III Renormalization:

A. 1 d Example (ibid., Chapter 13)
B. 2d Example (ibid., Chapter 14)

The Concept

The idea of renormalization is very simple. Imagine that we compare two problems in d dimensions: one with N spins called $\square_r$ the other with say $N/2^d$ spins called $\square_R$. The new lattice, R, is interleaved in the old one r. In fact, the new variables are functions of the neighboring old ones. In making this happen we have changed the lattice constant of the system by a factor of two. We can fix up the transformation so that the free energy of the two systems are identical and so are the long-ranged correlations. I'll show you how this might be done in a moment.

Then what we have done is changed the problem, but kept the answers the same. Since we have some sort of universality in the problems we have been describing, we might imagine that we have a mechanism for changing the form of the problem (the Hamiltonian or the couplings) but not the nature of the answer. This sounds like, and is, a powerful idea.

One Dimensional Ising Model Example

Return to our calculation of the partition function

$$z = \sum_{\sigma_1 .. \sigma_N} e^{K\sigma_1\sigma_2} e^{K\sigma_2\sigma_3} \ldots . e^{K\sigma_N\sigma_1}$$

$$= \sum_{\sigma_1 .. \sigma_N} (\sigma_0 | \mathbf{T} | \sigma_1)(\sigma_1 | \mathbf{T} | \sigma_2)(\sigma_2 | \mathbf{T} | \sigma_3)(\sigma_3 | \mathbf{T} | \sigma_4)$$

$$\ldots(\sigma_{N-2} | \mathbf{T} | \sigma_{N-1})(\sigma_{N-1} | \mathbf{T} | \sigma_0)$$

Imagine a new problem in which all the even spins are held constant, but renamed according to

$$\sigma_0 = \mu_0 \quad \sigma_2 = \mu_1 \quad \sigma_4 = \mu_2$$

while all the odd-numbered σ's serve as summation variables. The sum now looks like

$$= \sum_{\mu_0 \mu_1 \ldots} \sum_{\sigma_1} (\mu_0 | \mathbf{T} | \sigma_1)(\sigma_1 | \mathbf{T} | \mu_1) \sum_{\sigma_3} (\mu_1 | \mathbf{T} | \sigma_3)(\sigma_3 | \mathbf{T} | \mu_2) \ldots .$$

$$\ldots \sum_{\sigma_{N-1}} (\mu_{N/2-1} | \mathbf{T} | \sigma_{N-1})(\sigma_{N-1} | \mathbf{T} | \mu_0)$$

The sums over σ's are matrix multiplication so

$$z = \sum_{\mu_0 \mu_1 \ldots} (\mu_0 | \mathbf{T}' | \mu_1)(\mu_1 | \mathbf{T}' | \mu 2) \ldots .(\mu_{N/2-1} | \mathbf{T}' | \mu_0).$$

$$\mathbf{T}' = \mathbf{T}\mathbf{T}$$

The new T has a new coupling constant K'

The New Coupling

$$const\ \ e^{K'\sigma_0\sigma_1} = \sum_{\sigma} e^{K\sigma_0\sigma}\ e^{K\sigma\,\sigma_1}.$$

$$const\ \ e^{K'} = 2\cosh(2K)$$

$$const\ \ e^{-K'} = 2$$

$$e^{2K'} = \cosh(2K)$$

What happens? Imagine that the old coupling is large. Then to exponential accuracy

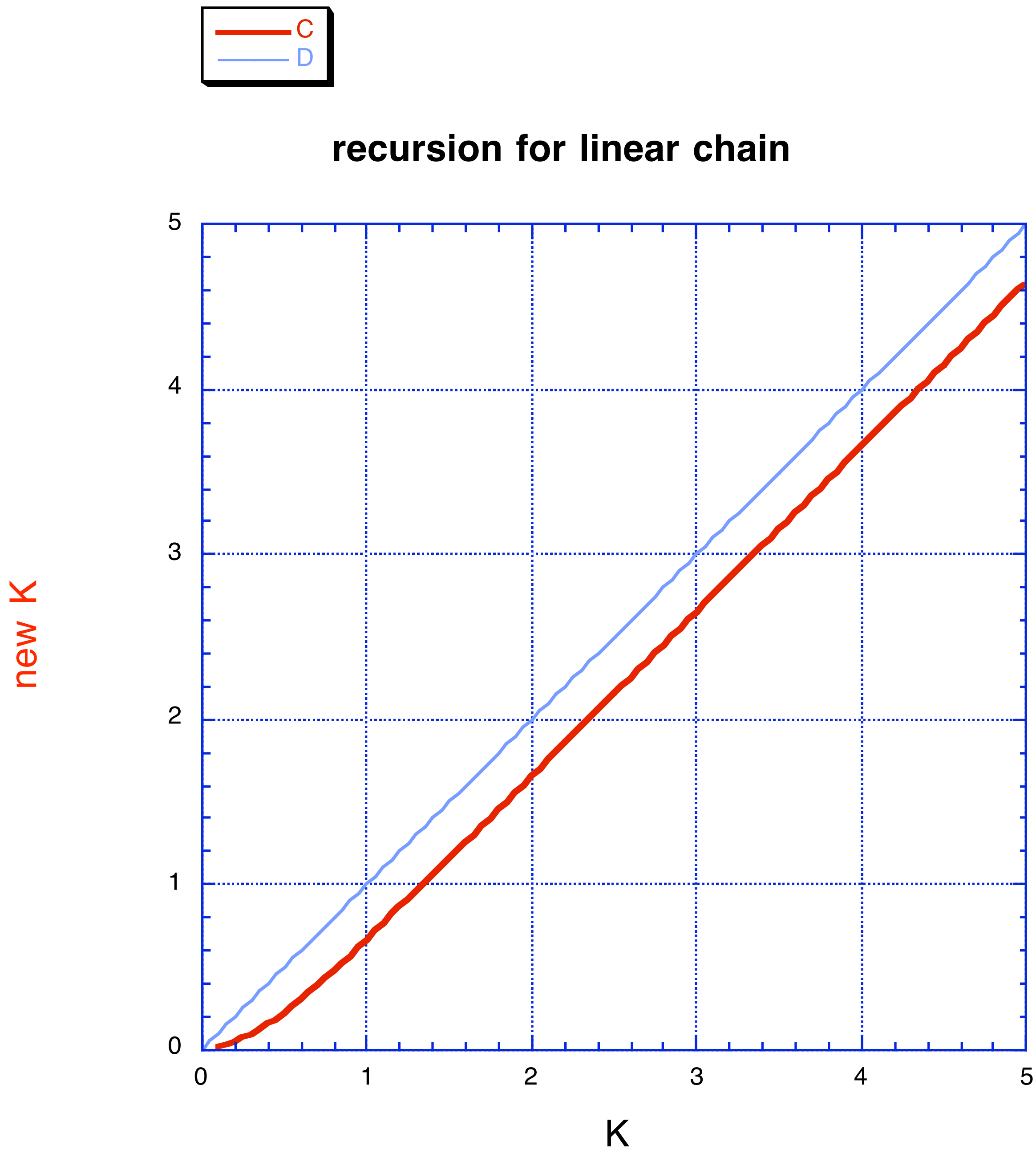
$$e^{2K'} = e^{2K}/2 \quad or \quad K' = K - (\ln 2)/2$$

Therefore in each iteration the coupling gets smaller and smaller until, finally, one reaches a small value of the coupling....and the problem gets easy to solve.

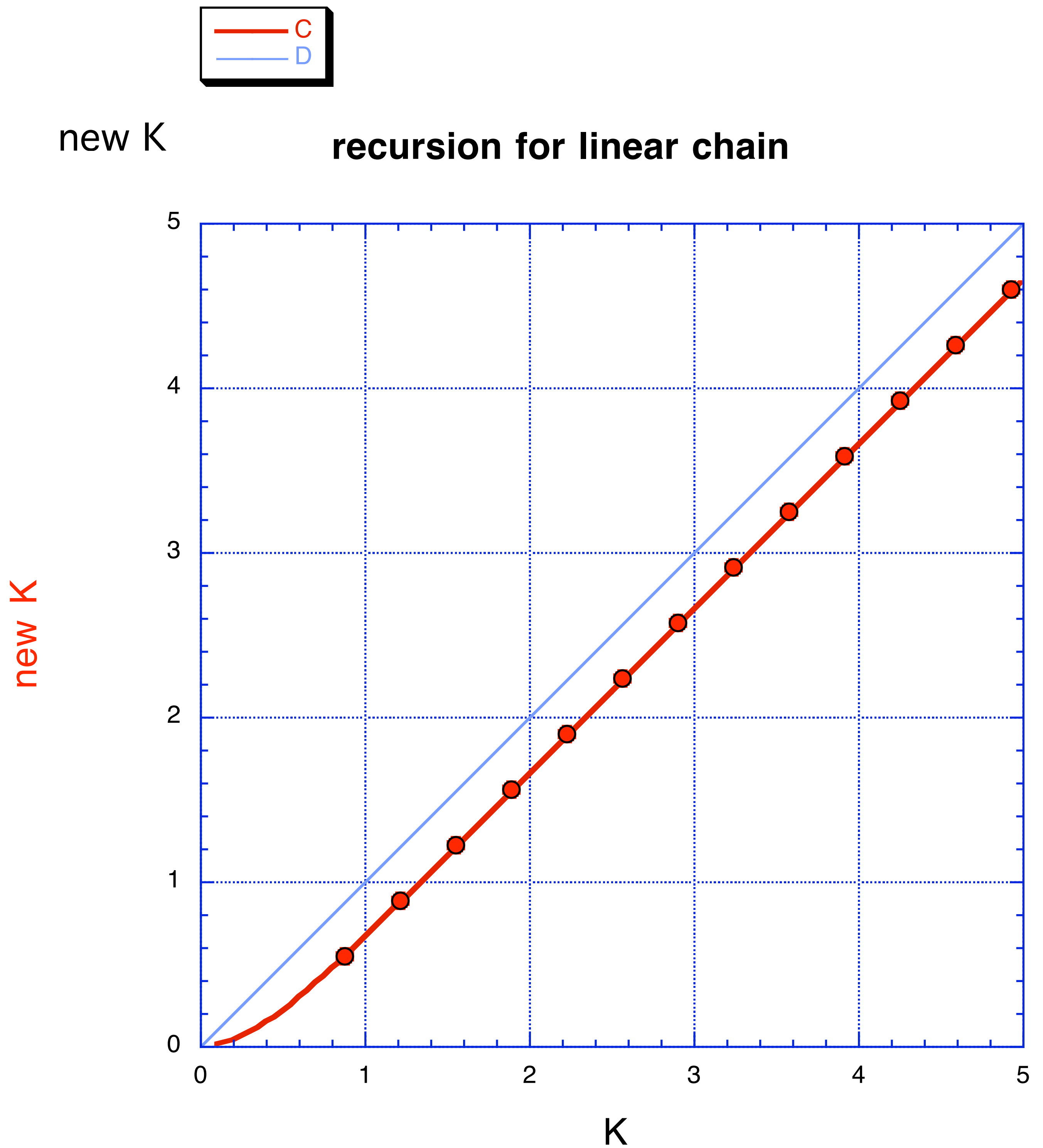
In the meantime, one can look at the coherence length for large values of K. This length has the form $\xi = a_0 f(K)$ where f is an unknown function. If we renormalize the correlation length retains the exact same value but the coupling decreases while the lattice constant doubles. We have

$$\xi = a_0 f(K) = (a_0)' f(K') = (2a_0) f(K') \quad so\ \ that \quad f(K') = F(K)/2$$

Inspection shows that for large K, f(K)=e^{2K} so that the coherence length grows exponentially with K.



$$K' = 0.5 \ln \cosh 2K$$


C
D
new K
recursion for linear chain
5
4
3
2
1
0
new K
0
1
2
3
4
5
K

Phenomenology

The phases of a statistical system are determined by the result of many successive renormalizations transforms upon the system. By some process of summation, we go from our "starting problem" with lattice constant a_0 and N degrees of freedom to another situation with a new, larger, lattice constant

$$(a_0)' = \ell a_0 \quad with \quad \ell > 1$$

and fewer degrees of freedom $N' = N / \ell^d$ where d is the dimension of the system. As this happens, we may generate new kinds of interaction terms with new coupling constants. If {K} is the set of all possible coupling constants, nearest neighbor, next neighbor, four-spin , etc. Then we can view the renormalization process a generating a transformation to a new set of couplings $\{K'\} = R_\ell(\{K\})$. We then imagine doing this many, many different times. Given either a classical or a quantum system there are several qualitatively different situations or "phases" possible

Phases

1. Normal (weak coupling) Phase: After many transformations each having

$(a_0)' = \lambda a_0 \quad with \quad \lambda > 1$ the couplings will get weak. The system has now fallen into its normal phase in which there are only finite length correlations among the different parts of the system and every physical quantity is an analytic function of all couplings. Depending on how the renormalization is set up the system may end up with various different values of couplings, but they will all show the same sort of scale invariance: all correlations will extend over zero or finite distances.

2. Ordered (strong coupling) phase. This is what we get at a first order transition. At least one coupling gets stronger and stronger. Thus the system shows a behavior in which it is in an ordered phase and some correlations extend over an infinite distance. The are several different nearby phases (e.g. positive $\langle\sigma\rangle$ and negative). Which phase is realized depends upon such things as whether a particular coupling goes to plus infinity or minus infinity. In turn that may depend sensitively upon the initial couplings.

3. critical phase. characteristic of “second order” phase transition point. System is between order and disorder. Couplings approach a finite values, However these are several couplings (magnetic field) or ($T-T_c$) in which a small positive initial value may produce an entirely different outcome from a small negative value. If these special, “relevant” couplings are set to their critical values, then after many renormalizations the couplings approach some set of finite values, depending upon the RG transform. When that happens we say that the system has approached a “fixed point”, and argue that the fixed point represents a scale invariant critical behavior.

Stability

The nature of the phases are largely determined by stability considerations.

1. No transition: Weak coupling phase: quite stable. Any small changes in the coupling strength send the system to the same behavior.

2. First order transition: Strong coupling phase: unstable. Imagine a large value of K and a weak magnetic field in any dimension above one. Under renormalization any tiny magnetic field will approach $+\infty$ or $-\infty$ depending upon whether it is initially positive or negative. This instability and infinity is characteristic of a first order transition.

....Critical Phase....

3. critical phase. characteristic of "second order" phase transition point. System is between order and disorder. Couplings approach finite values (unstably). The word "unstable" means that if we vary the couplings slightly the eventual destination is vastly different from the one in the critical phase. If the temperature is slightly higher than the critical one, or if the magnetic field is slightly different from zero, the system approaches the weak coupling phase. On the contrary, if the magnetic field is zero but the couplings are slightly stronger than their critical values, the system approaches a strong coupling phase. Thus, the critical phase is very sensitive to tiny variations in the coupling.

For Example, d=2 Ising Model, $\sigma=\pm 1$

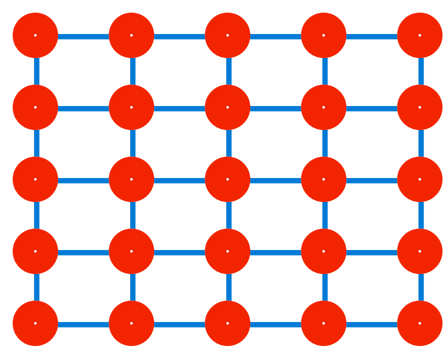
Spin in a magnetic field d-dimensions:

$$-\beta H\{\sigma\} = \sum_{<n.n>} K\sigma_r\sigma_s + \sum_r h_r\sigma_r$$

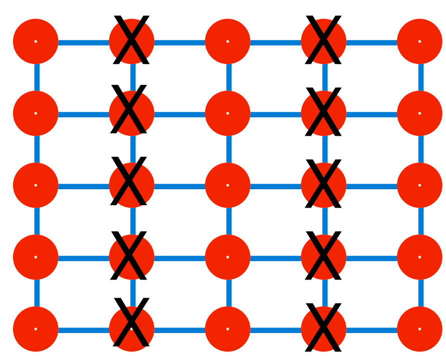
I want to describe the simplest transformation which gives the three kinds of fixed points described above. I describe an approximation due to A.A. Migdal, later analyzed by me. The basic approach is to take a square lattice with spins σ_{ij} and nearest neighbor coupling constants K_x in the x-direction and K_y in the y-direction. We work to construct the partition function

$$z = \sum_{\sigma} e^{-\beta H\{\sigma\}}$$

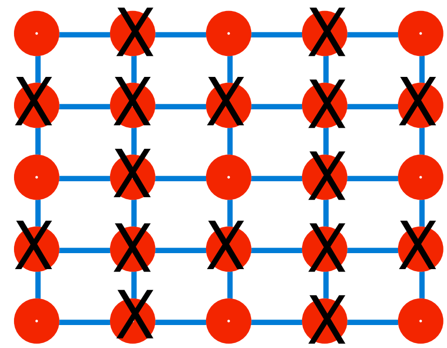
on a square lattice



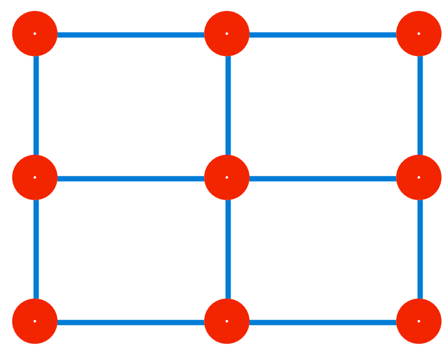
We first sum over all spins with odd values of i holding the ones with even values fixed



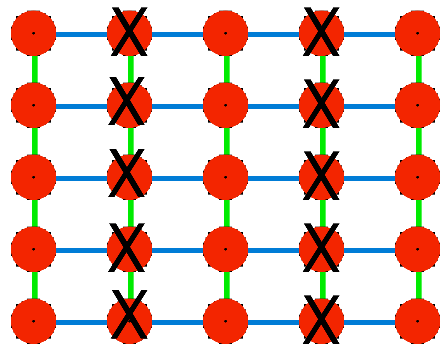
We then sum over all spins with even i and odd values of j holding the ones with even j and i values fixed.



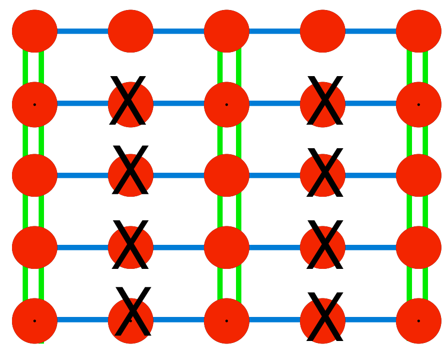
After this is all done we relabel the spins so as to represent a situation with doubled lattice constant.



However the green bonds connecting the x's prevent us from doing the sum



Move these bonds to equivalent positions where they won't harm our calculation. Do the calculation and find

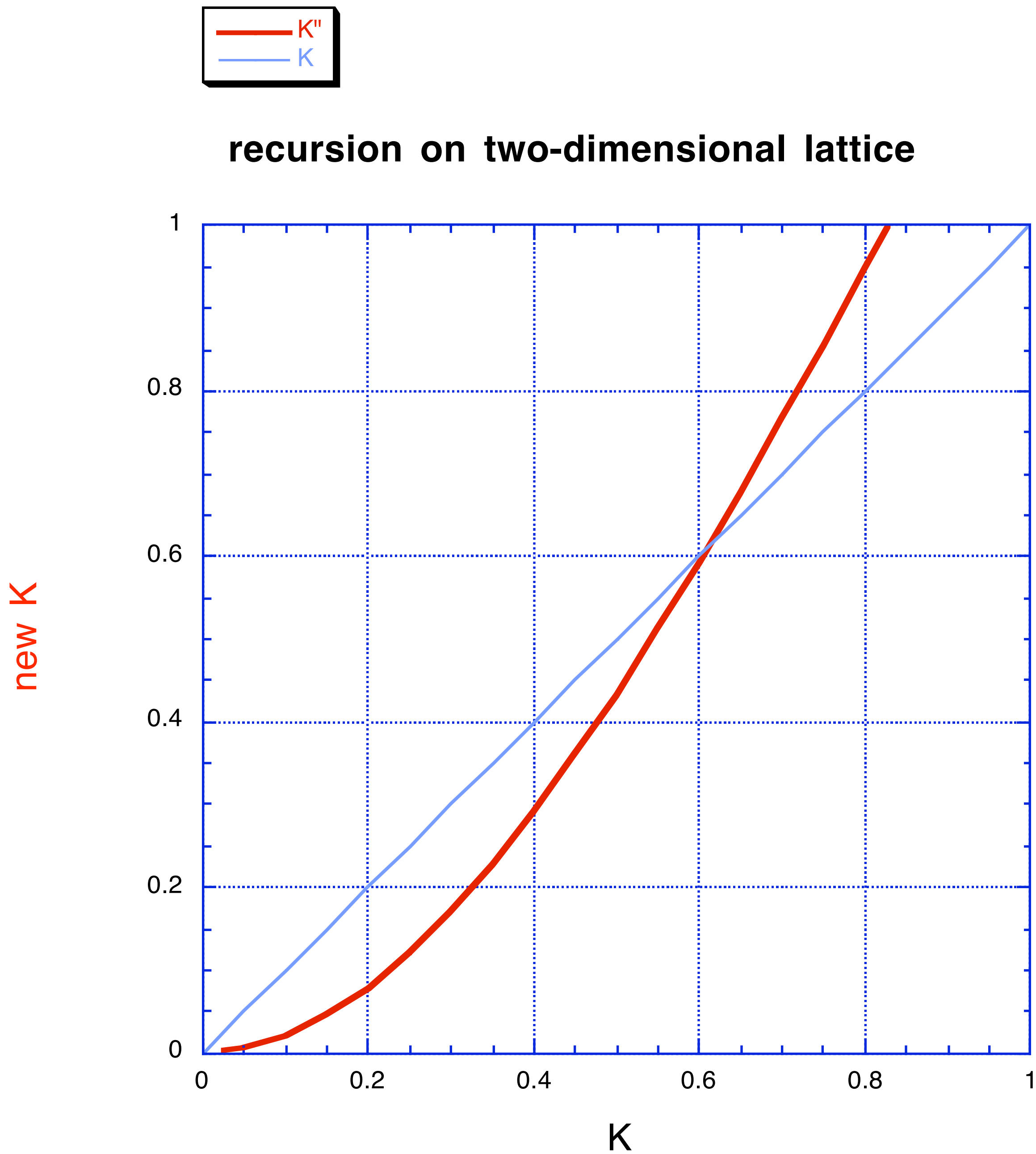


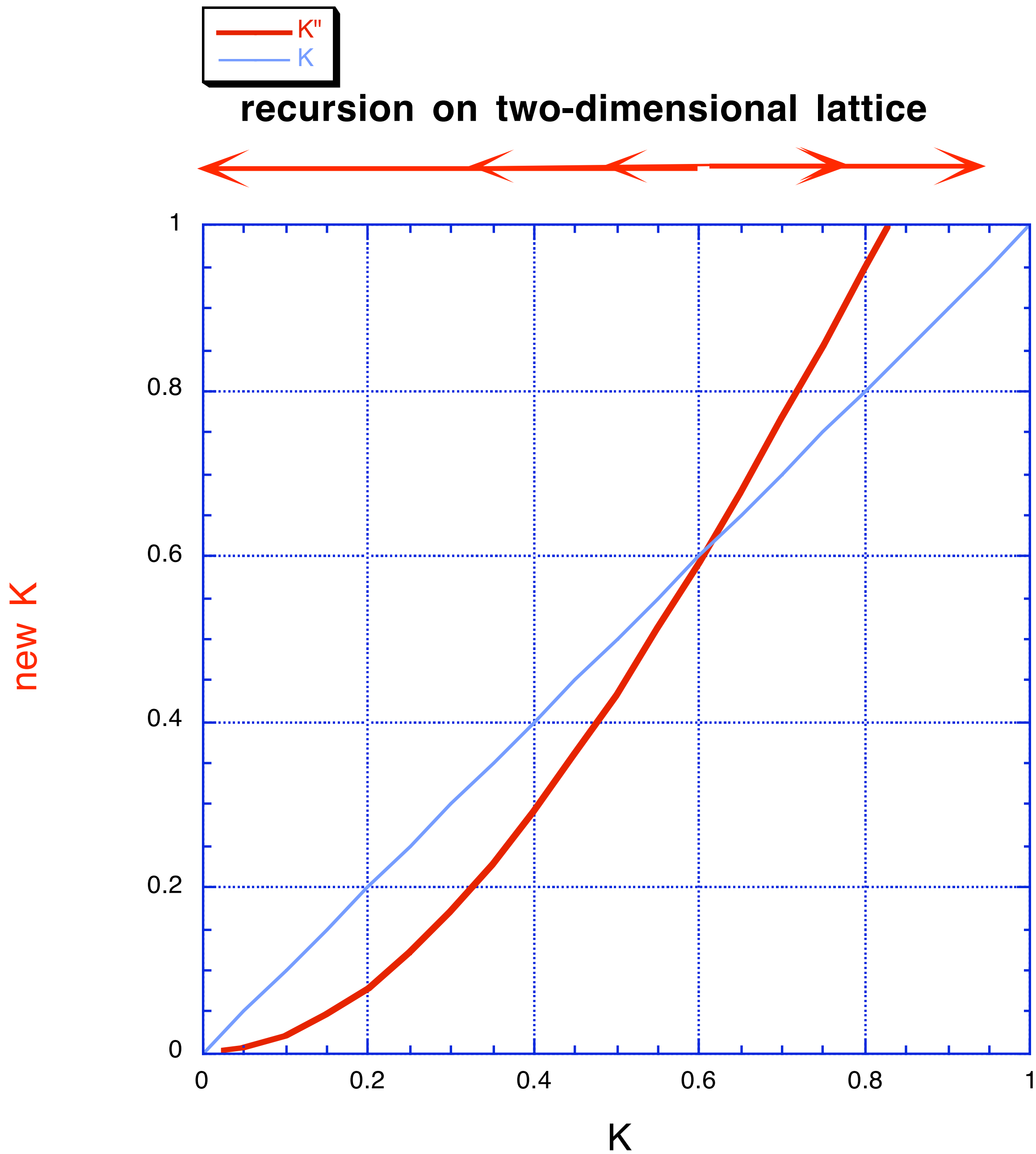
$$K_y' = 2K_y \quad and \quad K_x' = 0.5 \ \ln(\cosh 2K_x))$$

repeat in the other direction and find that the next set of couplings obey:

$$K_y'' = .5 \ \ln(\cosh 4K_y) \quad and \quad K_x'' = 2(.5 \ \ln(\cosh 2K_x))$$

Now let us draw a picture of what happens to these couplings.


K"
K
recursion on two-dimensional lattice
1
0.8
0.6
0.4
0.2
0
new K
0
0.2
0.4
0.6
0.8
1
K


K"
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recursion on two-dimensional lattice
1
0.8
0.6
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new K
0
0.2
0.4
0.6
0.8
1
K

Stability Near the Critical Point (again)

The system can approach any on the three critical points, weak couple, (K=0), strong coupling (K=∞) and critical ($K=K_C$). If you are near the weak coupling point, any little couplings that you put away will after several renormalization transforms, approach zero. Thus the system is completely stable against perturbations. Physically, we see no sign of remaining correlation at large distances. Near the critical coupling, the are two kinds of perturbations which can grow--a magnetic field perturbation--which breaks the essential spin flip symmetry--and a temperature perturbation, shown in the last plot which is a deviation from criticality which grows in each iteration of the RG. This transition is unstable against perturbations. The magnetization and energy variables conjugate to the growing perturbations have very long-ranged correlations. In the first order transition is at K approaches infinity and h=0. In this transition, a small magnetic field will grow and grow, eventually sending you away from the transition. It is known that this growth produces weak but long-ranged correlations of the magnetization variables.

Stability Near the Critical Point (again!)

There are thus three different kinds of behaviors for field variables near critical points:

1. Stable. They die away after many renormalizations. They are irrelevant to what happens at the critical point. Nothing much depends upon them.

2. Unstable. These variables grow exponentially with the number of renormalizations. So does the rescaled lattice constant length. Hence they each grow algebraically with the length. These variables, usually $(T-T_c)$ and magnetic field are set to zero so that we can be at the critical point. Because they are zero, no property of the critical point is directly affected by these fields.

Hence nothing affects the critical behavior. It is universal except for the cases in which there is a third kind of variable.

3. Marginal. Variables like this neither grow nor decay with successive renormalizations. They remain constant and cause a variability (non-universality) in critical behavior. They are rare, and do not appear in most problems.