$$\nabla\Theta(x) = \pi[\rho_R(x) - \rho_L(x)] = \pi\Pi(x)$$

$$H = \int \frac{dx}{2\pi} v_F [(\pi\Pi(x))^2 + (\nabla\Phi(x))^2]$$

$$S = \int \frac{dx d\tau}{2\pi} [\tfrac{1}{v_F}(\partial_\tau(x,\tau))^2 + v_F(\partial_x(x,\tau))^2]$$

- Phonon Hamiltonian

- Link with 2d stat mech

Interactions

$$\rho(x)\rho(x') \approx (\nabla\Phi(x))^2$$

$$H = \int \frac{dx}{2\pi}[uK(\pi\Pi(x))^2 + \frac{u}{K}(\nabla\Phi(x))^2]$$

- u velocity of sound
- K dimensionless parameter,

K< 1 : repulsive

K>1 : attractive

Properties

- Only collective excitations

- Thermodynamics

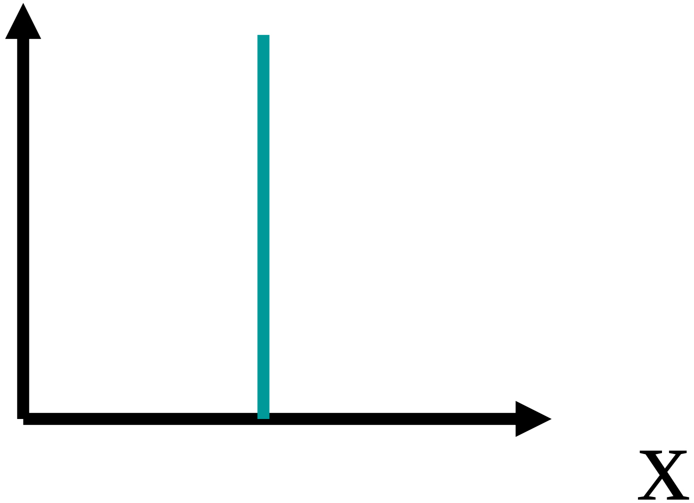
$$C_V = \frac{T}{u}\left(\frac{L\pi}{3}\right) \qquad \kappa/\kappa_0 = K\frac{v_F}{u}$$

- Looks like a Fermi liquid for q~0

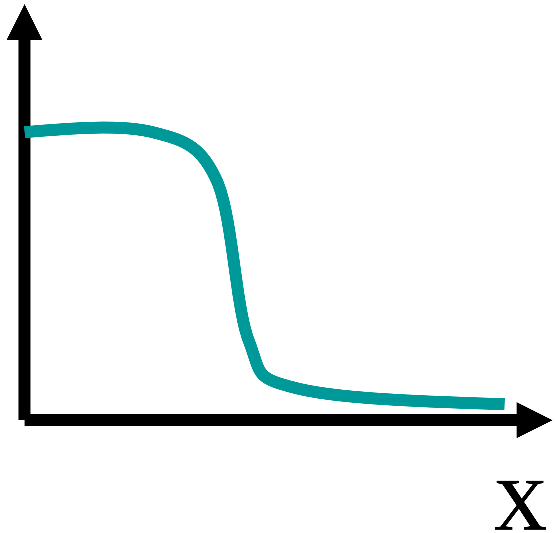
Fermion operator

$$[\rho_R(p), \psi_R(x)] = -e^{ipx}\psi_R(x)$$

$\rho(x)$



$\Phi(x)$



$$\psi(x) \approx e^{\int_{-\infty}^{x} dy\, i\pi\Pi(y)} = e^{i\Theta(x)} \qquad \psi_r(x) = \frac{e^{irk_F x}}{\sqrt{2\pi\alpha}} e^{-i[r\Phi(x)-\Theta(x)]}$$

Correlation functions

$$\rho(x) = (\psi_R^*(x) + \psi_L^*(x))(\psi_R(x) + \psi_L(x))$$

$$\rho(x) = \frac{-1}{\pi}\nabla\Phi + e^{i2k_F x + 2\Phi(x)}$$

$$\langle \rho(x)\rho(0) \rangle = \frac{1}{x^2} + \cos(2k_F x)\left(\frac{1}{x}\right)^{2K}$$

Non universal decay of correlation functions

$$O_{SU}(x) = \psi_R^*(x)\psi_L^*(x) \approx e^{i2\Theta(x)}$$

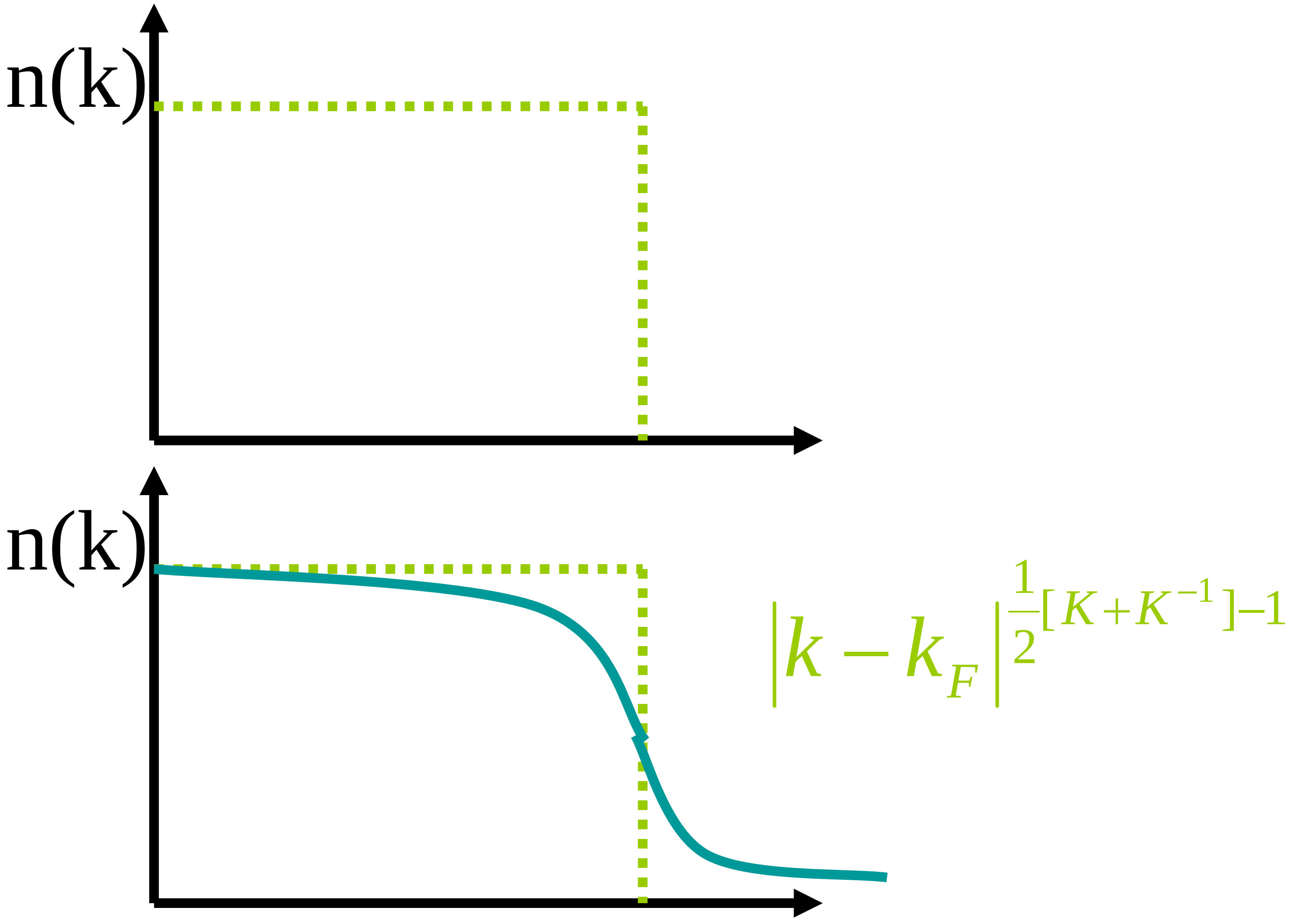
$$\langle O_{SU}(x) O_{SU}(0) \rangle = \left(\frac{1}{x}\right)^{1/2K}$$

Single particle excitations

$$\langle \psi_R(x)\psi_R^*(0) \rangle = \left(\frac{1}{x}\right)^{\frac{1}{2}[K+K^{-1}]} \quad e^{iArg(\tau/x)}$$

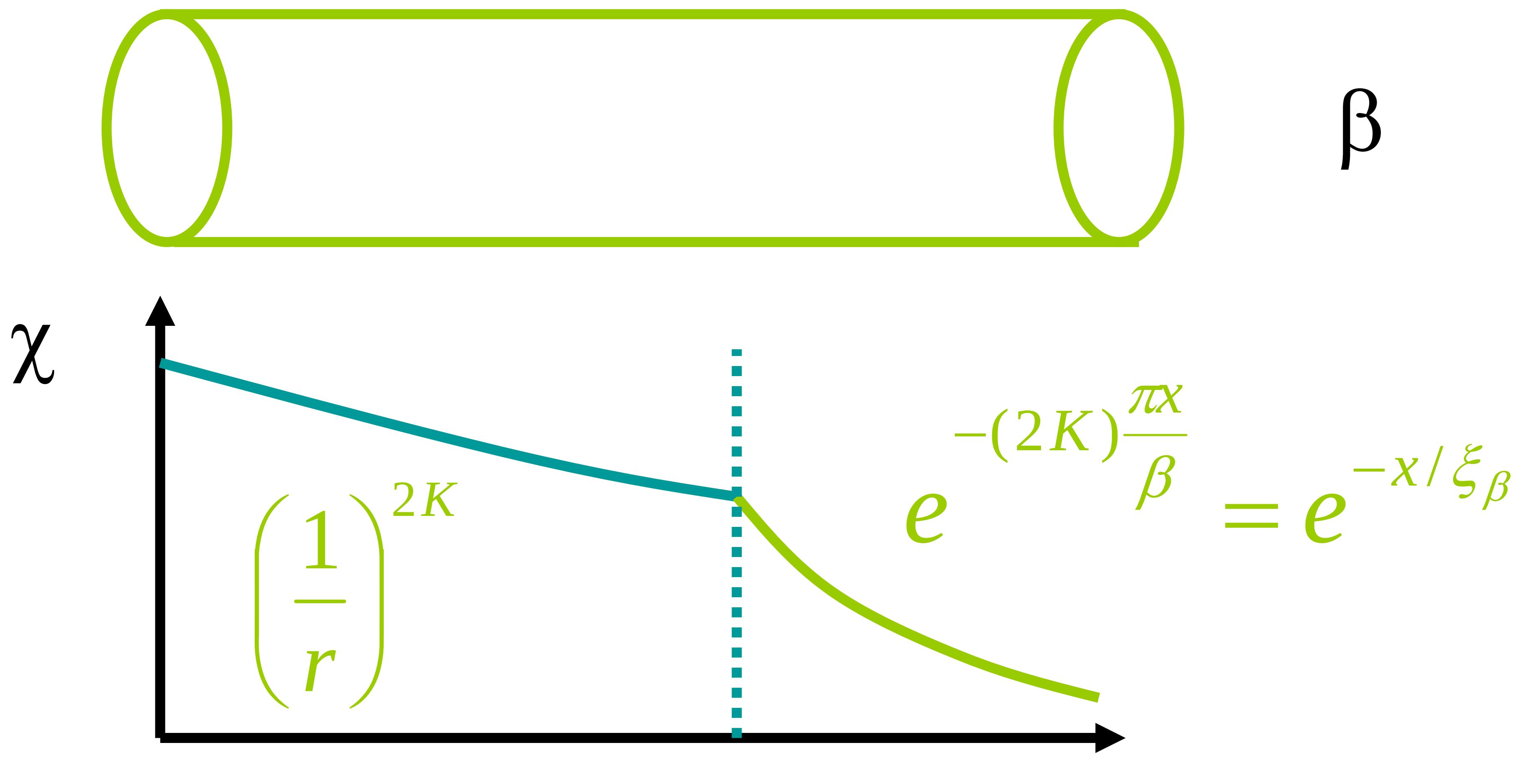
$$K = 1 \qquad \langle \psi_R(x)\psi_R^*(0) \rangle = \frac{1}{x - v_F\tau}$$

No Landau Quasiparticles



Finite temperature

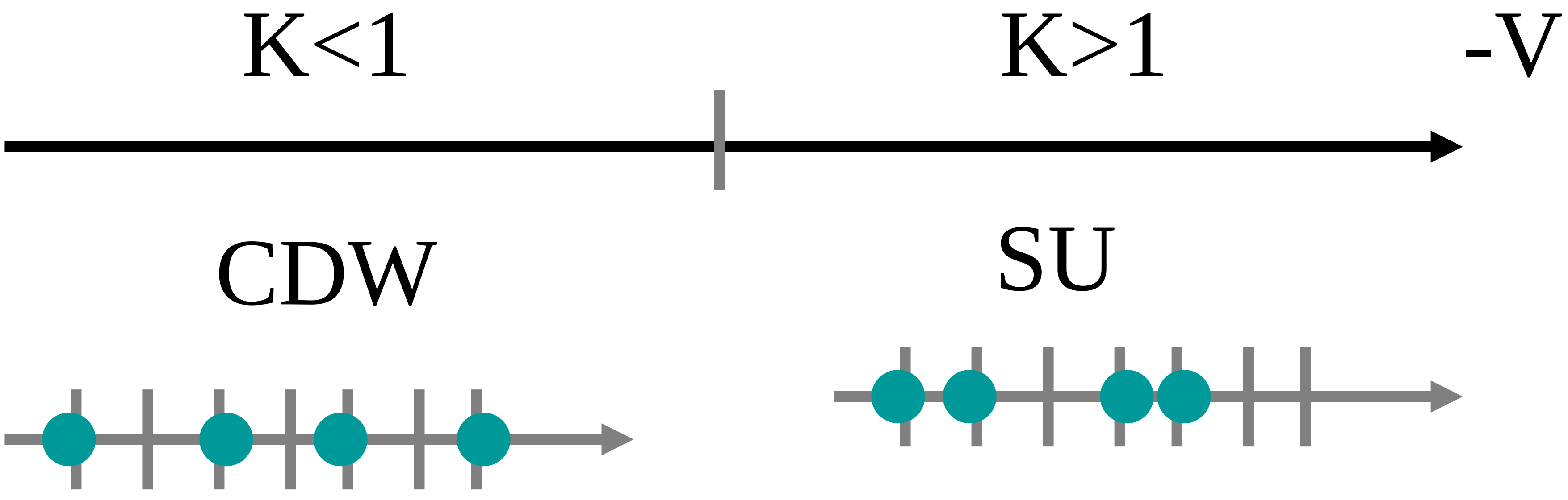
Conformal theory



Phase diagram

$$\chi(q,\omega) = \int dx d\tau \; e^{i(qx+\omega\tau)} \chi(x,\tau) \qquad \chi \approx \omega^{\eta-2}$$

Most divergent fluctuations



System with spin

Same treatment

$$\rho_{\uparrow} \rightarrow \nabla\Phi_{\uparrow} \qquad \rho_{\downarrow} \rightarrow \nabla\Phi_{\downarrow}$$

More convenient

$$\rho = \frac{1}{\sqrt{2}}(\rho_{\uparrow} + \rho_{\downarrow}) \qquad \sigma = \frac{1}{\sqrt{2}}(\rho_{\uparrow} - \rho_{\downarrow})$$

$$H_{kin} = H_{\uparrow} + H_{\downarrow} = H_{\rho} + H_{\sigma}$$

$$H_{\mathrm{int}} = U \sum_i \rho_\uparrow \rho_\downarrow = U(\rho + \sigma)(\rho - \sigma)$$

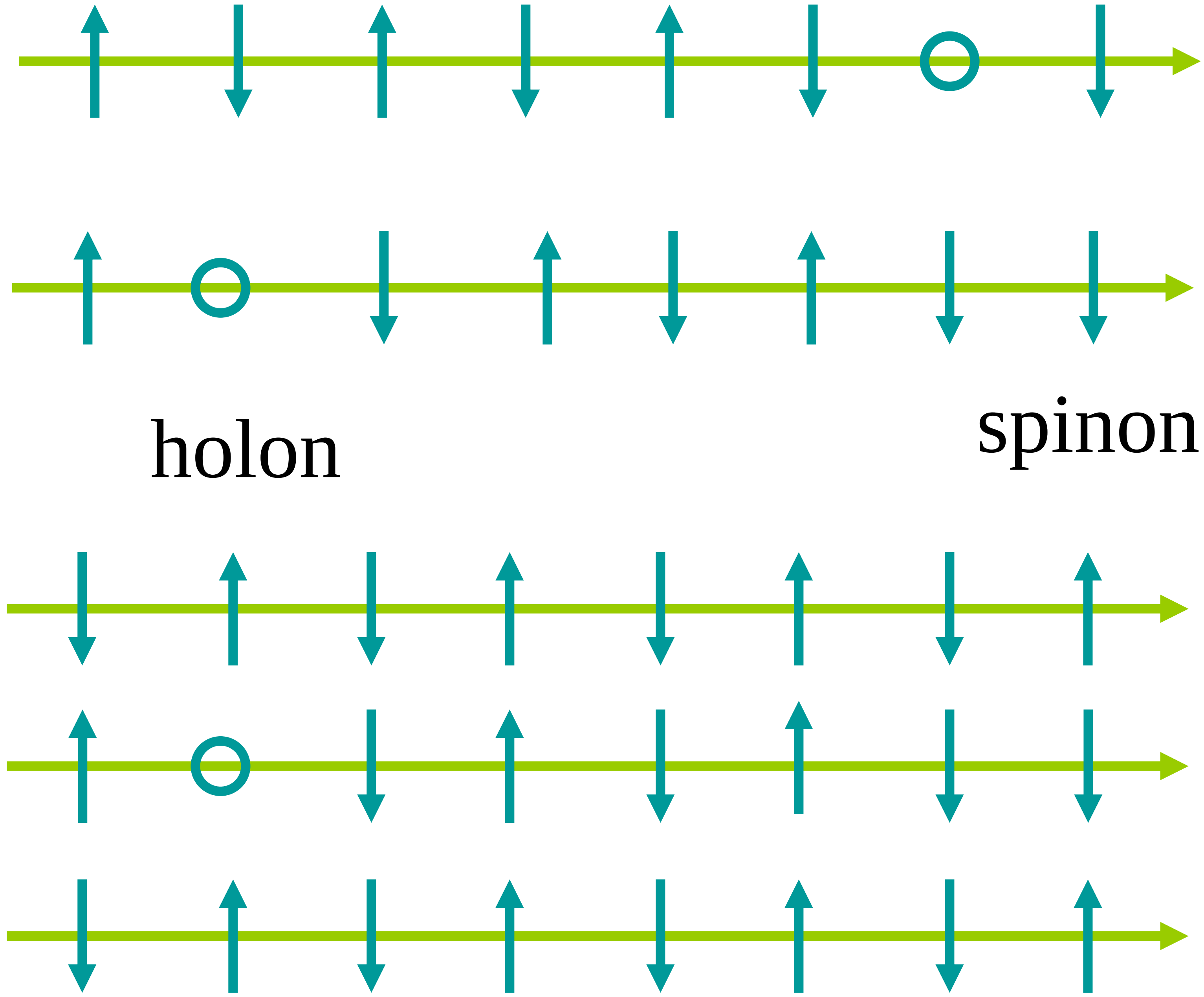
$$= U(\rho\rho - \sigma\sigma)$$

$$H = H_\rho + H_\sigma$$

(u_ρ, K_ρ) Charge excitations

(u_σ, K_σ) Spin excitations

Charge-spin separation


holon
spinon

Correlation functions

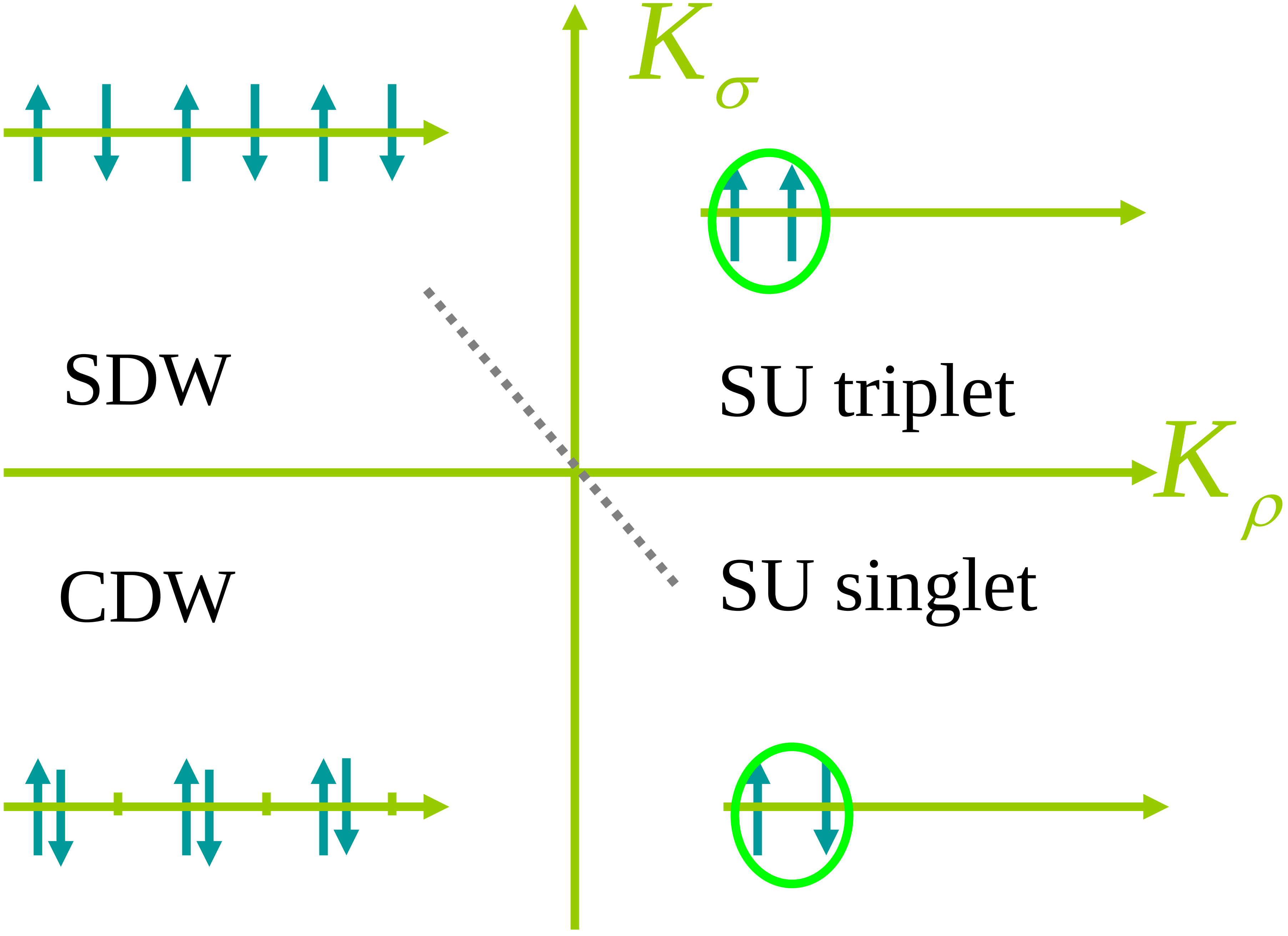
$$\langle S(x)S(0)\rangle = \tfrac{1}{x^2} + \cos(2k_F x)\left(\tfrac{1}{x}\right)^{K_\sigma + K_\rho}$$

- Perturbation (small U)

$$u_\rho K_\rho = u_\sigma K_\sigma = v_F$$

$$u_\rho / K_\rho = v_F + U/\pi$$

$$u_\sigma / K_\sigma = v_F - U/\pi$$



K_σ
SDW
SU triplet
K_ρ
CDW
SU singlet

Spin sector more complicated (gap)

$$H = \int \frac{dx}{2\pi}[uK(\pi\Pi(x))^2 + \frac{u}{K}(\nabla\Phi(x))^2] + g\int dx \cos(\sqrt{8}\Phi(x))$$

Anomalous correlation functions

$$n(k) \approx \left|k - k_F\right|^{\frac{1}{4}[K_\rho + K_\rho{}^{-1}] - \frac{1}{2}}$$ photoemission

$$\chi_{2k_F} \approx T^{K_\rho - 1}$$ NMR

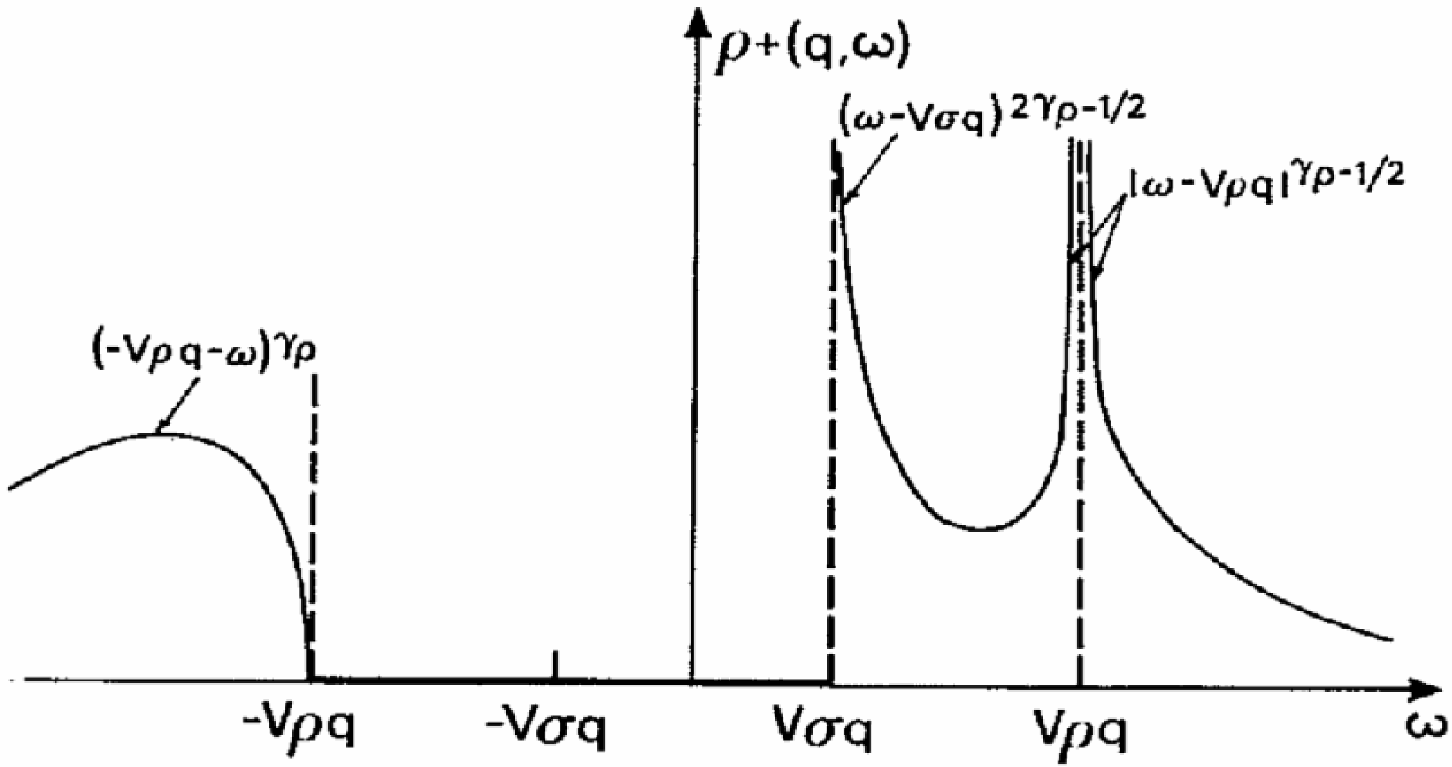


FIG. 3. Spectral function $\rho_+(q,\omega)$ for the spin-$\frac{1}{2}$ Luttinger liquid for $q > 0$.

J. Voit

Luttinger liquid concept

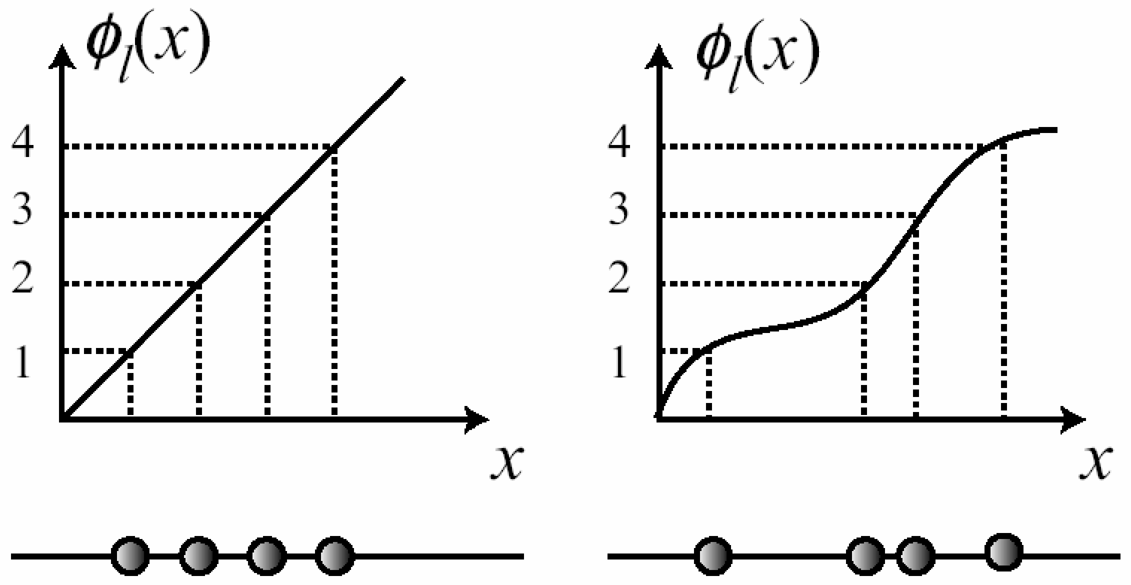
- How much is perturbative

- Nothing provided the correct u,K are used

- Low energy properties: Luttinger liquid (fermions, bosons, spins…)

General Derivation (Haldane)

$$\rho(x) = \sum_i \delta(x - x_i)$$
$$= \sum_n |\nabla \phi_l(x)| \delta(\phi_l(x) - 2\pi n)$$



$$\phi_l(x) = 2\pi\rho_0 x - 2\phi(x)$$

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right]\sum_p e^{i2p(\pi\rho_0 x - \phi(x))}$$

$$\psi^\dagger(x) = [\rho(x)]^{1/2} e^{-i\theta(x)}$$

$$[\frac{1}{\pi}\nabla\phi(x), \theta(x')] = -i\delta(x - x')$$

$$\psi_B^\dagger(x) = [\rho_0 - \frac{1}{\pi}\nabla\phi(x)]^{1/2}\sum_p e^{i2p(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)}$$

$$\psi_F^\dagger(x) = [\rho_0 - \frac{1}{\pi}\nabla\phi(x)]^{1/2}\sum_p e^{i(2p+1)(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)}$$

Interacting Bosons

$$H_K = \int dx \frac{1}{2m} (\nabla \psi^\dagger(x))(\nabla \psi(x))$$

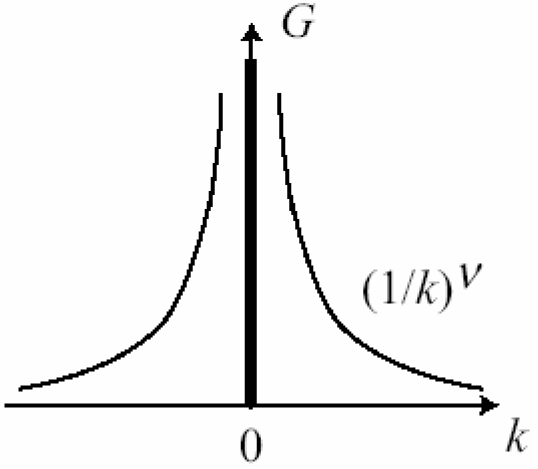
$$H_K = \int dx \frac{\rho_0}{2m} (\nabla \theta(x))^2$$

$$H_{\text{int}} = \int dx V_0 \frac{1}{2\pi^2} (\nabla \phi)^2$$

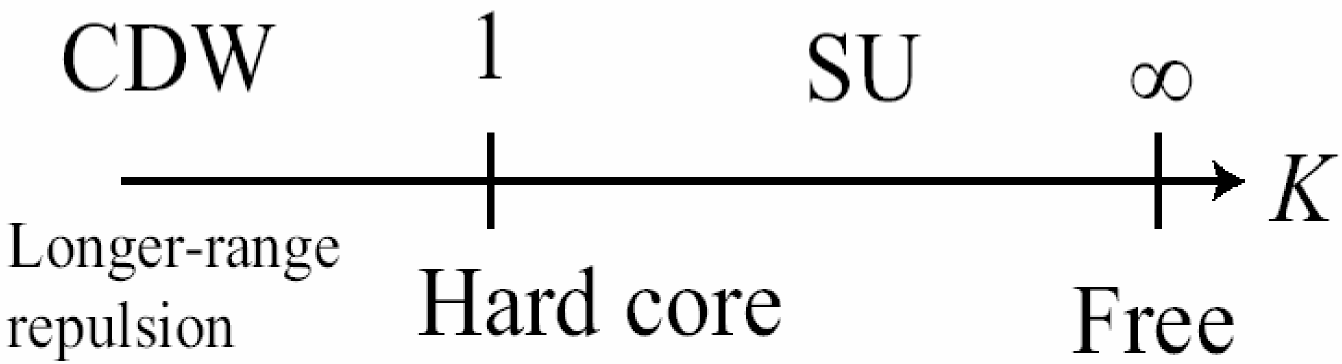
$$H = \frac{\hbar}{2\pi} \int dx [\frac{uK}{\hbar^2} (\pi \Pi(x))^2 + \frac{u}{K} (\nabla \phi(x))^2]$$

$$\langle \psi(r)\psi^\dagger(0)\rangle = A_1 \left(\frac{\alpha}{r}\right)^{\frac{1}{2K}} + \cdots$$

$$\langle \rho(r)\rho(0)\rangle = \rho_0^2 + \frac{K}{2\pi^2}\frac{y_\alpha^2 - x^2}{(y_\alpha^2 + x^2)^2} + A_3 \cos(2\pi\rho_0 x)\left(\frac{1}{r}\right)^{2K} + \cdots$$



No condensate



Fermions

$$\langle \rho(x)\rho(0)\rangle = \frac{1}{x^2} + A_1 \cos(2k_F x)\left(\frac{1}{x}\right)^{K_\rho+1}$$

$$+ A_2 \cos(4k_F x)\left(\frac{1}{x}\right)^{4K_\rho}$$

Hubbard

$$U = 0 \quad K_\rho = 1$$

$$U = \infty \quad K_\rho = 1/2$$

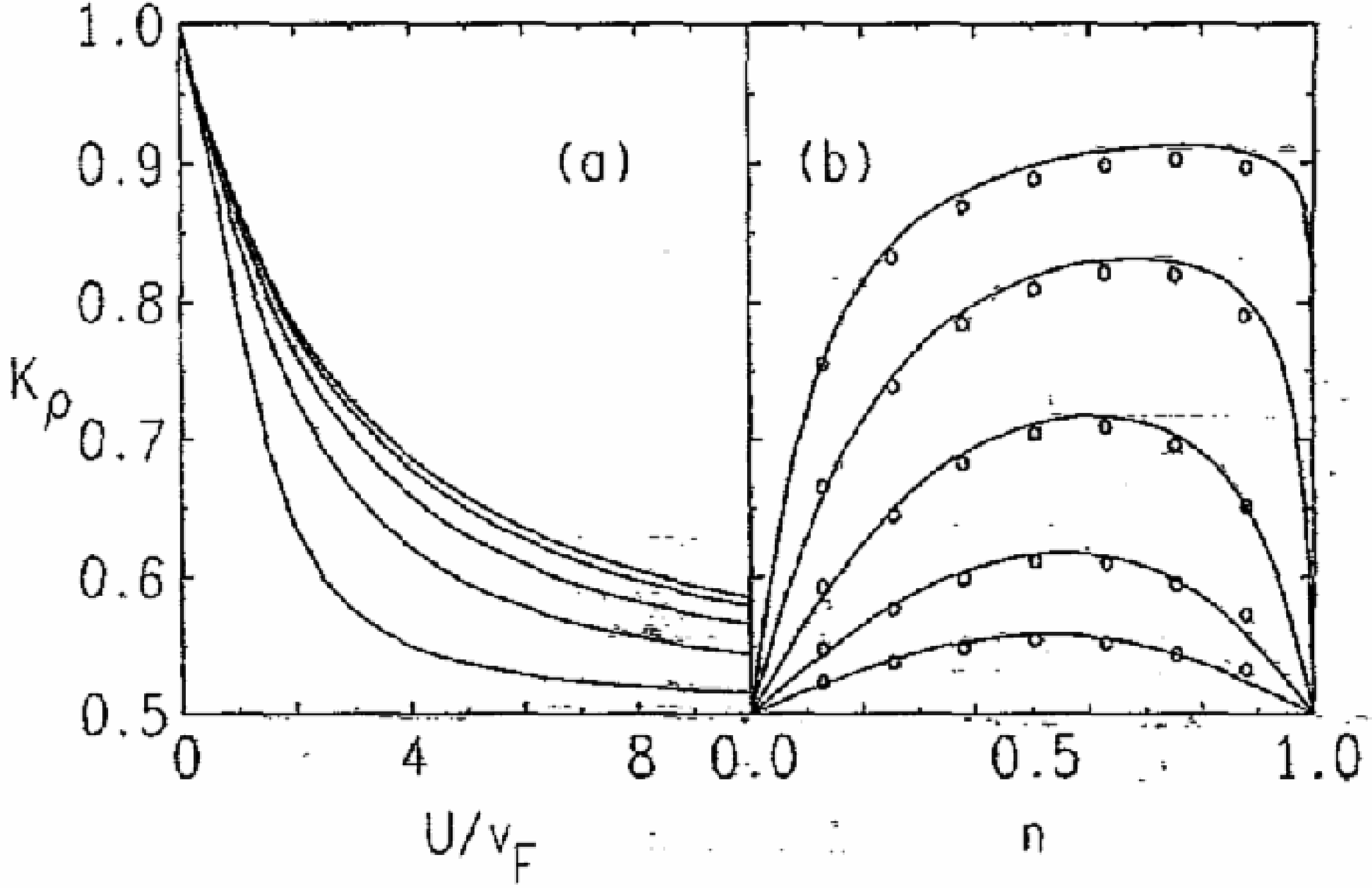
How to compute (u,K)

- Perturbation
- Exact solutions (Bethe Ansatz): thermodynamics

$$C_V \to u \quad \kappa \to u/K \quad D = uK$$

$$(E(L) - E(\infty))/L \propto cu$$

- Numerics



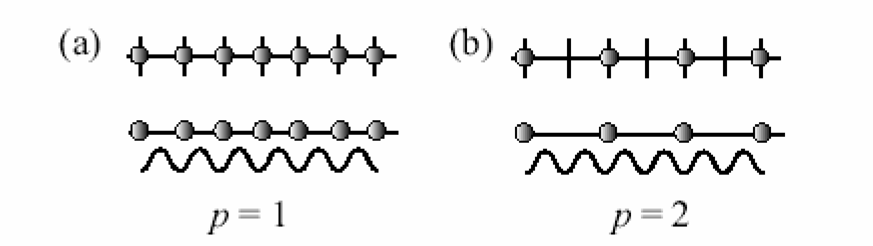
H.J. Schulz PRL 64 2831 (90)

Conclusions

- Good control of d=1 massless phases
- Massive phases
- Luttinger Liquid: crucial starting point to study perturbations (lattice, disorder, etc.)

Examples

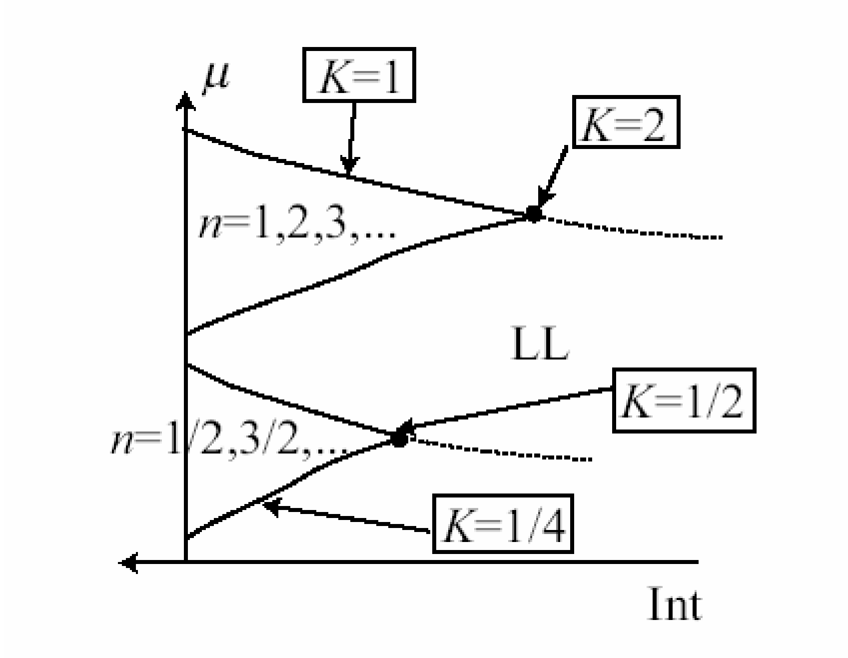
Lattice



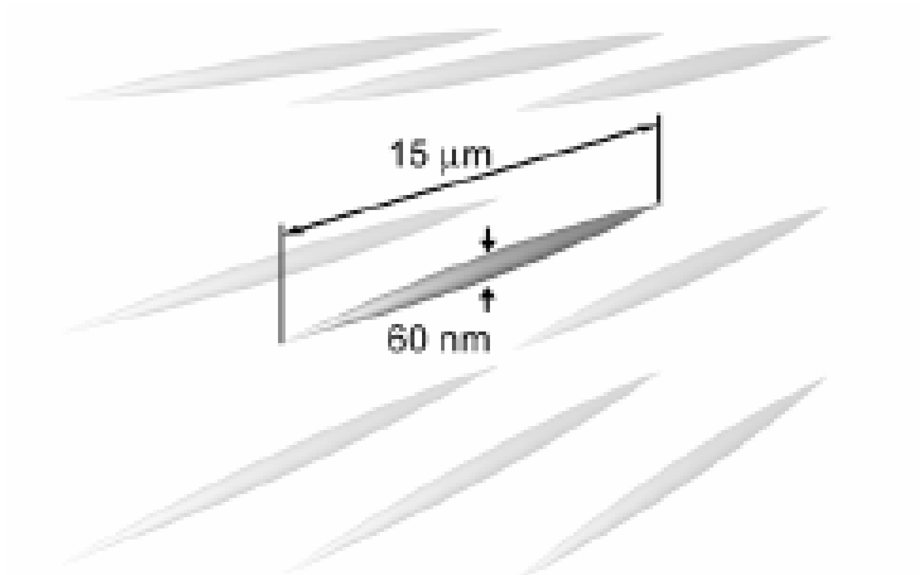
$$H_L \propto V_n^0 \int dx \ \cos(2p\phi(x))$$

Mott insulator:

ϕ is locked



Coupled one dimensional bosons



Cold Bosons

N_0 ~ 10 to 10^3 atoms

T. Stoferle *et al.* PRL **92** 130403 (2004)

Interchain hopping

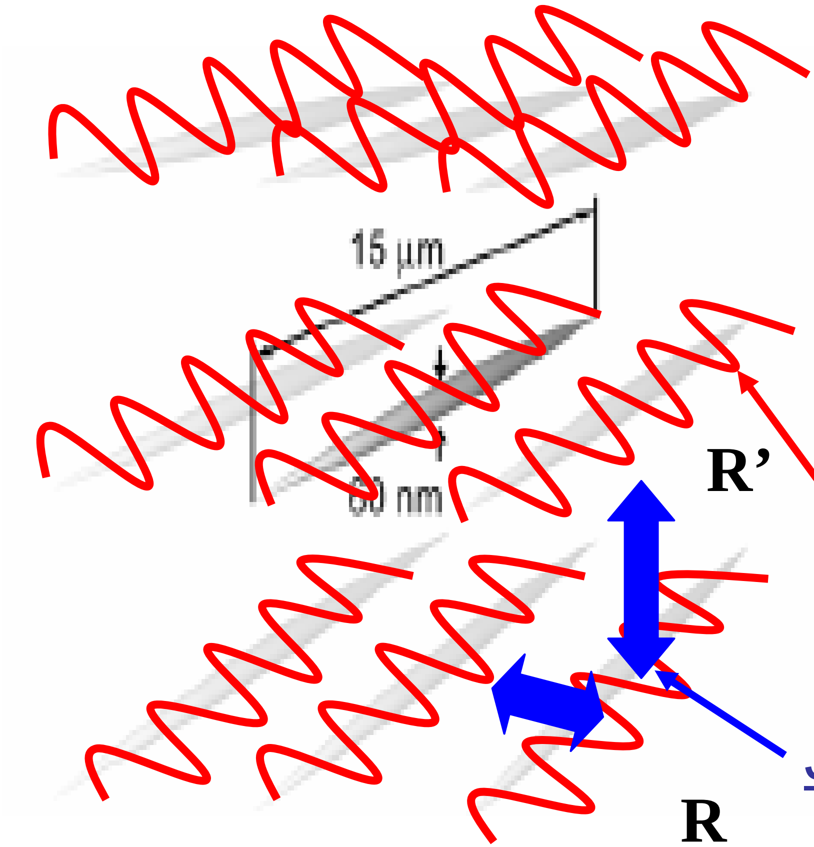
$$-t_{\perp}\sum_{\langle\alpha,\beta\rangle}\int dx\psi_{\alpha}^{\dagger}(x)\psi_{\beta}(x)=-t_{\perp}\rho_0\int dx e^{i(\theta_{\alpha}(x)-\theta_{\beta}(x))}$$

Wants to order θ on each chain

Competes with Mott potential that wants to order ϕ

Mott vs. Josephson

$$
\begin{aligned}
H_{\text{eff}} &= \frac{\hbar v_s}{2\pi}\sum_{\mathbf{R}}\int_0^L dx\,\left[\frac{1}{K}\left(\partial_x\phi_{\mathbf{R}}(x)\right)^2 + K\left(\partial_x\theta_{\mathbf{R}}(x)\right)^2\right] \\
&+ \frac{\hbar v_s g_u}{2\pi a^2}\sum_{\mathbf{R}}\int_0^L dx\,\cos\left(2\phi_{\mathbf{R}}(x)+\delta\pi x\right) \\
&- \frac{\hbar v_s g_J}{2\pi a^2}\sum_{\langle\mathbf{R},\mathbf{R}'\rangle}\int_0^L dx\,\cos\left(\theta_{\mathbf{R}}(x)-\theta_{\mathbf{R}'}(x)\right) \qquad (1)
\end{aligned}
$$

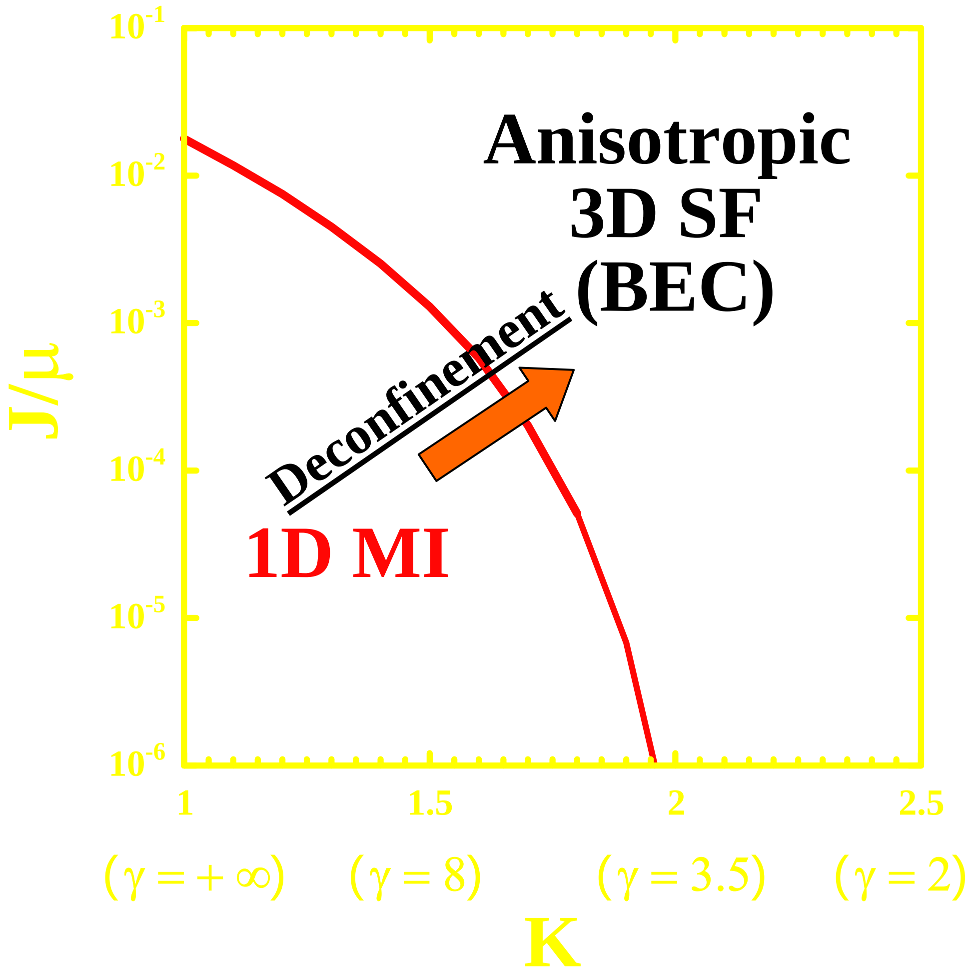


"Mott" potential: localizes atoms

Josephson coupling: delocalizes atoms

AF Ho, M.A. Cazalilla, and TG, PRL **92** 130403 (2004)

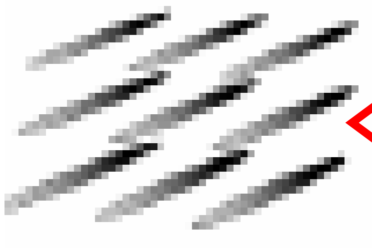
Phase diagram



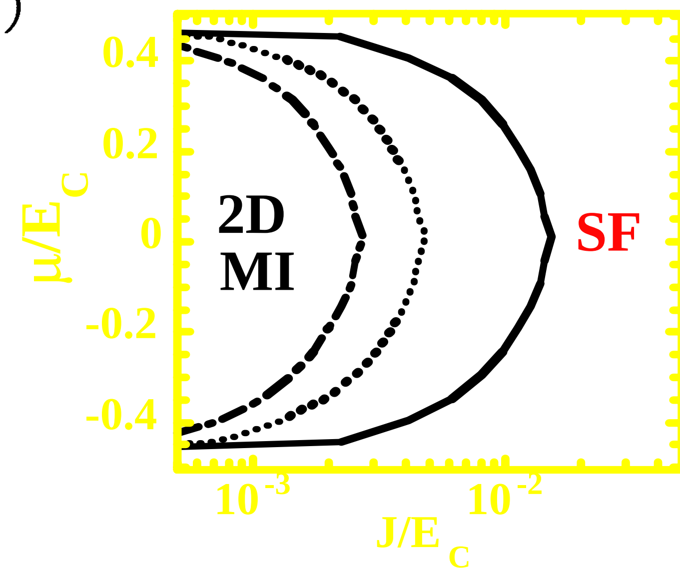
Mesoscopic effects

- Charging enery of a tube $E_C = \sim \pi\, v_s/K\, L$

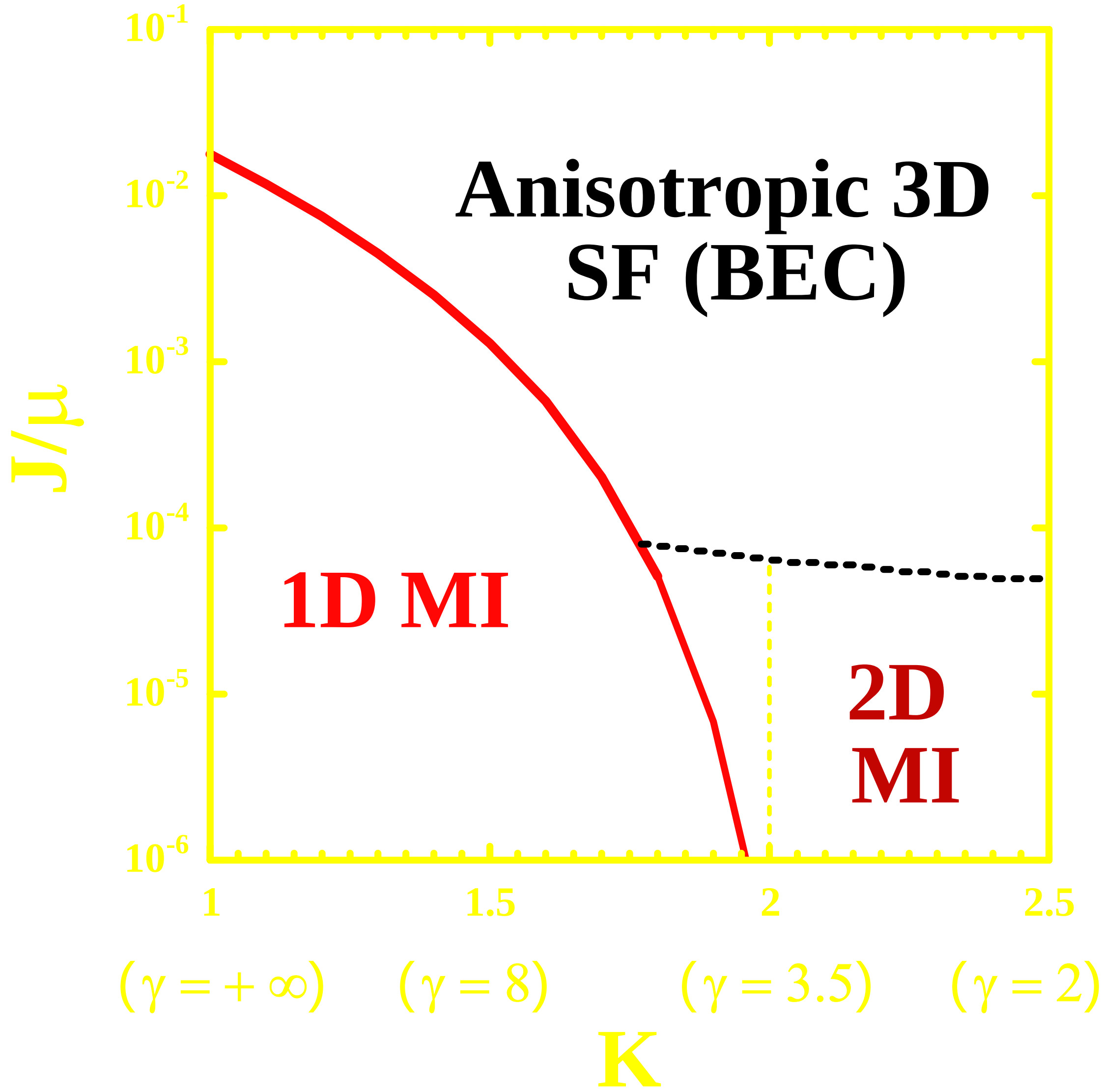
$$H_{\mathrm{QP}} = -E_J \sum_{\langle \mathbf{R},\mathbf{R}' \rangle} \cos\left(\theta_{0\mathbf{R}} - \theta_{0\mathbf{R}'}\right) + \frac{E_C}{2} \sum_{\mathbf{R}} (N_{\mathbf{R}} - N_0)^2 - \mu \sum_{\mathbf{R}} N_{\mathbf{R}}, \quad (1)$$



Array of atomic 'quantum dots'

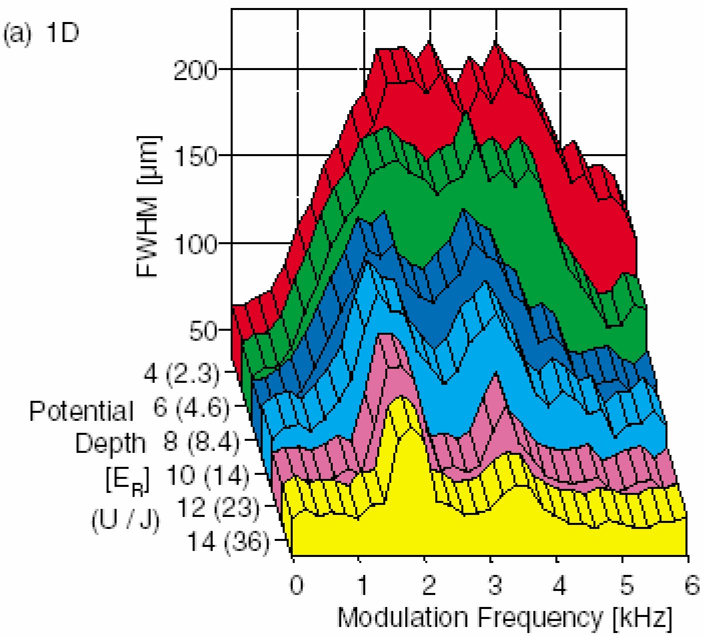


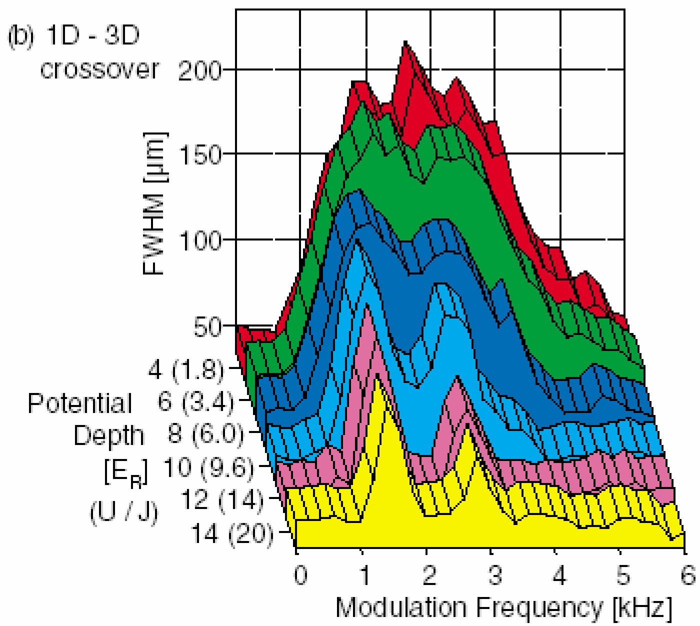
Phase diagram for finite tubes

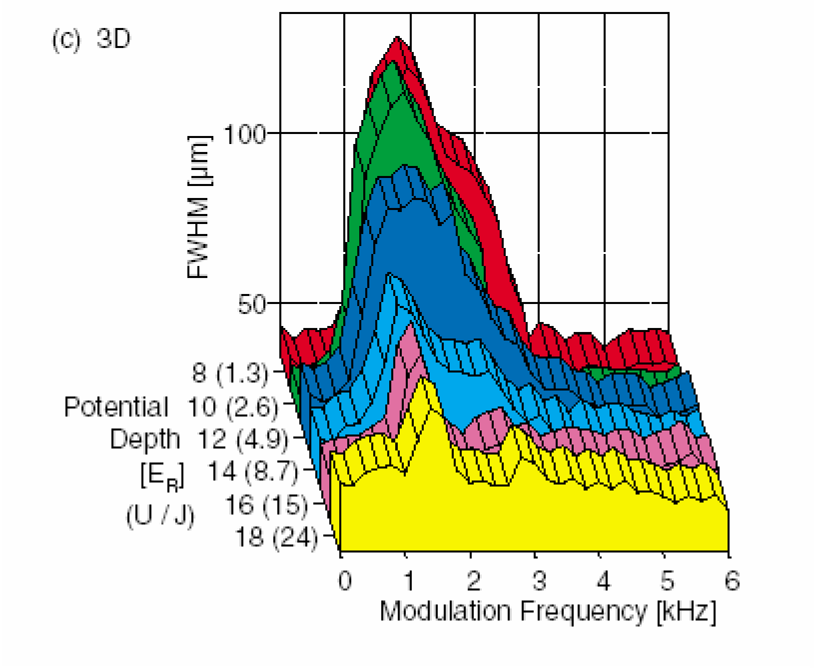


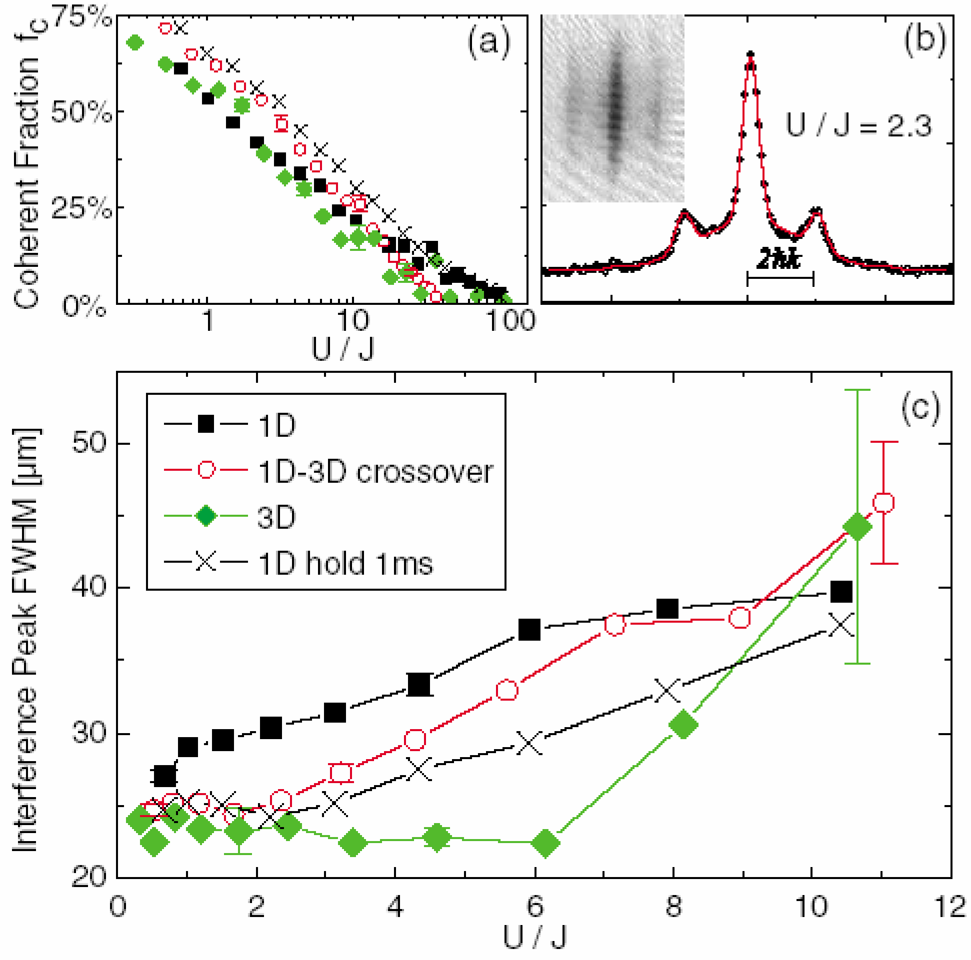
Experiments

T. Stoferle *et al.* PRL **92** 130403 (2004)









Large quantum depletion

Disorder, no interactions

$$H = H_0 + \int dx V(x) \rho(x)$$

$$= H_0 + \int dx \eta(x) [\psi_R^* \psi_R + \psi_L^* \psi_L] + \int dx [\xi(x) \psi_R^* \psi_L + \xi^*(x) \psi_L^* \psi_R]$$

Backscattering gives localization

Compressible system $\psi \sim e^{-r/\xi_{loc}}$

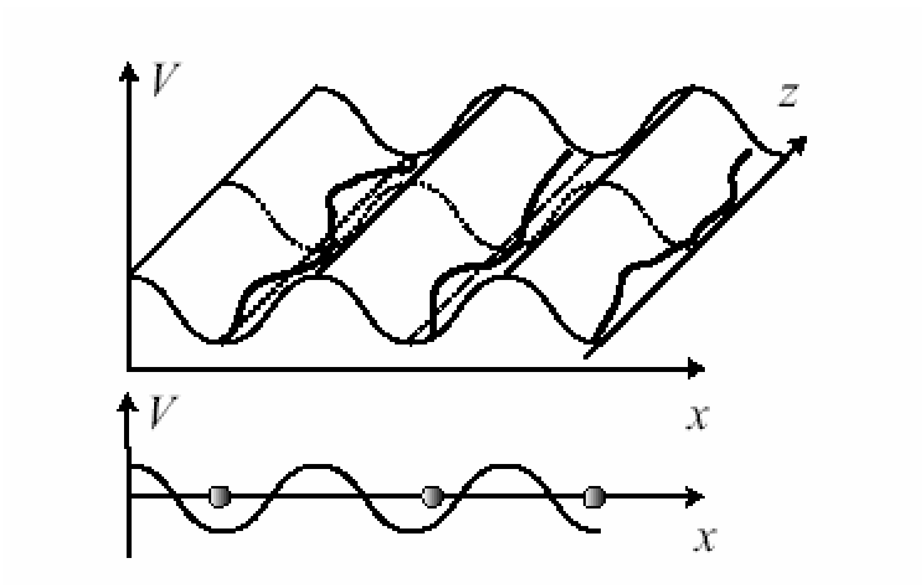
Localization length is mean free path

$$\sigma(\omega) \sim \omega^2 \log^2(\omega)$$

Bosonized representation

$$S = \int \frac{dx d\tau}{2\pi K}\left[\tfrac{1}{u}(\partial_\tau \phi(x,\tau))^2 + u(\partial_x \phi(x,\tau))^2\right]$$

SΩx dxd¿V ☎ ½☎; ¿ dxd¿V ☎ e i Á☎;¿



Disordered
Elastic System