Loewner Evolution

Maps and Shapes in two Dimensions

presented by

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I. Critical Phenomena
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Abstract

In statistical physics, a very considerable effort has gone into studying the shapes of two-dimensional objects. especially fractals. Shapes of critical clusters are important in percolation and phase transitions, and in many dynamical problems.

Conformal mappings in the complex plane provide another mechanism for producing shapes. This one is particularly natural for physics since analytic functions automatically obey Laplace's equation. This method has been directly used by Hastings and Levitov to study DLA.

Loewner devised a method to use conformal maps for constructing fractal line patterns. This method is described and explained. More recently Loewner's method has been used to solve shape problems in percolation, self-avoiding walks, and critical phenomena. These applications are described.

References

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B. Duplantier Conformally Invariant Fractals and Potential Theory, PRL. 84, 1363 (2000); M. B. Hastings and L. S. Levitov, Laplacian growth as one-dimensional turbulence, Physica D 116, 244 (1998); arXiv:cond-mat/9607021.

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Preface

Long ago, a bunch of us cracked the problem the theory of critical phenomena, or phase transition theory. We were able to describe how information about the system's local order could be transferred to one part of the system to another, and especially how this information could be transferred over long distances near the critical point. The description was in terms of local quantities like density $\rho(r)$ and magnetization $\sigma(r)$ at the point r and of correlations among these.

A typical problem was the Ising model in which there was a lattice and at each lattice site there was a spin- variable, $\sigma(r)$, which took on the values plus one and minus one. Spins at neighboring sites were coupled together so that they tended to line up. The local alignment was strong at low temperatures and weak at high temperatures.

The system has three phases: At high temperatures a disordered (paramagnetic phase) in which far distant points had no correlation. At low temperatures a ordered (ferromagnetic phase) in which correlation among spins extend over the entire system with a majority spins being (say) positive.

At some intermediate temperature there is a critical phase in which large islands (domains) of spins tend to be lined up, but with the alignment getting weaker and weaker average as the clusters get larger.

By 1975 or so we thought we were done. We understood how $<\sigma(\mathbf{r})>$ might depend upon the thermodynamic variables measuring distance from the critical point and how correlations like $<\sigma(\mathbf{r})$ $\sigma(\mathbf{s})>$ might behave.

We also understood something about the basic global symmetries in the problem: the usual rotational and translational invariance, the spin-flip symmetry. At the critical point we saw in addition universality (rather different problems showed identical large-scale behavior), scaling (the large scale behavior was the same at all scales), and conformal invariance (a slight mysterious symmetry under complex analytic transformations of the space).   So many of us went on to other things.

But the old problem was not dead. We had left behind a crucial issue. How can one characterize the shapes of the spin clusters that form in critical phenomena? We did not even realize that we had left something behind.

Percolation.

Probability p that a site will be occupied (black)
versus empty (red). Small p--small isolated clusters
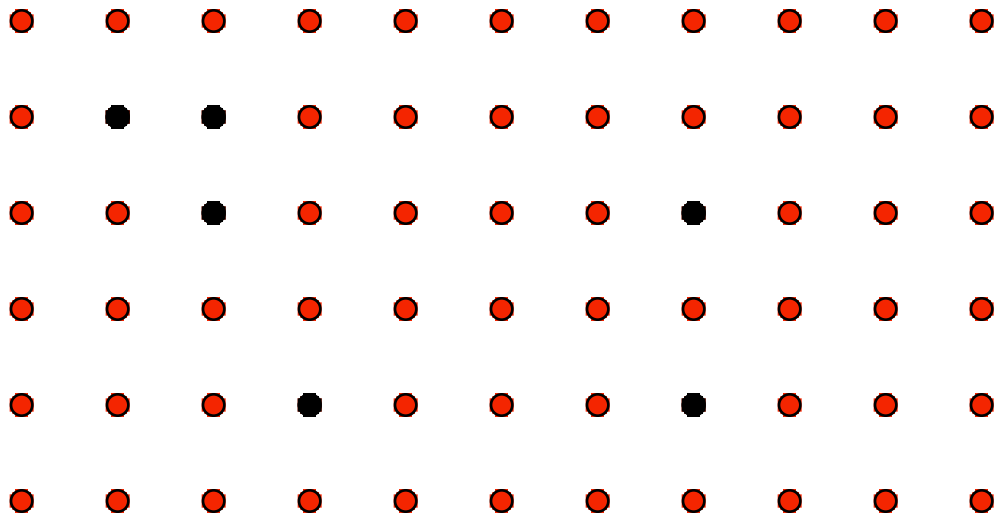
P close to one, same story. If p is close to one half,
lots of large clusters of both colors are formed
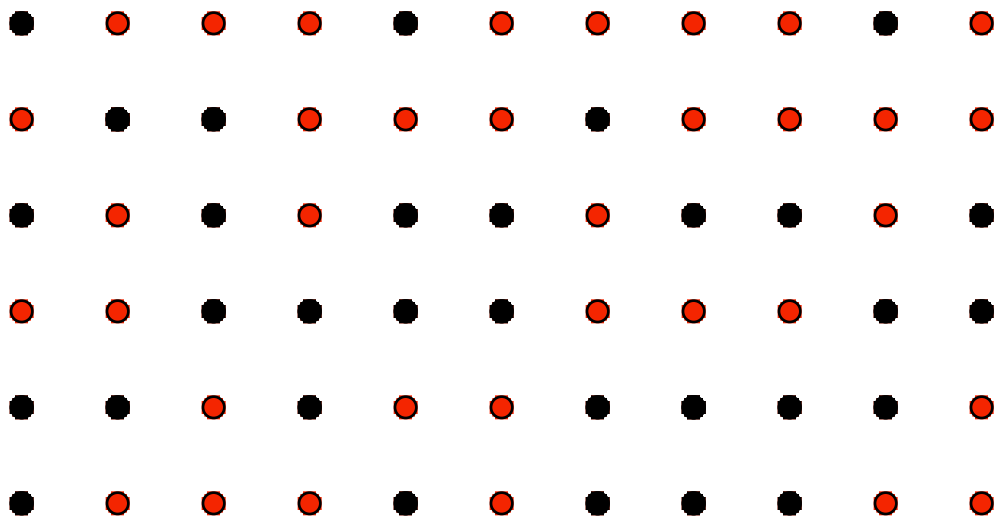
In 1981 Witten & Sander developed a dynamical model called DLA. This model was intended to construct scale-invariant, (fractal) objects. It did so, but despite a huge amount of work, we developed little understanding of the universality, scaling, or conformal properties of that model.

Late in the 20th century, a new point of view was developed which enabled one to calculate the shapes of "domains", i.e. correlated regions, in two dimensions. This point of view was somewhat mathematical in character and was in some sense invented by the mathematician Oded Schramm. I'll try to tell about it and how it fits in to the other two development. trans,trans.

On to DLA:

DLA is a model invented by Witten and Sander to describe how tiny bits of soot may come together and form one large, fractal aggregate. It was one of the first models to be initially expressed as a computer algorithm trans,trans.

Conformal Maps

We don't teach much complex analysis to physics students. So I'll say some elementary things here.

A point $\vec{r} = (x,y)$ can also be written as z=x+iy. A bunch of points define a shape. Take z=$e^{i\theta}$ = cos θ + i sin θ with θ between 0 and 2 π and you get a circle.


z
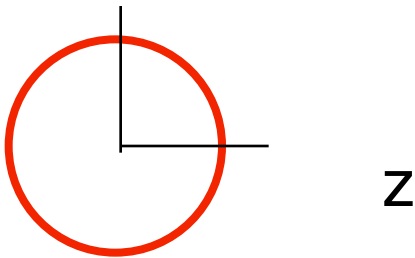
Functions of a complex variable automatically obey Laplace's equation at every point where there is no singularity. If w=1/z=1/(x+iy) =(x-iy)/(x^2+y^2) then we automatically know that

$$[(\partial_x)^2 + (\partial_y)^2] \, [x/(x^2+y^2)] \; =0$$

except at z=0.
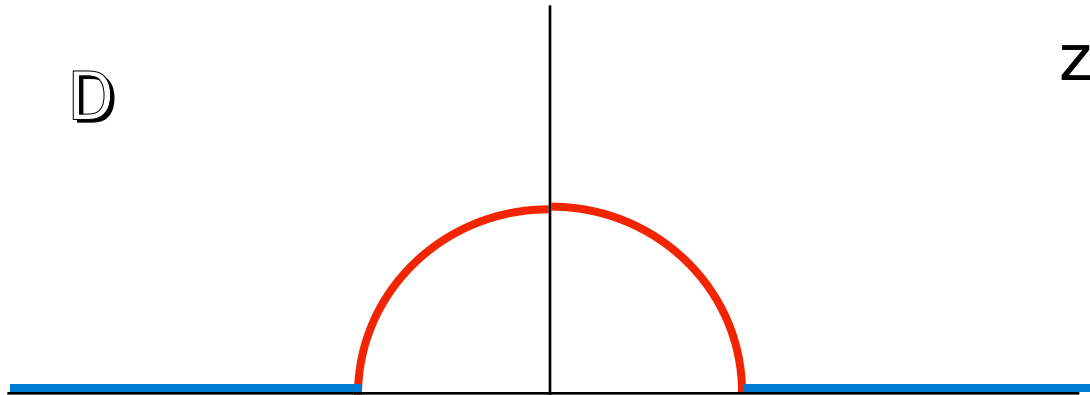
Here is another curve. It sits in the z-plane. Let w=f(z)=(z+1/z)/2. What do you get? Take the curvey part described by $z=e^{i\theta}$ with θ real and in $[0,\pi]$. Plug this into f. All these points together give a line segment, $w=\cos\theta$. Now take the right hand straight part given by $z=e^u, u>0$.   This gives a line segment, w=cosh u, extending from 1 to infinity. All together you get the shape shown below.
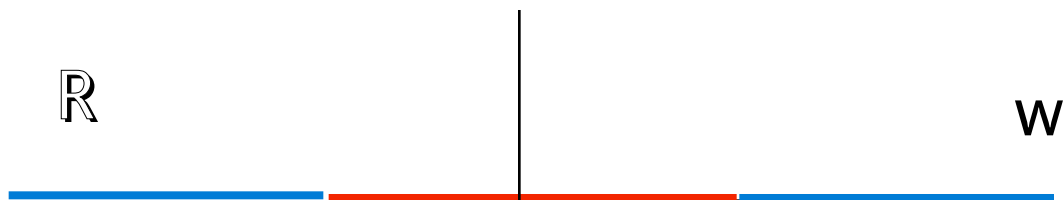
What we have just done? We have constructed a *conformal mapping*, which takes the colored curve in the z-plane into the line in the w-plane. It also takes the region above the curve in z, $\mathbb{D}$, into the region above the line in w, $\mathbb{R}$, in a one to one manner.

Definition: Conformal Map

A *conformal mapping*, is a function f(z) analytic within a simply connected region $\mathbb{D}$ of the complex z-plane, and which has the property that df/dz is never zero in $\mathbb{D}$. It then provides a one-to-one mapping of the interior of $\mathbb{D}$ into the interior of another simply connected region $\mathbb{R}$ and likewise maps the curves which bounds these regions into one another. Riemann proved that for finite regions the mapping is unique. To go backwards from $\mathbb{R}$ to $\mathbb{D}$ you use the inverse function g(w) which obeys g(f(z))=z. This too is unique.

In the complex plane analysis and geometry are the same thing.

Going Backwards

w=f(z)=(z+z⁻¹)/2   takes $\mathbb{D}$ into $\mathbb{R}$.
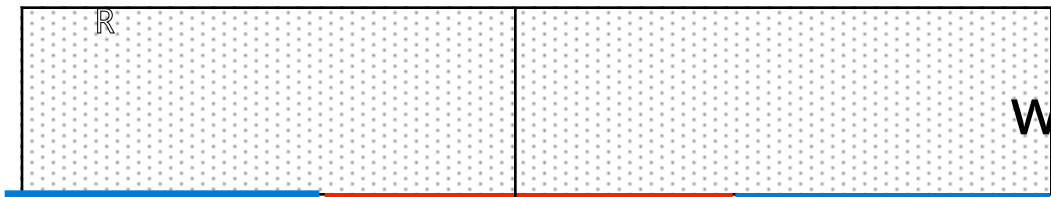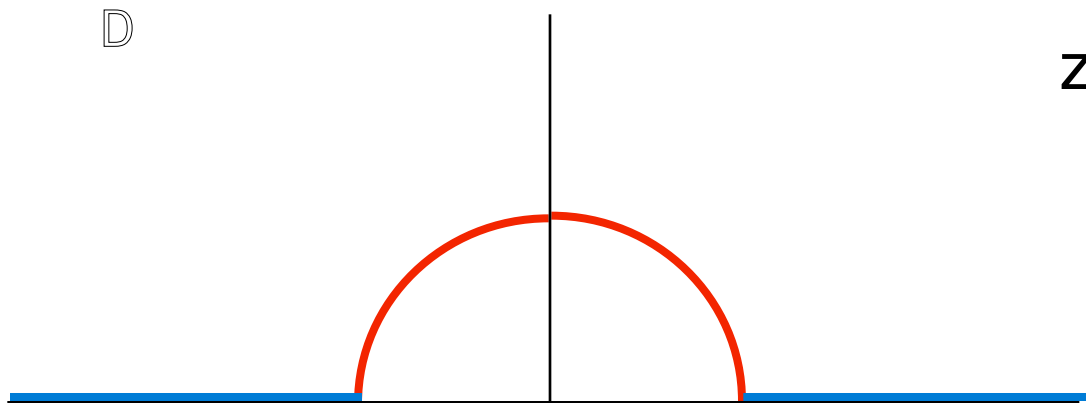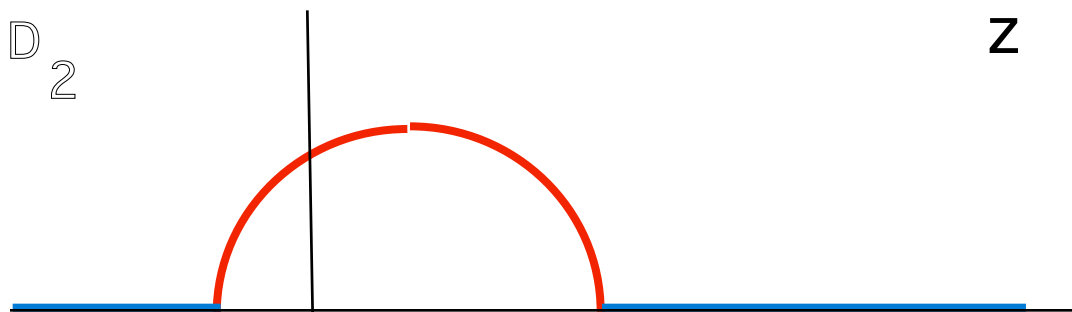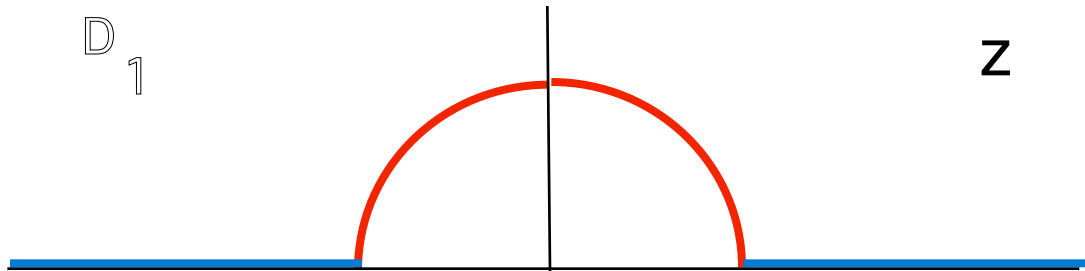

Solve for z. $z = w + \sqrt{w^2 - 1}$ z=g(w) Then g takes $\mathbb{R}$ into $\mathbb{D}$.

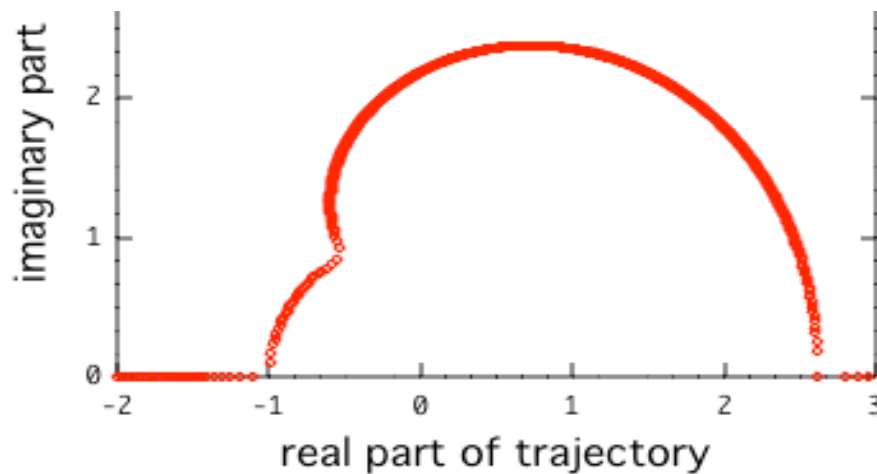Note that non-analyticity at w=1,-1, z=0 and vanishing derivatives z=1,-1 at all give interesting behavior.

Composition Properties

$w=f_1(z)=(z+z^{-1})/2$ takes $\mathbb{D}_1$ to $\mathbb{R}$.

$w=f_2(z)=[(z-0.5+(z-0.5)^{-1}]/2$ takes $\mathbb{D}_2$ to $\mathbb{R}$.

$\mathbb{D}_1$ z

$\mathbb{D}_2$ z

$w=f_2(f_1(z))=$ takes a larger region to $\mathbb{R}$.

Diffusion Limited Aggregation, Again

One can equally well form DLA aggregates by doing a bump map many many times. On picks a map of the form
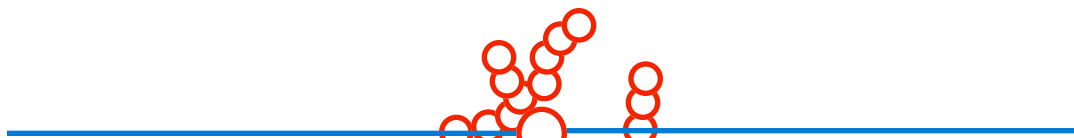
$$w = f_a(z) = (z - a + \frac{r^2}{z-a})/2,$$

which makes bumps with radius r and position a. (r and a must be real.) The one puts together a bunch of bumps by successively forming the maps one after the other:

$$F(z) = f_{a_5}(f_{a_4}(f_{a_3}(f_{a_2}(f_{a_1}(z)))))$$

One bump:

many bumps tend to clump

trans

Basic math behind this: (Hastings and Levitov) The probability for finding a random walker someplace is defined by a diffusion process, which thus obeys the discrete analog of

$$[(\partial_x)^2 + (\partial_y)^2]\, p(x,y)=0$$

with the relative probability, p(x,y), being zero on the aggregate. The actual probability of hitting is the value of the probability on a site next to the aggregate, which is proportional to the normal gradient of the probability at the aggregate.

Using complex analysis we know precisely how to construct such a function. Take our strange shaped object. Let the outside of that object be the region $\mathbb{R}$ in the z plane. Construct the map which takes $\mathbb{R}$ into the upper half of the w-plane. The map gives p: w=f(z)=f(x+iy)=q(x,y)+ip(x,y).

To check note that f(z) obeys the Laplace equation, is real on the boundary and has the right behavior at infinity.

Physical models:

DLA: bumps appear at random positions in w, all equal in size. (Hastings and Levitov)

Noise-reduced DLA: many hits in the neighborhood of a given w are required before a bump is raised. Chao Tang: result is different.

Other models: bump size =1. New bump is produced at a "distance" $\Delta w = \kappa^{0.5}$ from last bump and a random direction. This random walk produces fractals very different from DLA. Structure of fractals differs depending on value of κ (Oded Schramm had the basic insights, further developed by Lawler and Warner, and then Hastings translated them into this language.)

On to Loewner

There is a connection between the work of Hastings and Levitov and the much earlier work of C. Loewner who was studying the generation of singularities in conformal maps. He looked at the map $f_t(z)$ and asked himself how he could produce smooth deformations of this map. As part of this he turned to consideration of the equation

$$\frac{d}{dt} f_t(z) = \frac{2}{f_t - \xi(t)}$$ with the initial condition $f_0(z)=z$

Here f and z are complex, t and ξ are real. The time development of f(z), as generated by this equation is called the Loewner evolution.

This equation has been used to prove some important and fundamental propitious of analytic functions. However, we wish to look at it from a geometrical point of view: $w=f_t(z)$ is a mapping which takes you from a point z in region, $\mathbb{D}$, contained in the upper half of the z-plane to another region $\mathbb{R}$ which is the upper half of the w plane. The geometry of $\mathbb{D}$ is the primary focus of our interest.

At time zero the geometry is simple, f=z and $\mathbb{D}=\mathbb{R}$.

The simplest case

The simplest case, $\xi(t)=\xi(0)=\xi_0$, is quite instructive. The equation is: $df/dz=2/(f-\xi_0)$ or $d((f-\xi_0)^2)/dt=4$ so that $[f_t(z)-\xi_0]^2=4t+c(z)$ where c is a "constant" of integration. The initial condition, $f=z$ then, implies $f_t(z)-\xi_0 = \sqrt{(z-\xi_0)^2 + 4t}$. This solution has a branch cut, at which the square root changes sign, running up a line Re $z=\xi_0$ from $z=\xi_0$ to $z=\xi_0+2i\sqrt{t}$. The region $\mathbb{D}$ is the upper half plane less the branch cut.
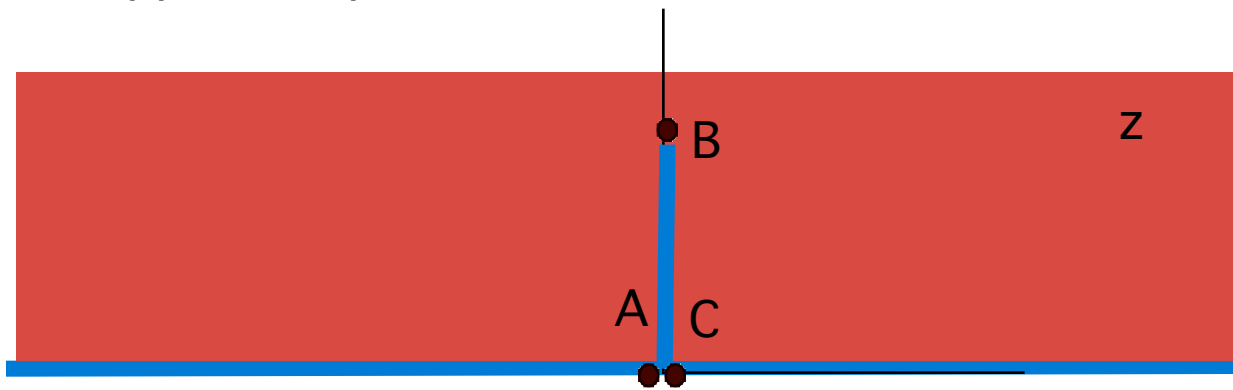
One can work with these line segments in just the same way as Hastings worked with bumps. One again the function of a function composition works to give more complex structures from simpler ones.
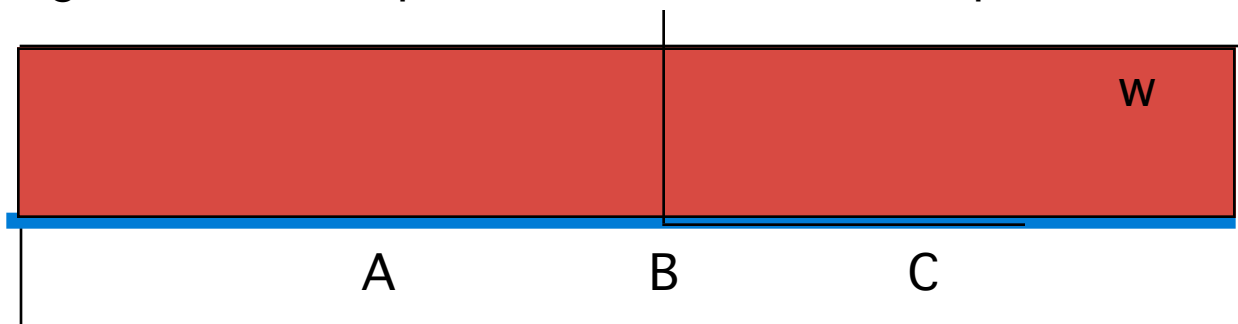
More On Loewner

What a bore! All this work and you get a line segment. And it gets worse. If $\xi(t)$ is any smooth curve with $\lim_{t \to s} \dfrac{\xi(t) - \xi(s)}{(t-s)^{1/2}} = 0$ then the geometrical object generated is just a curvey line. For example, for $\xi(t) = t$, the trace looks like trans

However, all this dullness is, if seen the right way, quite exciting.   The Loewner equation can be seen as a machine for producing a curve called a trace which depends upon the forcing function $\xi(t)$. This machine has quite fantastic qualitative properties, which depend almost entirely upon the singularity structure of $\xi(t)$. This is exciting in itself.

The form of the trace has simple covariance properties under the operation of translation, rotation, and scale change.

The curve in the plot is labeled $U(t) = t$.
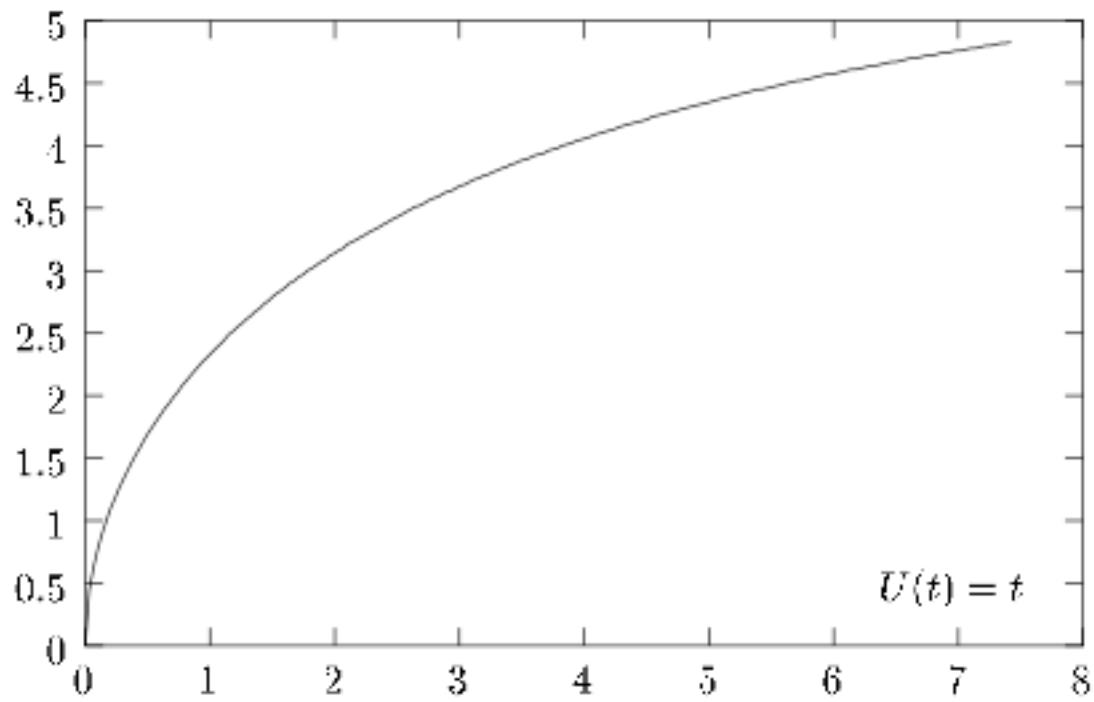
xy plot of trace

Wouter Kager   7/03

$\xi(t) = t$

For smooth forcing functions

The trace is non-self-interecting and does not intersect the real axis.

Essentially any such curve can be generated in this fashion

If the forcing is sufficiently singular, one can generate other shapes. If we have $\xi(t) = 2[\kappa(1-t)]^{\beta}$ then if $\beta < 1/2$ the trace will intersect itself.

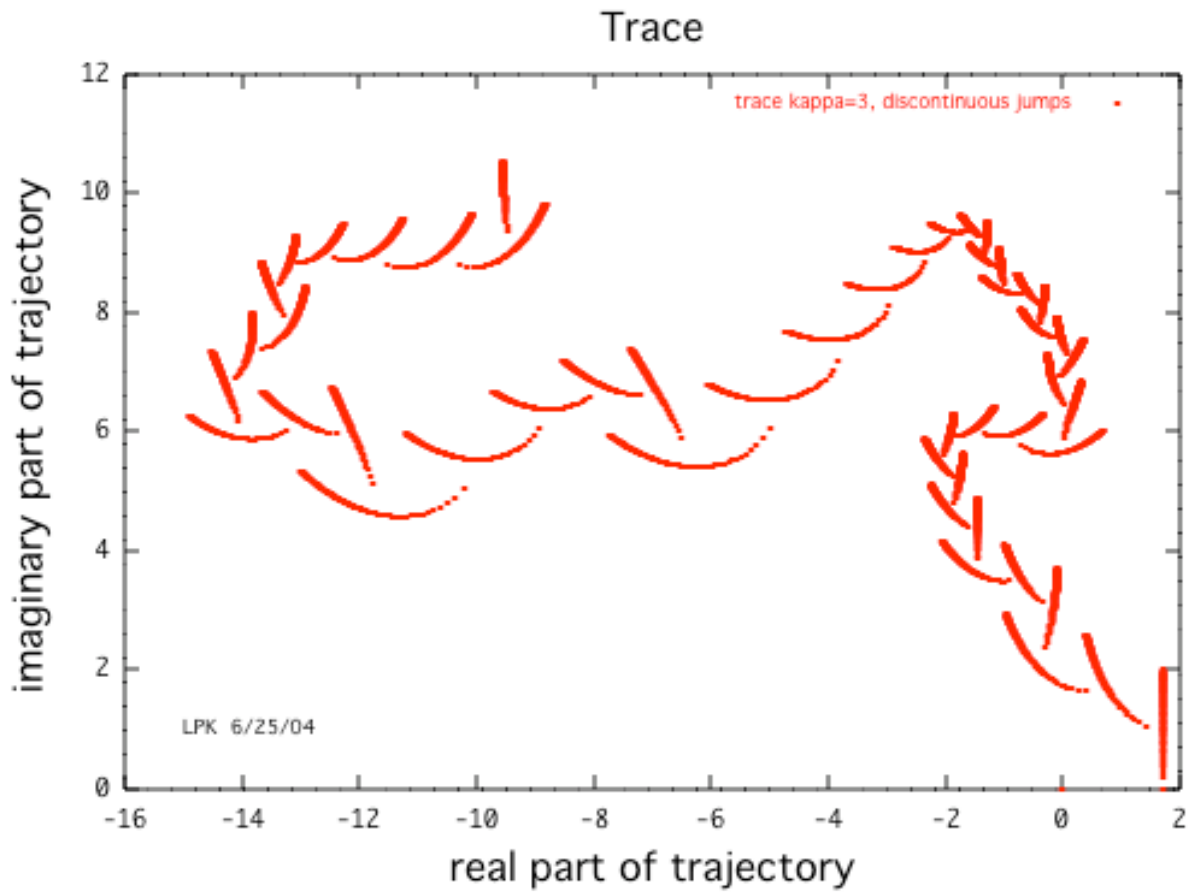•For $\beta = 1/2$ and $\kappa < 4$ we get a logarithmic spiral which circles around and infinite number of times. trans
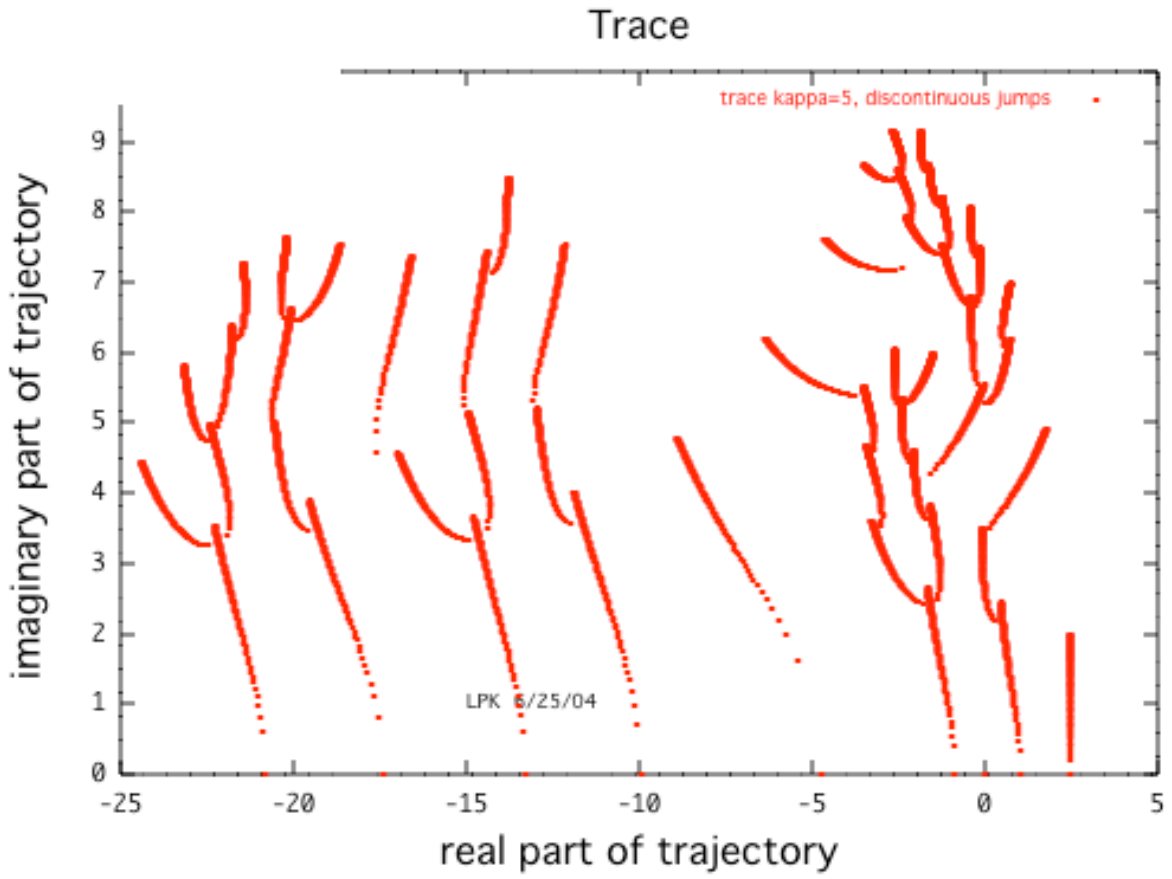
•For $\beta = 1/2$ and $\kappa > 4$ we get an intersection with the real line trans,trans,trans,

• For $\beta = 0$, i.e. discontinuous forcing, one can get discontinuous traces. trans,

We have had lots of fun with this*, but let me move on to other things

*coworkers: Bernard Nienhuis, Isabelle Claus, Wouter Kager, Panos Oikonomou, Marko Kleine Berkenbusch

Trace

trace kappa=3, discontinuous jumps

imaginary part of trajectory

real part of trajectory

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Trace

trace kappa=5, discontinuous jumps

imaginary part of trajectory

real part of trajectory

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On to Stochastic Evolution...

The real fun comes when the forcing function is stochastic. The first hint of this came from Hastings and Levitov and DLA.

To Stochastic Loewner Evolution... SLE

Look at the traces induced by the stochastic version of the Loewner equation in which the forcing is the Gaussian random process $\xi(t)=\sqrt{\kappa}\ B(t)$ where B represents normalized Brownian motion. Specifically
$$<[\xi(t)-\xi(s)]^2 > = \kappa\ |t-s|$$
There are several qualitatively different forms of the traces depending upon the values of κ.

The results are spectacular.

We have already seen that if we chose $\xi(t)$ to be a smooth real-valued function, the mapping $f_t(z)$ would give a conformal map from the upper half-plane, with a simple curve cut away. However because a Brownian walk contains small-scale singularities, we cannot be sure *a priori* what will happen. Indeed the RMS separation $[\xi(t)-\xi(s)]$ is $(\kappa\ |t-s|)^{1/2}$ which is just marginal for singularity production. What happens has been sorted out by some elegant mathematics, which shows that the result depends on κ.
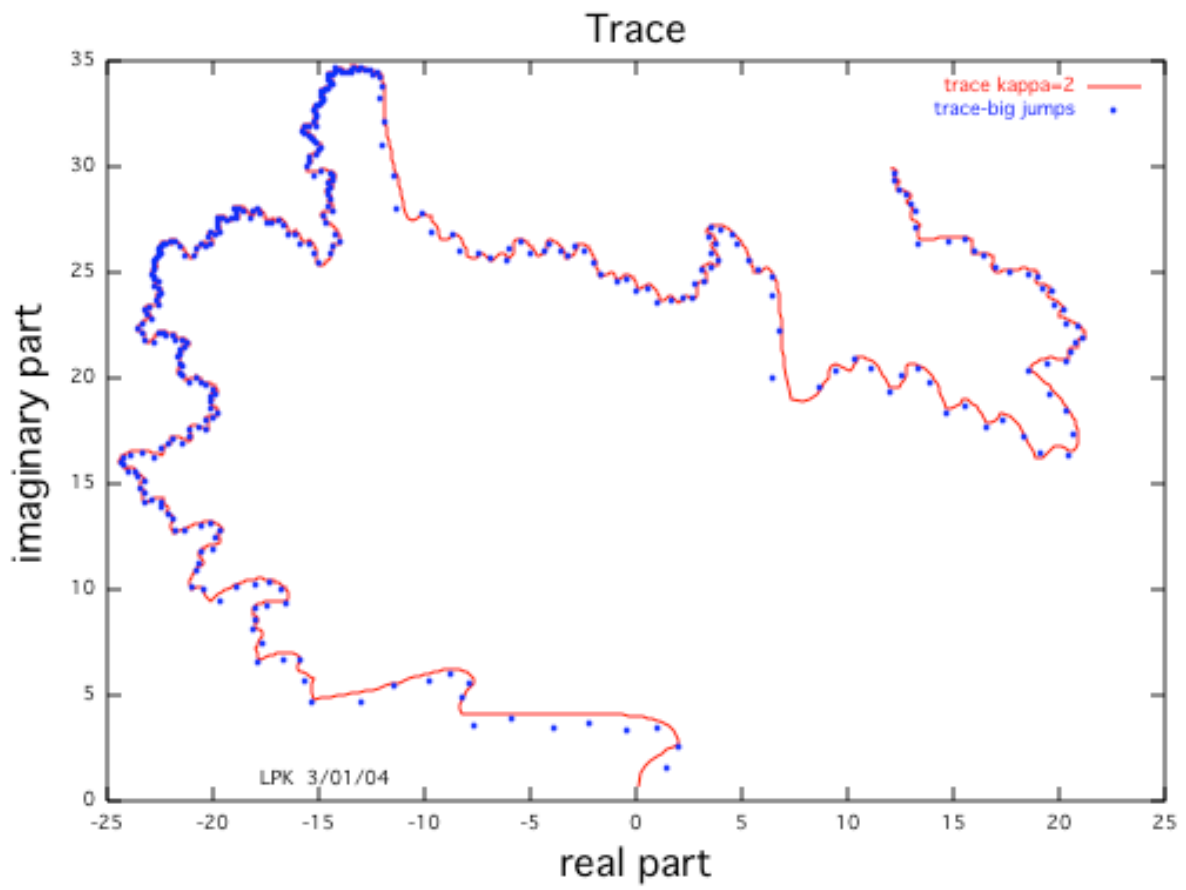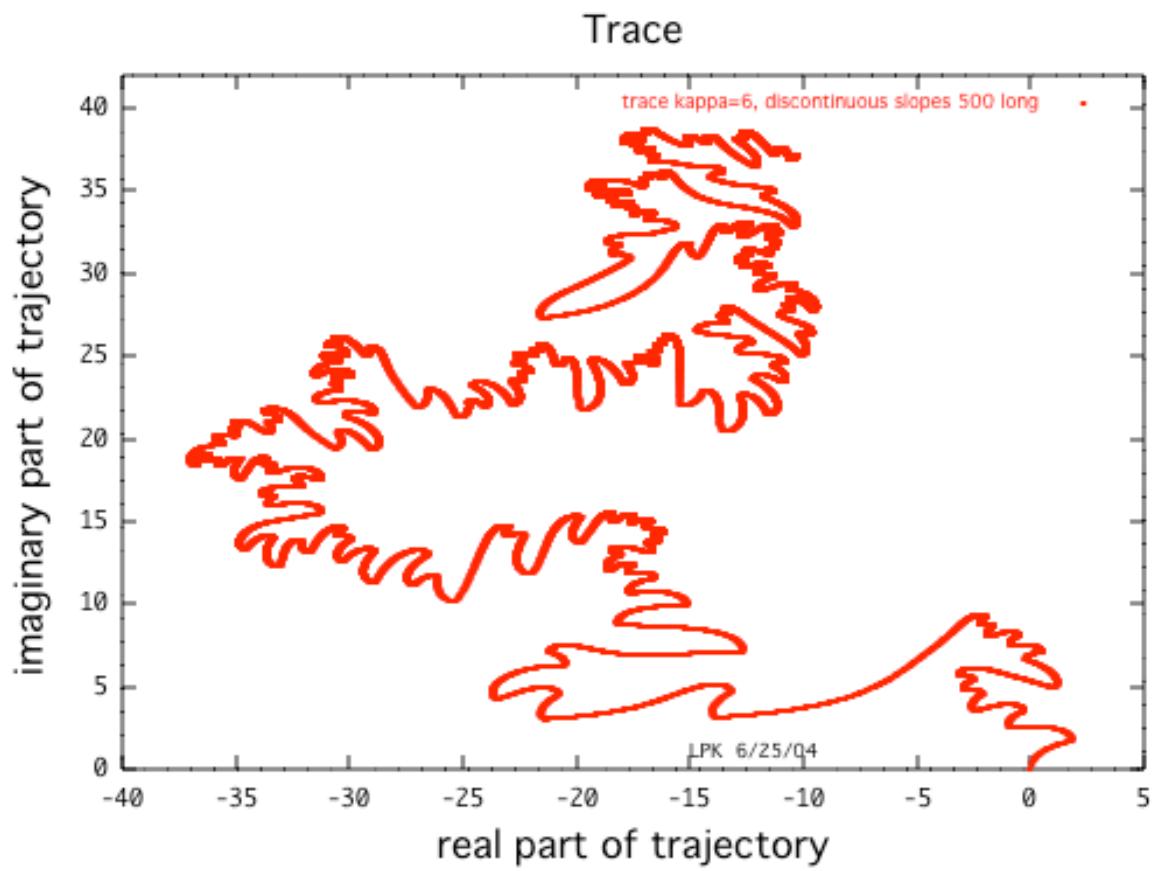
What Happens

is that the SLE process cuts away some portion of the z-plane. For $0 < \kappa \leq 4$ the SLE trace is a non-self-intersecting curve
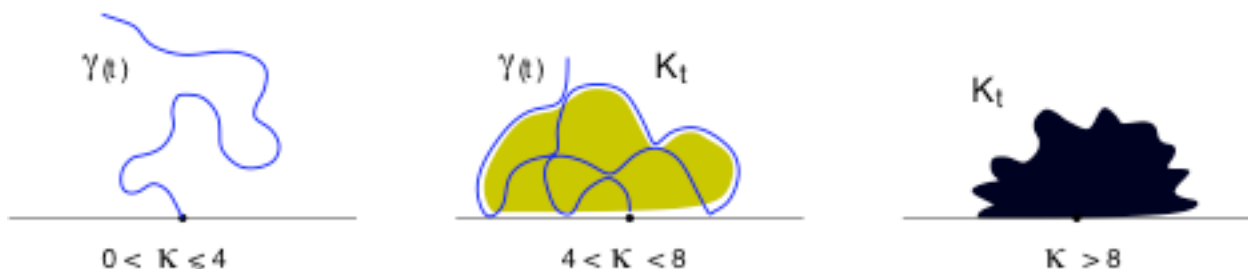
for $4 < \kappa \leq 8$ a self-intersecting one and for $8 < \kappa$ a filled in region of the plane. These geometrical objects turn out to be closely related to scale invariant objects formed in critical phenomena and other physically interesting scale-invariant processes.

• There is are theorems which show that the ensemble of all traces formed by different Brownian walks for $\kappa = 8/3$ have the same members with the same weights as the class of all self-avoiding walks. Or more precisely, the scaling limiting of the latter is identical to the SLE ensemble! Thus SLE, which is relatively easily analyzed, describes this walk-problem, which is not.

Trace

Trace

trace kappa=6, discontinuous slopes 500 long

LPK 6/25/04

imaginary part of trajectory

real part of trajectory

More problems solved

• Similarly, the percolation problem has a solution related to $\kappa = 6$ SLE. In the scaling limit, critical percolating clusters have the same ensemble of shapes as the exterior boundaries of the intersecting SLE paths.

These results are theorems. Next comes a bunch of very plausible speculations about the relation between SLE and the critical limit of phase transition problems. Concretely the speculation is that the ensemble of all SLE shapes give the shapes of the outline of clusters of the critical points of different phase transition problems. Specifically, $\kappa=3$ corresponds to the Ising model.

Finally Loewner evolution automatically give information about the electric field singularities which emerge when LE and SLE geometrical objects are taken to be charged conductors. Judging from DLA, these questions about charging are quite relevant to the deeper geometrical meaning of these objects. The work of Duplantier on the electrical properties of critical clusters actually came before SLE but it has a similar spirit.