Nonlinear σ model for disordered systems: an introduction

Igor Lerner

THE UNIVERSITY OF BIRMINGHAM

School of Physics & Astronomy

OUTLINE

- What is that & where to take it from?
- How to use it (1 or 2 simple examples)?
- Including interactions

3rd Windsor School on Condensed Matter Theory, 2004

What is that

$$F = \int \mathrm{d}^d r \, \mathrm{Tr} \left[\frac{\pi \nu D}{8} (\nabla Q)^2 + \frac{i \pi \nu \omega}{4} \Lambda Q \right]$$

Nonlinearity condition:

$$Q^2 = 1 \qquad \mathrm{Tr}\, Q = 0$$

How to get here – and where from?

How to use this?

Why to be bothered?

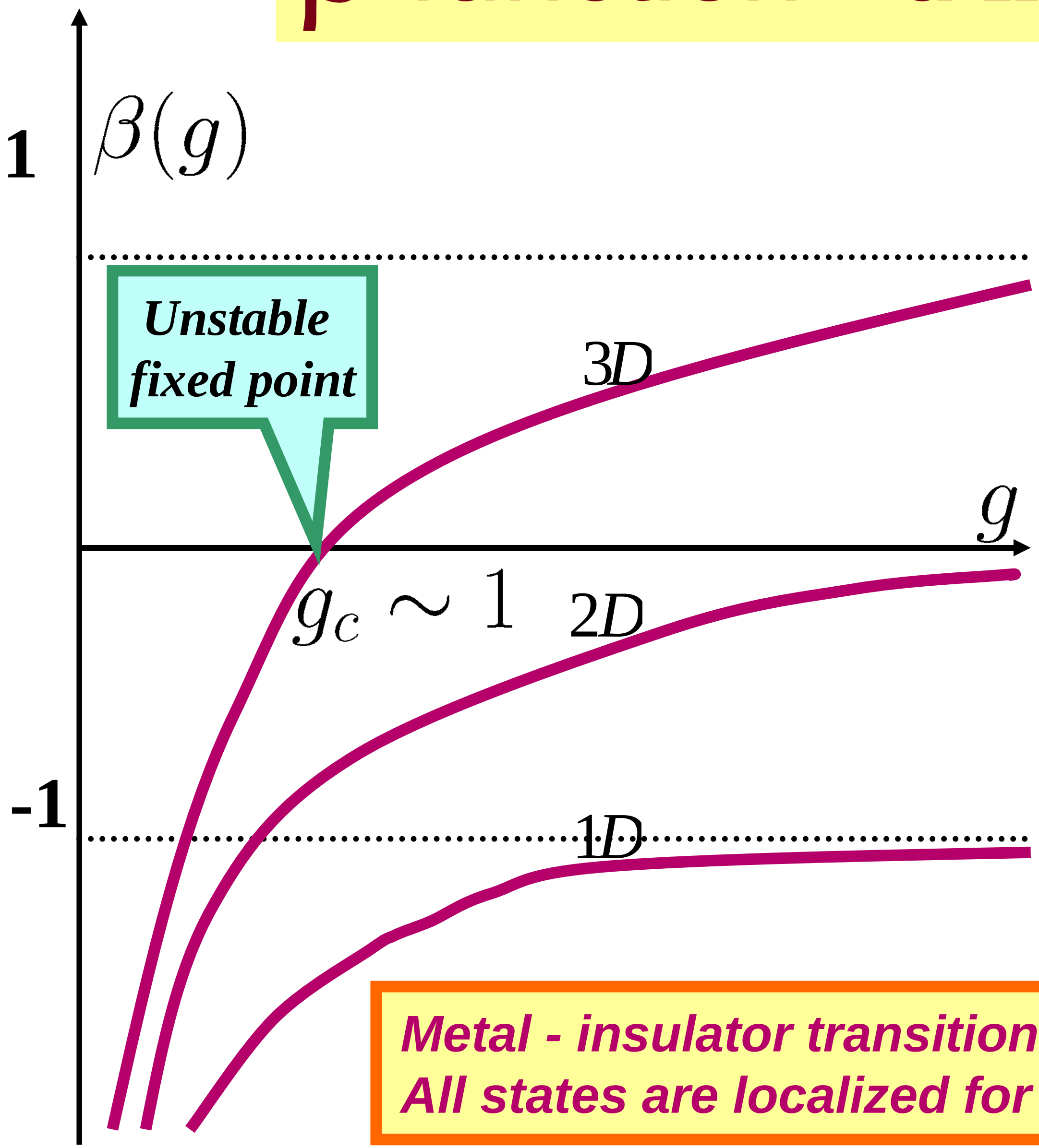
Why to be bothered?

The first application (1979-1981):

putting some foundation under the Gang-of-Four* speculations

***Abrahams, Anderson, Licciardello, and Ramakrishnan, 1979**

β-function ≡ d ln*g*/d ln*L*



Pert. Theory in 2D:

$g/g_0 = 1 - g_0^{-1} \ln L/\ell$

$+A\, g_0^{-2} \ln^2 L/\ell$

RG within the NLσM:

$g/g_0 = 1 - g_0^{-1} \ln L/\ell$

$+\ldots A_n\, g_0^{-n} \ln^n L/\ell$

$+A\, g^{-4} \ln L/\ell + \ldots$

Metal - insulator transition in 3D
All states are localized for 2D and 1D

And more …

NLσM describes, apart from g(L) itself

- mesoscopic distribution functions (of UCF *etc*)
- correlations of energy levels and wavefunctions
- long-time asymptotics of different observables
- statistics of rare events

and changing of the above with increasing disorder.

Also, it's a natural tool for describing some non-perturbative (ie non-analytic in g^{-1}) effects

- and more …

Where to get it from?

Starting Point: The TOE model

THE THEORY OF EVERYTHING

$$H = \sum_i \frac{p_i^2}{2m} + \sum_i V_i + \frac{1}{2}\sum_{ij} U_{ij}$$

interactions

lattice +disorder

A field-theoretical approach :

- addressing only low-energy modes
- averaging over *weak* disorder

Start with a toy model (to get to the TOE one)

A generic, after diagonalization, 1-particle Hamiltonian

$$H = \sum_{\alpha} \varepsilon_\alpha a^\dagger_\alpha a_\alpha \quad \Rightarrow \quad H\left|\alpha\right\rangle = \varepsilon_\alpha \left|\alpha\right\rangle$$

Green's function: $(\varepsilon - H)\, G = I$

$$\hat{G} = \left(\varepsilon - \hat{H}\right)^{-1} \sum_\alpha \left|\alpha\right\rangle\left\langle\alpha\right| = \sum_\alpha \frac{\left|\alpha\right\rangle\left\langle\alpha\right|}{\varepsilon - \varepsilon_\alpha}$$

$$G^{\pm}_{\varepsilon}(\mathbf{r},\mathbf{r}') \equiv \left\langle\mathbf{r}\right|\hat{G}^{\pm}\left|\mathbf{r}'\right\rangle = \sum_\alpha \frac{\left\langle\mathbf{r}\,|\,\alpha\right\rangle\left\langle\alpha\,|\,\mathbf{r}'\right\rangle}{\varepsilon - \varepsilon_\alpha \pm i\delta} \equiv \sum_\alpha \frac{\varphi^*_\alpha(\mathbf{r})\varphi_\alpha(\mathbf{r}')}{\varepsilon^{\pm} - \varepsilon_\alpha}$$

Gaussian Integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{i(\varepsilon^+ - \varepsilon)x^2} = \sqrt{\frac{\pi}{i(\varepsilon^+ - \varepsilon)}} \quad \Rightarrow \quad \frac{1}{\varepsilon^+ - \varepsilon} = \int \frac{\mathrm{d}c^* \mathrm{d}c}{2\pi}\, \mathrm{e}^{ic^*(\varepsilon^+ - \varepsilon)c}$$

Represent $\det(\varepsilon - H)^{-1} = \Pi_\alpha (\varepsilon - \varepsilon_\alpha)^{-1}$ as a Gaussian integral:

$$\det(\varepsilon - H)^{-1} = \prod_\alpha \int \frac{\mathrm{d}c_\alpha^* \mathrm{d}c_\alpha}{2\pi}\, \mathrm{e}^{ic_\alpha^*(\varepsilon^+ - \varepsilon_\alpha)c_\alpha}$$

$$\equiv \int \mathcal{D}c^* \mathcal{D}c\, \mathrm{e}^{i \sum_\alpha c_\alpha^*(\varepsilon^+ - \varepsilon_\alpha)c_\alpha}$$

$\mathcal{D}c^*\mathcal{D}c$ is a symbolic notation for the product over all $\mathrm{d}c_\alpha$

Functional Integral

Transform the exp:

$$\sum_{\alpha} c_\alpha^*(\varepsilon^+ - \varepsilon_\alpha)c_\alpha = \sum_{\alpha\beta} c_\alpha^*(\varepsilon^+ - \varepsilon_\alpha)c_\beta \int \mathrm{d}^d r \varphi_\alpha^* \varphi_\beta$$

$$= \int \mathrm{d}^d r \underbrace{\sum_\alpha c_\alpha^* \varphi_\alpha^*}_{\psi^*(\mathbf{r})} \left(\varepsilon^+ - \hat{H}\right) \underbrace{\sum_\beta c_\beta \varphi_\beta}_{\psi(\mathbf{r})} \equiv \underbrace{\int \mathrm{d}^d r\, \psi^*(\mathbf{r}) \left(\varepsilon^+ - \hat{H}\right) \psi(\mathbf{r})}_{\text{“Action” } S^+}$$

“Partition function”:

$$Z^+ \equiv \det\left(\varepsilon^+ - \hat{H}\right)^{-1} = \int \mathcal{D}\psi^* \mathcal{D}\psi \, \mathrm{e}^{iS^+}$$

Integration over “all fields” means integration over all c & c^*

Green's Functions

obtained by variable's shift in a Gaussian integral: $x \to x + h/\alpha$

$$\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \mathrm{e}^{-\alpha x^2 + 2hx} = \mathrm{e}^{h^2 \alpha}$$

Apply to the functional integral, a product of Gaussian ones:

$$Z_h^+ \equiv \int \mathcal{D}\psi^* \mathcal{D}\psi \mathrm{e}^{iS + i(\psi^* \cdot h + h^* \cdot \psi)} = Z^+ \mathrm{e}^{-ih^* \cdot G^+ \cdot h}$$, using

$$\psi \mapsto \psi - G^+ \cdot h \qquad \psi^* \mapsto \psi^* - h^* \cdot G^+ \qquad (\varepsilon^+ - \hat{H}) G^+ = I$$

Thus,

$$\hat{G} = \frac{i}{Z} \frac{\partial^2 Z_h}{\partial h \, \partial h^*} \bigg|_{\substack{h = 0 \\ h^* = 0}} = \frac{i}{Z} \int \psi^* \psi \, \mathrm{e}^{iS} \mathcal{D}\psi^* \mathcal{D}\psi$$

Fermions do it like Bosons

… but upside down

$$Z^+ \equiv \det\left(\varepsilon^+ - \hat{H}\right) = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \, \mathrm{e}^{iS^+}$$

$$\hat{G} = \frac{i}{Z} \int \overline{\psi}\psi \, \mathrm{e}^{iS} \mathcal{D}\overline{\psi}\mathcal{D}\psi \qquad S = \int \mathrm{d}^d r \overline{\psi}(\mathbf{r})(\varepsilon - H)\psi(\mathbf{r})$$

For our 1-particle toy model both fermionic and bosonic representations are equivalent.

Having the TOE model as a target, we will deal with fermions from now on.

From the Toy to Anderson model

by diagonalizing

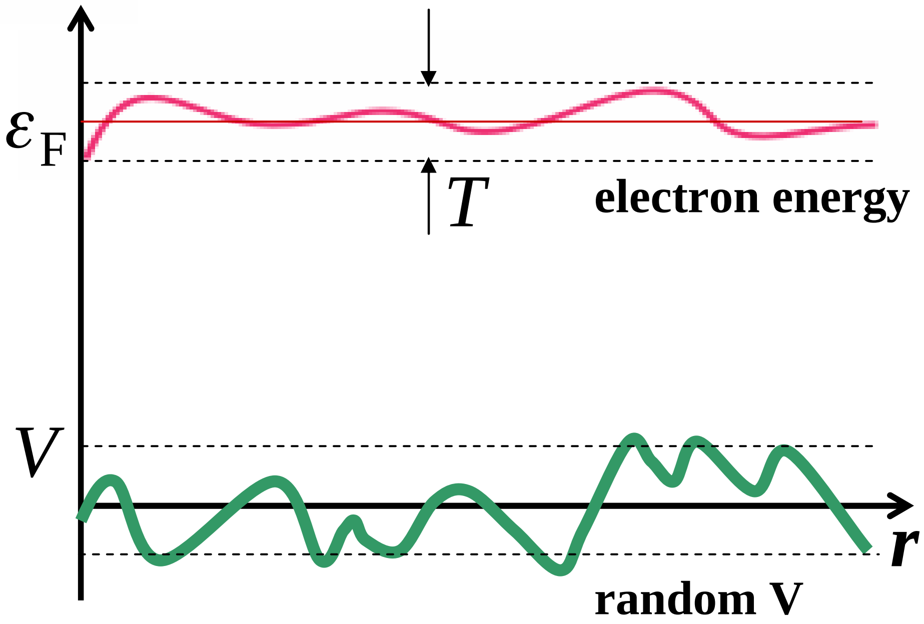
$$H = \sum_\alpha \varepsilon_\alpha a^\dagger{}_\alpha a_\alpha \quad \Longleftarrow \quad H = \sum_i \varepsilon_i a_i^\dagger a_i + J \sum_{\langle ij \rangle} a_i^\dagger a_j$$

Gaussian white noise:

$$\langle V(\mathbf{r}) V(\mathbf{r}') \rangle = \frac{1}{2\pi\nu\tau_{\mathrm{el}}} \delta(\mathbf{r} - \mathbf{r}') \quad \Longleftarrow \quad H = \frac{\hat{p}^2}{2m} + V(\mathbf{r})$$

A typical value of a short range ($r_c \sim \lambda_F$) potential is

$$V^2{}_{\mathrm{typ}} \sim \frac{1}{2\pi\nu{\lambda_F}^d\tau_{\mathrm{el}}} = \frac{{\varepsilon_F}^2}{2\pi\varepsilon_F\tau_{\mathrm{el}}}$$



Averaging Green's Function(s)

$$\langle G \rangle = \int G_V P(V) \mathcal{D}V$$

with the white-noise Gaussian potential of the previous slide

Explicit form of the Gaussian distribution:

$$P\{V(\mathbf{r})\} \sim \exp\left[-\pi\nu\tau_{\mathrm{el}} \int \mathrm{d}^d r V^2\right]$$

Gaussian integration of G_V which is exp of V would be straightforward, if only G_V were not a fraction:

$$G_V = \frac{\int \overline{\psi}\psi\, \mathrm{e}^{iS} \mathcal{D}\overline{\psi}\mathcal{D}\psi}{\int \mathrm{e}^{iS} \mathcal{D}\overline{\psi}\mathcal{D}\psi}$$

$$S = \int \mathrm{d}^d r \overline{\psi}(\mathbf{r}) \left[\varepsilon - \hat{\varepsilon}_p - V(r)\right] \psi(\mathbf{r})$$

Replica Method (or Trick)

$$\overline{\psi} \mapsto (\overline{\psi}_1, \ldots, \overline{\psi}_n) \equiv \overline{\Psi}$$

n "replicas" of the field

Then

$$G = \frac{i}{n} \frac{\int \overline{\Psi} \cdot \Psi \, \mathrm{e}^{iS_n} \mathcal{D}\overline{\Psi} \mathcal{D}\Psi}{Z_n}, \qquad Z_n = Z_1^n$$

As $Z_n \to 1$ as $n \to 0$, in this "replica" limit the denominator of G can be averaged independently of the numerator:

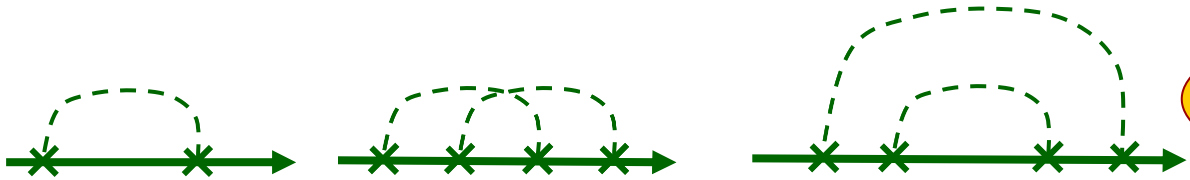
$$\langle G \rangle = \lim_{n \to 0} \frac{1}{n} \frac{\langle \ldots \rangle_V}{\langle \ldots \rangle_V}$$

The averaged 1-particle G is trivial, but $\langle GG \rangle$ is as easy:

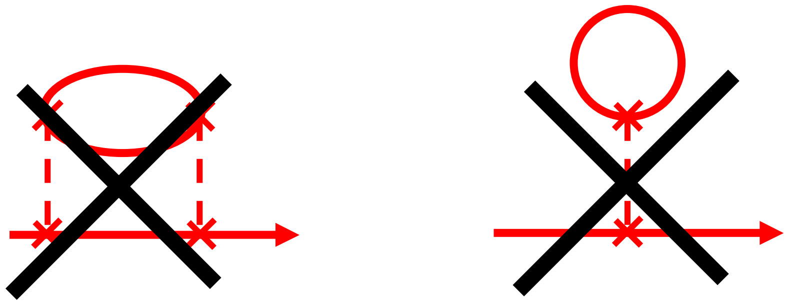
$$\left\langle G^+(\varepsilon + \tfrac{1}{2}\omega) G^-(\varepsilon - \tfrac{1}{2}\omega) \right\rangle = \lim_{n \to 0} \frac{1}{n^2} \frac{\langle \ldots \rangle_V}{\langle \ldots \rangle_V}$$

Why it works ?

Perturbatively, this is a method, not a trick. One expands ***S*** to get diagrams to be compared with those of direct diagrammatic technique

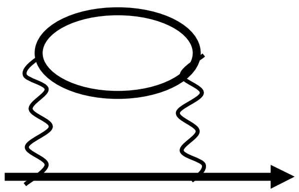


exist in the impurity techniques



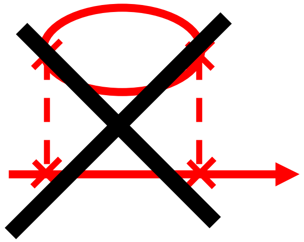
exist in QFT but not in the impurity techniques

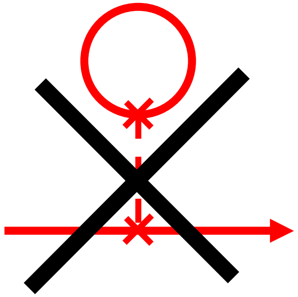
Killed by the replica trick: each closed loop has an extra ***n***



An interaction loop does survive

Alternatives





ALSO KILLED IN SUSY TECHNIQUES AS FERMION AND BOSON LOOPS HAVE OPPOSITE SIGNS

NEVER APPEAR IN KELDYSH TECHNIQUES

Why Replicas?

- SUSY – by far the best for non-perturbative calculations for non-interacting electrons **cannot** be generalised for interactions in any meaningful way
- Keldysh techniques would probably be better but replicas are considerably easier

How it works?

One just calculates the Gaussian integral:

$$\langle\ldots\rangle_V \mapsto (\ldots)\int \mathcal{D}V\, \mathrm{e}^{-i\overline{\Psi}\cdot V\cdot\Psi - \pi\nu\tau_{\mathrm{el}}V^2} = (\ldots)\exp\left[-\frac{(\overline{\Psi}\cdot\Psi)^2}{4\pi\nu\tau_{\mathrm{el}}}\right]$$

Thus, one arrives at a quartic in Ψ action S. Particularly, in calculating the product $\left\langle G^+(\varepsilon+\frac{1}{2}\omega)G^-(\varepsilon-\frac{1}{2}\omega)\right\rangle$ the action is

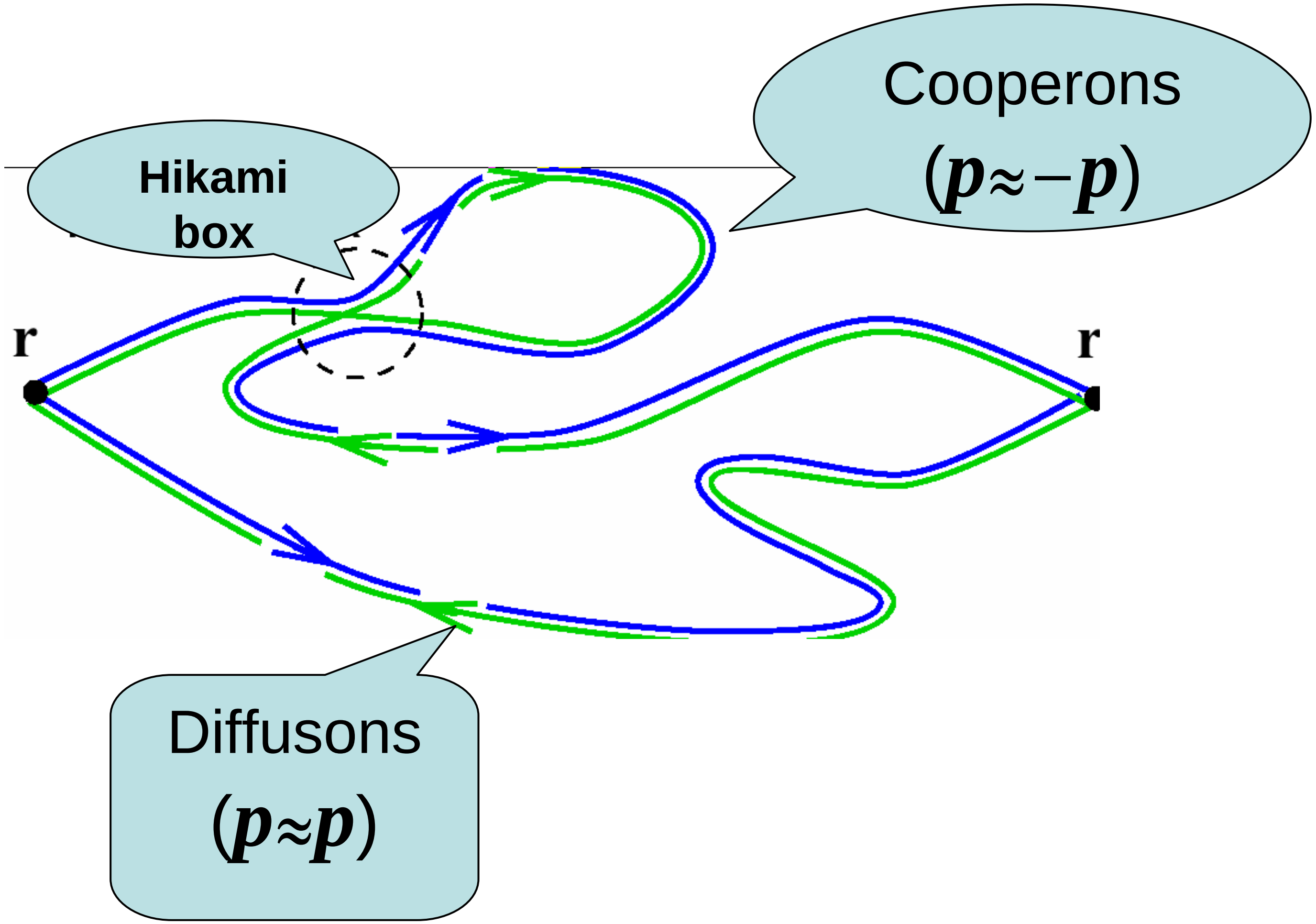
$$S = \int \mathrm{d}^d r \left\{ \overline{\Psi}\left(-\hat{\xi}+\frac{1}{2}\omega\Lambda\right)\Psi - \frac{1}{4\pi\nu\tau_{\mathrm{el}}}\left(\overline{\Psi}\cdot\Psi\right)^2 \right\}$$

$\xi \equiv \varepsilon_{\mathbf{p}} - \varepsilon_{\mathrm{F}}$, $\Lambda = \mathrm{diag}(\mathbb{1}_n, -\mathbb{1}_n)$, and Ψ is 2n-component:

$$\overline{\Psi} = \left(\overline{\psi}_1^+, \ldots, \overline{\psi}_n^+, \overline{\psi}_1^-, \ldots, \overline{\psi}_n^-,\right)$$

Slow Modes

The full action is too complicated. A proper FT describes interacting slow modes that should be extracted from $\left(\overline{\Psi} \cdot \Psi\right)^2$



Hubbard-Stratonovich transfomation

$q \equiv \frac{2}{\pi\nu}\psi \otimes \psi^\dagger$ – the "disorder" field that includes both slow channels: cooperon and diffuson

To get rid of $-\frac{1}{4\pi\nu\tau_{\rm el}}\left(\overline{\Psi}\cdot\Psi\right)^2 \propto -\mathrm{tr}\, q^2$, use

$$\int_{-\infty}^{\infty} e^{-Q^2+2qQ} dQ = e^{-q^2} \int_{-\infty}^{\infty} e^{-Q^2} dQ$$

Applying to the matrix field q above gives

$$Z = \int e^{iS}\, \mathcal{D}Q\, \mathcal{D}\overline{\Psi}\, \mathcal{D}\Psi$$

Q is $2n$x$2n$ matrix field and

$$iS \mapsto \int d^d r \mathrm{Tr}\left\{-\frac{\pi\nu}{8\tau_{\rm el}}Q^2 + i\overline{\Psi}\left[-\hat{\xi} + \tfrac{1}{2}\omega\Lambda + \frac{i}{2\tau_{\rm el}}Q\right]\Psi\right\}$$

Effective Functional

Integrate out Ψ using $\det A = \int \mathcal{D}\overline{\Psi} \cdot \Psi \exp\{i\overline{\Psi} A \Psi\}$

The remarkable identity det A = exp[tr ln A] gives i$S \to -F$ with

$$F = \frac{\pi\nu}{8\tau} \operatorname{Tr} Q^2 - \frac{1}{2} \operatorname{Tr} \ln \left[-\hat{\xi} + \tfrac{1}{2}\omega\Lambda + \frac{i}{2\tau} Q \right], \quad Z_n = \int \mathcal{D}Q \mathrm{e}^{-F}$$

the saddle-point approximation at $\omega = 0$:

$$Q = \frac{2}{\pi\nu} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \left(-\hat{\xi} + \frac{i}{2\tau_{\mathrm{el}}} Q \right)^{-1}$$

$$Q = U^\dagger \Lambda U$$

$$\Lambda = \operatorname{diag}(\mathbb{1}_n, -\mathbb{1}_n)$$

This is equivalent to

$$Q^2 = 1 \qquad \operatorname{Tr} Q = 0$$

It makes the first term in F irrelevant (const), leaving one to deal only with Tr ln (…)

Gradient Expansion

Substitute $Q = U^{+}\Lambda U$ into Tr ln $(U \ldots U^{+})$

$$F = -\frac{1}{2}\,\mathrm{Tr}\ln\left[\underbrace{-\hat{\xi} + \frac{i}{2\tau}\Lambda}_{G_0^{-1}} - U\left[\hat{\xi}, U^{+}\right] + \omega U \Lambda U^{\dagger}\right]$$

solution to the saddle -point approximation

Since $\xi \equiv \hat{p}^2/2m - \mu \approx v_F\,\mathbf{n}\cdot\boldsymbol{\nabla}$, one expands this in powers of ∇Q and ω, with expansion parameters $q\ell$ and $\omega\tau$

$$F = \int \mathrm{d}^d r\,\mathrm{Tr}\left[\frac{\pi\nu D}{8}\,(\nabla Q)^2 + \frac{i\pi\nu\omega}{4}\Lambda Q\right]$$

The lowest nonvanishing orders of the expansion

Finally, the nonlinear σ model

Limits of validity

Saddle-point + gradient expansion are legitimate provided that min is deep enough (dimensionless $F \gg 1$):

Dimensionless conductance

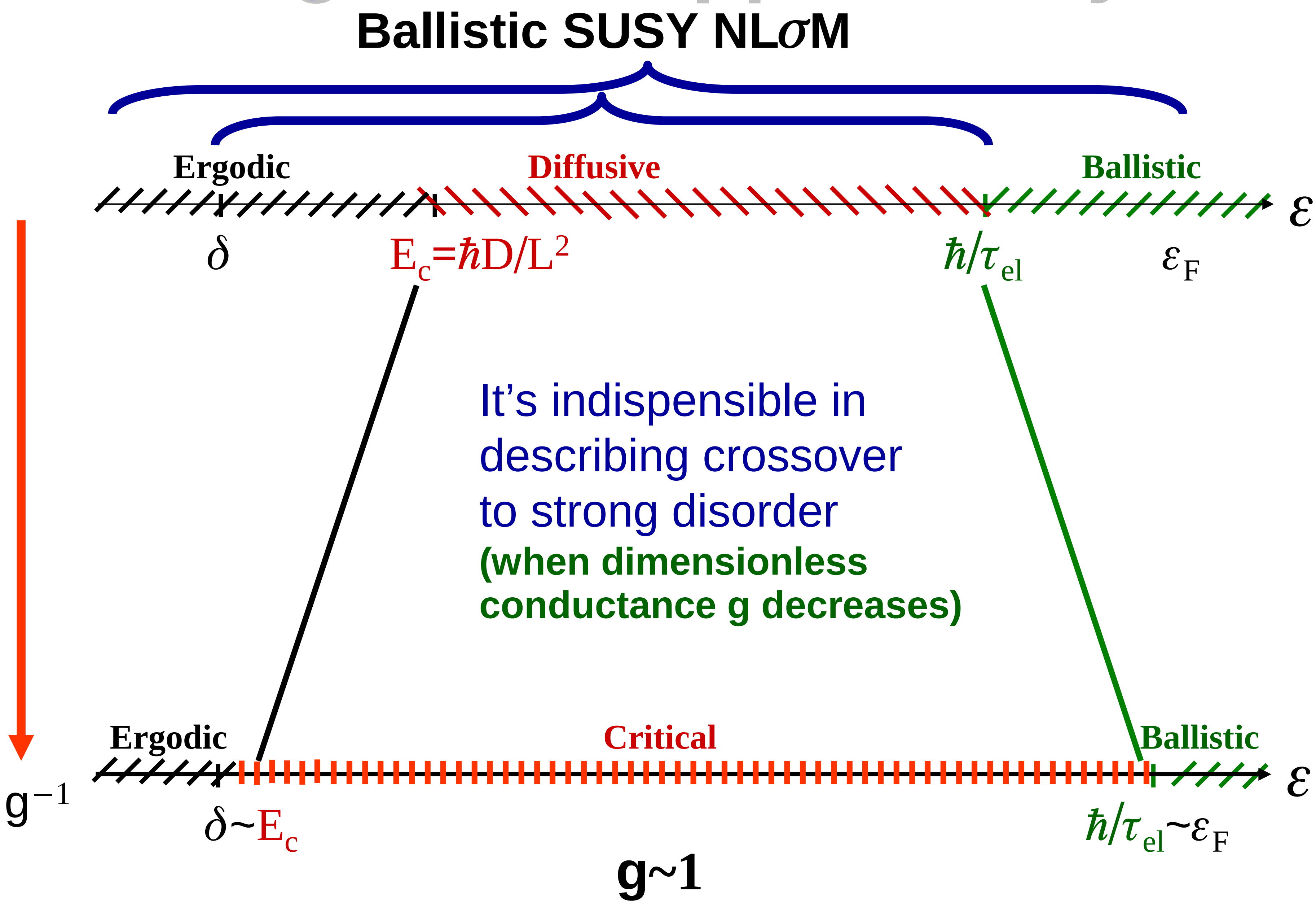
$$g_0 \equiv \nu D L^{d-2} \gg 1$$

Massive modes ($Q^2 \neq 1$) can be neglected at the same condition that justifies the gradient expansion:

$$q\ell \gg 1$$

This variant of the NLσM is not applicable to the ballistic regime, $L<\ell$

Regions of applicability



It's indispensible in describing crossover to strong disorder

(when dimensionless conductance g decreases)

Symmetry classes

Matrix elements	Ensemble	β	Realization in NLσM
real	orthogonal	1	symplectic space
complex	unitary	2	unitary space
quarternions $(2\times 2$ matrices)	simplectic	4	orthogonal space

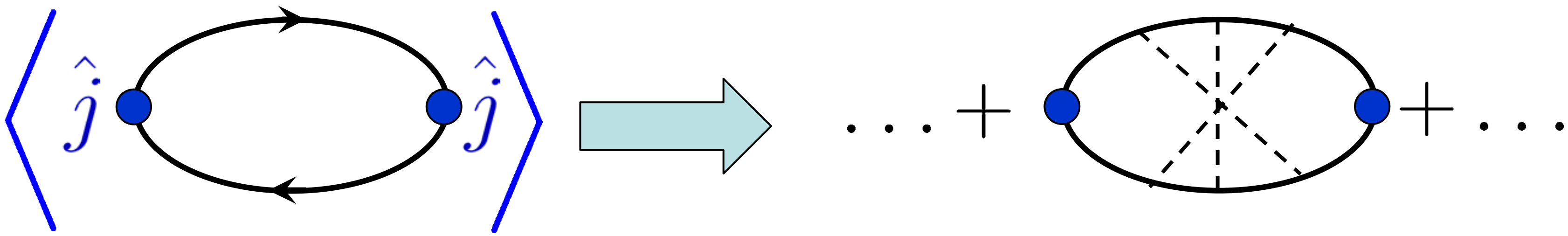
Diagrammatics in NLσM

Lecture 2: OUTLINE

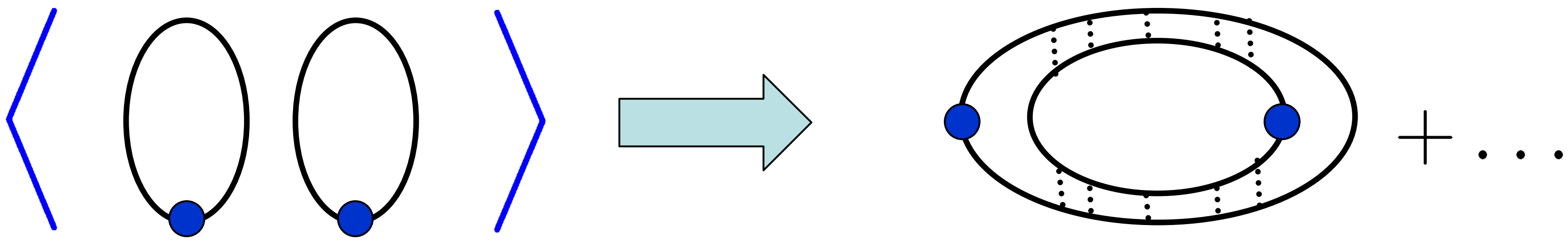
- Observable quantities in NLσM
- Parameterization & diagrams
- Level Sratistics
- Beyond perturbations with replicas

Observable quantities

Response functions



Conductivity $\sigma(\mathbf{r},\mathbf{r}')$

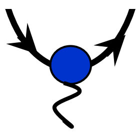


LDoS correlations

Observables & source fields

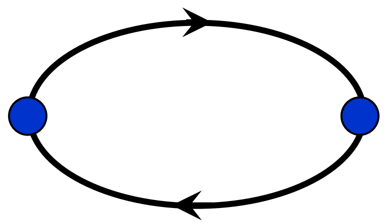
A product of any number of Green functions can be written as a pre-exponential factor in the ψ-ψ field theory

A better alternative is to introduce source fields that allow one to exponentiate these factors "in groups":



$$\equiv \bar{\psi} h \psi \mapsto \mathrm{tr}\,\left(h \cdot \psi \otimes \overline{\psi}\right) \mapsto \frac{\partial}{\partial h} \mathrm{e}^{\mathrm{tr}\,\left(h \cdot \psi \otimes \overline{\psi}\right)}\Big|_{h=0}$$

Any number of such groups is obtained by repeated dif. :



$$\propto \frac{\delta^2}{\delta\, h^2} \mathrm{e}^{iS + \mathrm{tr}\,\left(h \cdot \psi \otimes \overline{\psi}\right)}\Big|_{h=0}$$

As $\psi \otimes \overline{\psi} \to Q\,,$ this structure is preserved in NLσM

Conductance in NLσM

$\sigma(\boldsymbol{r},\boldsymbol{r}')$ is a response to an external $\boldsymbol{E}$ field introduced by

$$\boldsymbol{p} \to \boldsymbol{p} - e\boldsymbol{A}_\omega \quad \Longrightarrow \quad \boldsymbol{\nabla} \to \boldsymbol{\partial} \equiv \boldsymbol{\nabla} - e\boldsymbol{A}$$

Current density $\boldsymbol{j} \propto \delta/\delta A$ so that $\sigma(\boldsymbol{r},\boldsymbol{r}')$ is given by

$$\left\langle \sigma_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}') \right\rangle = \frac{1}{16\pi n^2} \frac{\partial^2 \ln Z[\boldsymbol{A}]}{\partial A_\alpha(\boldsymbol{r})\, \partial A_\beta(\boldsymbol{r}')} \Bigg|_{\substack{\boldsymbol{A}=0\\ n=0}} \qquad \text{where}$$

$$Z[\boldsymbol{A}] \equiv \int \mathcal{D}Q\, \mathrm{e}^{-F[\boldsymbol{A}]}, \quad F[\boldsymbol{A}] = \mathrm{Tr}\left[\frac{\pi\nu D}{8}(\boldsymbol{\partial}Q)^2 + \frac{i\pi\nu}{4}\omega\Lambda Q\right]$$

$$\text{Finally,} \quad G_{\alpha\beta} = \frac{1}{L^2}\int \sigma_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}')\, \mathrm{d}^d r\, \mathrm{d}^d r'$$

DoS in NLσM

Since DoS by itself is not affected by disorder, we consider DoS correlation function

$$R_2(\omega) = \frac{1}{\langle \nu \rangle^2} \langle \nu(\varepsilon + \omega)\nu(\varepsilon) \rangle - 1$$

Although no source field is required, it's better to have one

$$R_2(\omega) = \frac{1}{(\pi n L^d)^2} \Re\mathrm{e} \left. \frac{\partial^2 \ln Z[\Omega]}{\partial \Omega^2} \right|_{\substack{\Omega = 0 \\ n = 0}}, \quad F[\Omega] = F + \frac{i\pi\nu\Omega}{4} \mathrm{Tr}\, \Lambda Q$$

By $\Omega \to \Omega(\boldsymbol{r})$ this can be generalised to a spatial correlation function of two LDoS

How to carry out calculations?

Parameterization

First one resolves constraints $Q^2=1$ & Tr $Q=0$. Examples:

$$Q = \Lambda e^{W} \quad \text{or} \quad Q = \Lambda\left(W + \sqrt{1+W^2}\right) \quad \text{or} \quad Q = \left(1 - \frac{W}{2}\right)\Lambda\left(1 - \frac{W}{2}\right)^{-1}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathbb{I}_n \,, \quad W = \begin{pmatrix} 0 & B_n \\ -B_n^\dagger & 0 \end{pmatrix}$$

B_n is unconstrained

nxn matrix field

Expanding Q in W, one finds $F=F_0(W)+\delta F(W)$ where

$$F_0 = -\frac{\pi\nu}{8}\mathrm{Tr}\left[D\left(\boldsymbol{\nabla} W\right)^2 - i\omega W^2\right]$$

$$\delta F = \frac{i\pi\nu}{8}\mathrm{Tr}\,\Omega W^2 + \text{higher powers in } W$$

Note that higher powers depend on the parameterization!

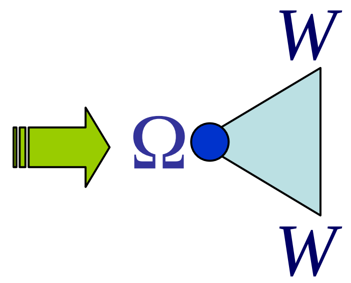
How it works perturbatively

- Expand e^{-F} in δF and δF in W
- Calculate Gaussian integrals $\langle (\delta F)^m \rangle_0$

Omitted matrix indices are vital for taking the n=0 replica limit

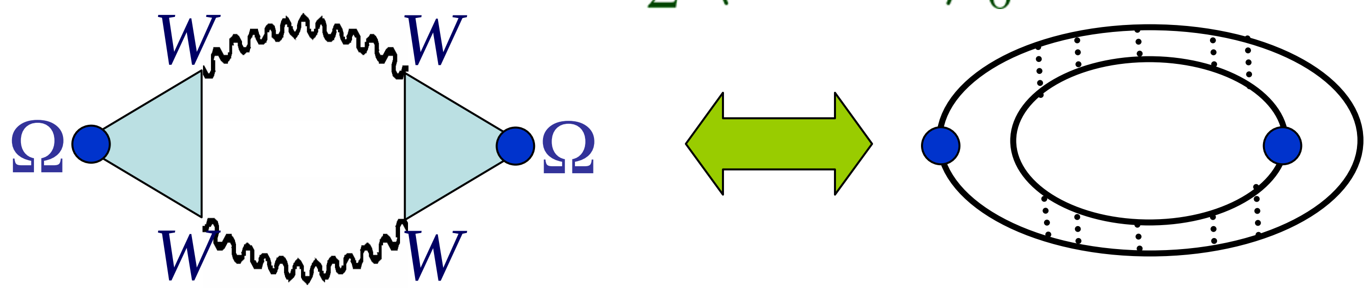
$$\langle WW \rangle_0 \equiv \frac{1}{Z}\int WW e^{-F_0} \propto \frac{1}{Dq^2 - i\omega} \equiv$$

In the lowest order $\delta F_\Omega = \frac{i\pi\nu}{8} \mathrm{Tr}\, \Omega W^2$



In this order R_2 is contributed by $\frac{1}{2}\left\langle (\delta F_\Omega)^2 \right\rangle_0$:

$R_2 =$



R_2 in the diffusive regime

$$R_2(\omega) = \frac{\Delta^2}{\pi^2}\Re\mathrm{e}\sum_{\boldsymbol{q}} \frac{1}{(D\boldsymbol{q}^2 - i\omega)^2}, \qquad \Delta \equiv \frac{1}{\langle \nu \rangle L^d}$$

In a finite system, $\boldsymbol{q}=(2\pi/\mathrm{L})(\mathrm{n}_x,\mathrm{n}_y,\mathrm{n}_z)$, integers $\mathrm{n}_i \geq 0$

$$\text{Dif. regime: } q^2 \ll \frac{\omega}{D} \equiv L_\omega^{-2} \Leftrightarrow E_c \equiv \frac{D}{L^2} \ll \omega, \Rightarrow \sum_{\boldsymbol{q}} \to L^d \int \frac{\mathrm{d}^d q}{(2\pi)^d}$$

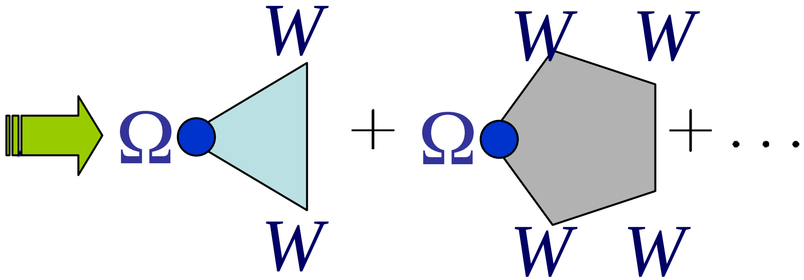
$$R_2(\omega) \sim \frac{L^{d-4}}{g^2}\Re\mathrm{e}\int \frac{\mathrm{d}^d q}{(q^2 - iL_\omega^{-2})^2} \simeq \frac{C_d}{g^2}\left(\frac{L_\omega}{L}\right)^{4-d} \equiv \frac{C_d}{g^{d/2}}\left(\frac{\Delta}{\omega}\right)^{2-d/2}$$

For d=2, this does not work:

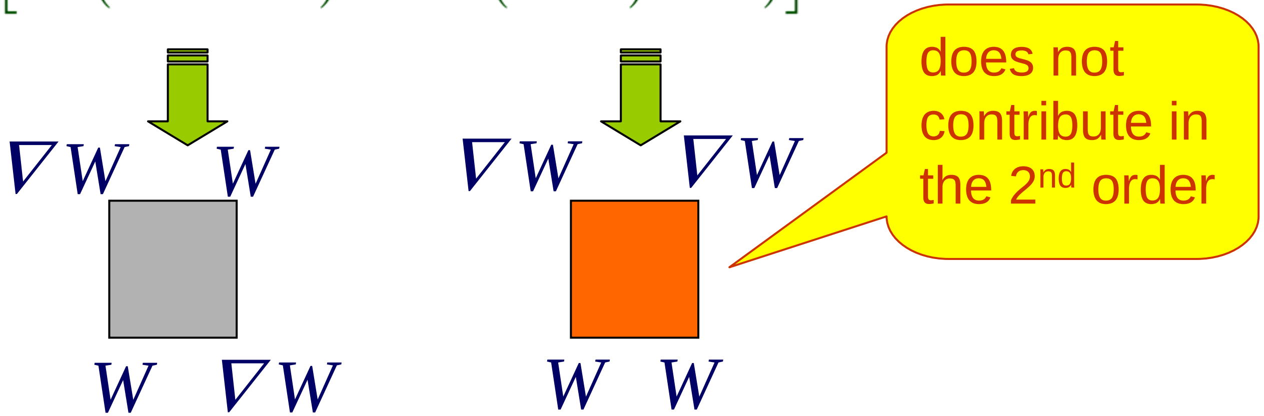
$$\Re\mathrm{e}\int \frac{\mathrm{d}^d q}{(q^2 - iL_\omega^{-2})^2} = 2\pi\Re\mathrm{e}\int_0^\infty \frac{q\mathrm{d}q}{(q^2 - iL_\omega^{-2})^2} = \pi\Re\mathrm{e}\int_{-\infty}^\infty \frac{\mathrm{d}z}{(z - iL_\omega^{-2})^2} = 0$$

Higher orders

$$\delta F_\Omega = \frac{i\pi\nu}{8}\left[\mathrm{Tr}\,\Omega W^2 - \frac{1}{2}\mathrm{Tr}\,\Omega W^4\right] + \ldots$$

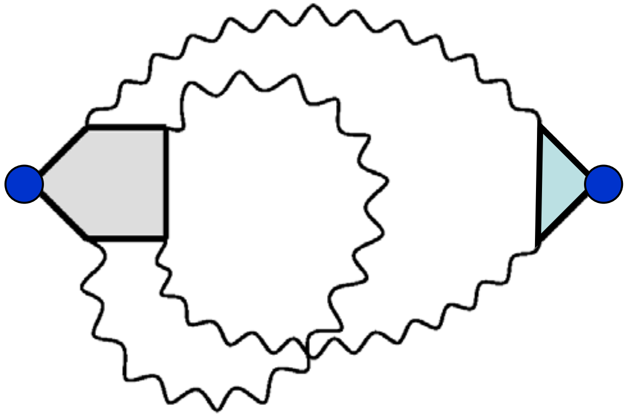


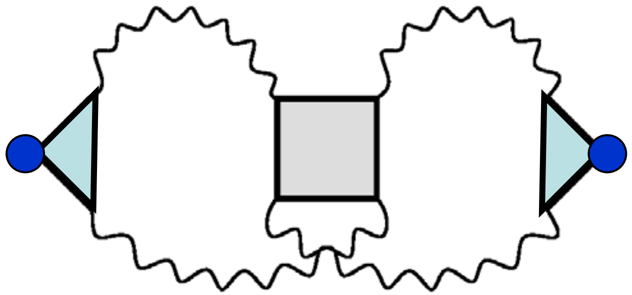
$$\delta F_{(\nabla Q)^2} = \frac{\pi\nu D}{16}\left[\mathrm{Tr}\,(\nabla W W^2) - \mathrm{Tr}\,(\nabla W)^2 W^2)\right] + \ldots$$

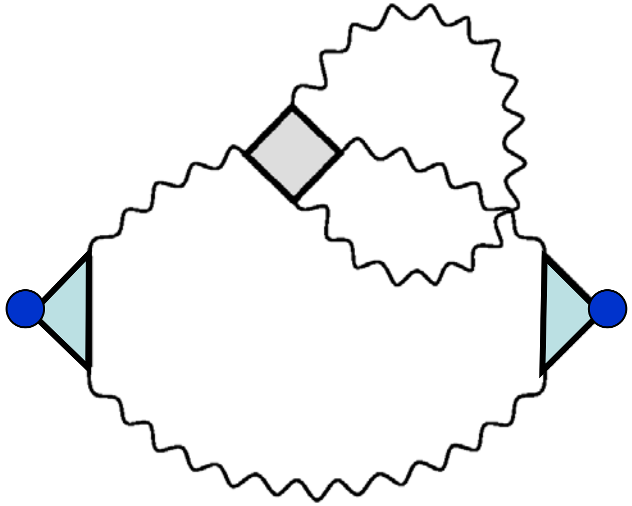


Contributions come from $\langle\, (\delta F_\Omega\,)^2\, \rangle_0$ and $\langle\, (\delta F_\Omega\,)^2 \delta F_{(\nabla Q)^2}\, \rangle_0$.
Both replica limit and angular integration severely cut the number of contributing diagrams

R_2 in the diffusive regime in 2D







Results:

$$R_2(\omega) \sim \frac{1}{g^2}\left(\frac{\Delta}{\omega}\right)$$

R_2 in the ergodic regime, $\omega \ll E_c$

$$R_2(\omega) = \frac{\Delta^2}{\beta\pi^2} \Re\mathrm{e} \sum_{\boldsymbol{q}} \frac{1}{(Dq^2 - i\omega)^2} \mapsto -\frac{\Delta^2}{\beta\pi^2\omega^2}$$

Higher order perturbative corrections $\sim(\Delta/\omega)^{\mathrm{m}}$:

Perturbation theory breaks down for $\omega \ll \Delta$

It is not that good at $E_c \gg \omega \gg \Delta$ either. Exact result

$$R_2(\omega) = -\left(\frac{\Delta}{\omega}\right)^2 \sin^2\left(\frac{\omega}{\Delta}\right)^2$$

unitary case $\beta=2$

Non-analytic in Δ/ω $\Rightarrow$ nonperturbative.

Exploring Replica Symmetry

- For $\omega < E_c$ the NLσM contains the ω term only:

$$Z_n(\omega) = \int \mathcal{D}Q \exp\left[-i\frac{\omega}{2}\text{Tr}\,\Lambda Q\right]$$

- The saddle-point: Q obeys $[Q,\Lambda]=0$, *i.e.* Q is a diagonal matrix with equal (as Tr Q=0) number of ±1 elements.

- ∴ the saddle-point is highly degenerate.

$$Z_n(\omega) = \sum_{p=0}^{n} (C_n^p)^2 \int \mathcal{D}Q \exp\left[-i\frac{\omega}{2}\text{Tr}\,\Lambda_p Q\right]$$

$\Lambda_p = \text{diag}(1_{n-p}, -1_p, 1_p, -1_{n-p})$

- Normal choice p=0; any p gives the same perturbatively

Breaking Replica Symmetry

- Calculating the integrals yields

$$Z_n(\omega) = \sum_{p=0}^{n} [F_n^p]^2 \cdot \frac{e^{i\omega(2p-n)}}{(2\omega)^{(n-p)^2+p^2}} \qquad F_n^p \equiv C_n^p \prod_{j=1}^{p} \frac{\Gamma(1+j)}{\Gamma(n+2-j)}$$

•Symmetry is broken by extending summation to $p=\infty$ **(**as $F_n^p=0$ for $n>p$), and **taking the replica limit** $n \to 0$.

•It works: $F_n^p \propto n^p$ as $n \to 0$ and only p=0,1 terms contribute:

$$Z_n(\omega) = \frac{e^{-i\omega n}}{\omega n^2} + n^2 \frac{e^{i\omega(2-n)}}{4\omega^{(n-1)^2+1}}$$

gives the exact result for S_2 and thus R_2.

Breaking replica symmetry leads to correct non-perturbative results

Summary

- It’s up and running
- It’s much easier to use than to derive, but this is much easier to declare than to demonstrate

Next time

- Including interactions (Coulomb and BCS)
- Mapping to various known models
- Describing superconductor-insulator transition

Announced at the last lecture:

- Including interactions (Coulomb and BCS)
- Mapping to various known models
- Describing superconductor-insulator transition

Lecture 3: changing gears to

Functional bosonization for Luttinger liquid

Hubbard-Stratonovich transfomation: change from fermionic to bosonic representation

$$\mathcal{S} = \int \mathrm{d}\boldsymbol{x} \overline{\psi}_\eta(\boldsymbol{x})(i\partial_\tau + \hat{\xi} - V(\boldsymbol{r}))\psi_\eta(\boldsymbol{x}) \qquad (\boldsymbol{x} \equiv \boldsymbol{r}, \tau)$$

$$+ \frac{1}{2} \int \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{x}' \bar{\psi}_\eta(\boldsymbol{x}) \bar{\psi}_{\eta'}(\boldsymbol{x}') V_0(\boldsymbol{x} - \boldsymbol{x}') \psi_{\eta'}(\boldsymbol{x}') \psi_\eta(\boldsymbol{x})$$

Note that we use the linearized spectrum

$$\xi = \frac{p^2}{2m} - \varepsilon_{\mathrm{F}} \approx v_{\mathrm{F}}(|p| - p_{\mathrm{F}})$$

but in d>1 it does not split e's into 2 species

$$\frac{1}{2}\mathrm{Tr}\left(\Phi U^{-1}\Phi + \frac{\pi\nu}{4\tau_{\mathrm{el}}} Q^2\right) - \frac{1}{2}\mathrm{Tr}\ln\left(-i\partial_\tau - \hat{\xi} + \frac{iQ}{2\tau_{\mathrm{el}}} + i\Phi\right)$$

NLσM would follow from the saddle point approximation

Pure Interaction in 1D

$$H = H_0 + H_{\text{int}}, \quad H_0 = \int \mathrm{d}x \hat{\psi}_\eta^\dagger(x) \hat{\xi} \hat{\psi}_\eta(x)$$

$$H_{\text{int}} = \frac{1}{2} \int \mathrm{d}x \mathrm{d}x' \hat{\psi}_\eta^\dagger(x) \hat{\psi}_{\eta'}^\dagger(x') V_0(x - x') \hat{\psi}_{\eta'}(x') \hat{\psi}_\eta(x)$$

In the simplest spinless case, keeping only forward-scattering local interaction, one gets from a linearised Tomonaga-Luttinger Hamiltonian a naïve bosonic one:

$$H = \frac{v}{4} \int_{-L/2}^{L/2} \frac{dx}{2\pi} {}^*_* \left[\frac{1}{g} (\tilde{\rho}_L + \tilde{\rho}_R)^2 + g (\tilde{\rho}_L - \tilde{\rho}_R)^2 \right] (x) {}^*_*$$

repulsion: $g<1$; free Fermions $g=1$ (see Giamarchi's lectures)

Hubbard-Stratonovich in D>1 is an entirely different "bosonisation" procedure than the operator one in 1D

The aims of this talk

- To introduce a "functional bosonisation" for the Luttinger liquid in the spirit of the approach used in the derivation of NLσM
- As an application, to derive the LDoS $\nu(\varepsilon, x)$ at an arbitrary distance x from a single impurity ("end") in the Luttinger liquid

Effective Functional

$$S[\psi^*, \psi\ ] = \int \mathrm{d}\xi \left[\psi_\eta^*(\xi)\partial_\tau\psi_\eta(\xi) - H(\psi^*, \psi) \right]$$

No spin!; η=(L,R) ≡±, ξ≡ (x,τ); $\int \mathrm{d}\xi \equiv \int_{-L/2}^{L/2} \mathrm{d}x \int_0^\beta \mathrm{d}\tau$

H is obtained from the Hamiltonian by substituting fermionic operators by continuous fields in the Matsubara representation

$$Z = \int \mathrm{e}^{-S[\psi^*,\psi\ ]} \mathcal{D}\psi^* \mathcal{D}\psi$$

Hubbard-Stratonovich transfomation

Introduce $\phi \propto \rho$ – the bosonic field to decouple the ρ^2 term

$$\mathrm{e}^{-\frac{1}{2}\rho * V_0 * \rho} = \frac{\int \mathcal{D}\phi \, \mathrm{e}^{i\rho*\phi} \mathrm{e}^{-\frac{1}{2}\phi * V_0^{-1} * \phi}}{\int \mathcal{D}\phi \, \mathrm{e}^{-\frac{1}{2}\phi * V_0^{-1} * \phi}}$$

This results in the effective action (with $\partial_{\pm} = \partial_\tau \mp i v_{\mathrm{F}} \partial_x$)

$$S[\phi, \psi] = \int \mathrm{d}\xi \, \psi_\eta^*(\xi) \left(\partial_\eta - i\phi\right) \psi_\eta(\xi) + \frac{1}{2} \int \mathrm{d}\xi \mathrm{d}\xi' \, \phi(\xi) \, V_0^{-1}(\xi - \xi') \, \phi(\xi')$$

Fermion-Boson Decoupling

Gauge transform eliminates the mixed term in the derivative

$$\psi_\eta \mapsto \psi_\eta e^{i\theta_\eta}\,, \quad \psi_\eta^* \mapsto \psi_\eta^* e^{-i\theta_\eta}\,, \qquad \partial_\eta \theta_\eta = \phi$$

Thus, fermions and bosons decouple

$$S_{\rm b} = \frac{1}{2}\int {\rm d}\xi {\rm d}\xi' \phi(\xi)\, V^{-1}(\xi - \xi')\, \phi(\xi')$$

$$S_{\rm f} = \int {\rm d}\xi\, \psi_\eta^*(\xi)\, \partial_\eta\, \psi_\eta(\xi)$$

Where is the trick? $V \neq V_0$ because of the Jacobian of the GT

Jacobian

$$\ln\ J[\phi] = \sum_{\eta=\pm} \mathrm{Tr}\ln\left|\frac{\partial_\eta - i\phi}{\partial_\eta}\right| = -\sum_{\eta=\pm}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\,(i\phi g_\eta)^n$$

$$\ln J = -\frac{1}{2}\int \mathrm{d}\xi \mathrm{d}\xi'\,\phi(\xi)\Pi(\xi-\xi')\phi(\xi')$$

(see Appendix)

$$\Pi(q,\Omega) = \frac{1}{\pi v_{\mathrm{F}}}\frac{v_{\mathrm{F}}^2 q^2}{\Omega^2 + v_{\mathrm{F}}^2 q^2}$$

RPA polarization operator is exact for the LL
(Dzyaloshinskii, Larkin, ‘73)

The Jacobian of the gauge transformation gives the screening!

$$V_0^{-1} \mapsto V^{-1} = V_0^{-1} + \Pi$$

Exact Green's function

Green's function is (since $\psi_\eta \mapsto \psi_\eta e^{i\theta_\eta}$):

$$\mathcal{G}_{\eta\eta'}(\xi;\xi') = \left\langle\!\!\left\langle \mathrm{e}^{i\theta_\eta(\xi)-i\theta_{\eta'}(\xi')}\psi_\eta(\xi)\psi^*_{\eta'}(\xi') \right\rangle\!\!\right\rangle$$

$$\langle\!\langle \ldots \rangle\!\rangle = \frac{1}{Z}\int \ldots \mathrm{e}^{-S}\mathcal{D}\psi^*\mathcal{D}\psi\ \mathcal{D}\phi$$

$$S = \frac{1}{2}\int \mathrm{d}\xi\mathrm{d}\xi'\phi(\xi)\,V^{-1}(\xi-\xi')\,\phi(\xi') + \int \mathrm{d}\xi\,\psi^*_\eta(\xi)\,\partial_\eta\,\psi_\eta(\xi)$$

$\partial_\eta\theta_\eta = \phi$ ⇒ we need $\langle\!\langle\theta_\eta\theta_{\eta'}\rangle\!\rangle$ from $\langle\!\langle\phi_\xi\phi_{\xi'}\rangle\!\rangle = V(\xi-\xi')$

$$\theta_- = \theta_+^* \equiv \theta = \theta_1 + i\theta_2$$

Bosonic Averaging

From $(\partial_\tau - iv_F\partial_x)\theta(\xi) = \phi(\xi)$, it follows that, e.g.

$$(\partial^2_{\tau\tau} + v_F^2\partial^2_{xx}) \langle\langle\theta_1(\xi)\theta_1(\xi')\rangle\rangle = -\partial^2_{\tau\tau}V(\xi),$$

This equation is solved by the Fourier transform:

$$\left\langle\theta_1(\xi)\theta_1(\xi')\right\rangle_\phi = \frac{1}{2}\ln\frac{|\sin(z_{\mathrm{F}} - z'_{\mathrm{F}})|}{|\sin(z - z')|^{1/g}}$$

$$z_{\mathrm{F}} = \pi T(\tau + ix/v_{\mathrm{F}}) \qquad z = \pi T(\tau + ix/v)$$

$$v = v_{\mathrm{F}}/g \qquad g \equiv \left(1 + V_0/\pi v_{\mathrm{F}}\right)^{-1/2},$$

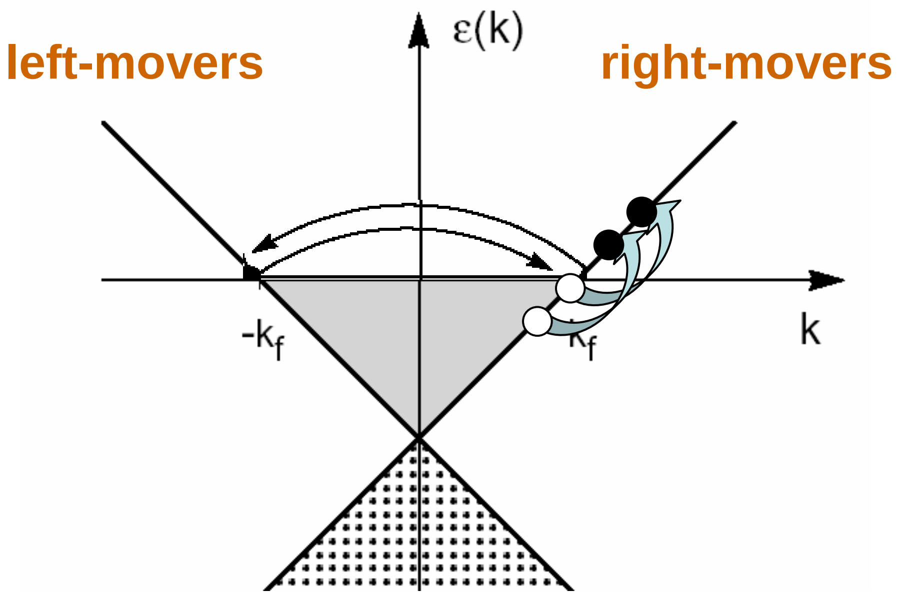
where $V_0(q) \longrightarrow V_0 = V_0(q \sim T/v_{\mathrm{F}})$

Pure Luttinger Liquid: Summary of the Approach

- Any correlation function can be calculated in terms of $\langle\langle\theta_a(\xi)\theta_b(\xi')\rangle\rangle$ and $\langle\langle\psi^*(\xi)\psi\ (\xi')\rangle\rangle$

- Calculating GF in this way reproduces the results for LDoS in the pure LL

Adding an Impurity



Back-scattering couples left- and right-movers: the bosonic excitations are no longer free:

Impurity adds a new "coupling" term:

$$H_{\mathrm{imp}} = v_{\mathrm{F}} \int \mathrm{d}x \lambda(x) \left[\hat{\psi}^{\dagger}_{+}(x) \hat{\psi}_{-}(x) + \hat{\psi}^{\dagger}_{-}(x) \hat{\psi}_{+}(x) \right]$$

Impurity Coupling

$$S_{\mathrm{imp}} = -v_{\mathrm{F}} \int \mathrm{d}\xi\, \lambda(x) \left[\psi_+^*(\xi)\psi_-(\xi) + \psi_-^*(\xi)\psi_+(\xi) \right]$$

couples fermionic and bosonic fields after the gauge transform:

$$S_{\mathrm{imp}} = -v_{\mathrm{F}} \int \mathrm{d}\xi \lambda \left[\mathrm{e}^{2\theta_2}\psi_+^*(\xi)\psi_-(\xi) + \mathrm{e}^{-2\theta_2}\psi_-^*(\xi)\psi_+(\xi) \right]$$

– the problem is no longer exactly solvable.

Green's function (that defines the tunnelling LDoS) is now

$$\mathcal{G}_{\eta\eta'}(\xi;\xi') = \frac{1}{Z_\lambda} \left\langle\!\left\langle \mathrm{e}^{i\theta_\eta(\xi) - i\theta_{\eta'}(\xi')} \psi_\eta(\xi)\psi_{\eta'}^*(\xi')\mathrm{e}^{-S_{\mathrm{imp}}} \right\rangle\!\right\rangle$$

$$Z_\lambda \equiv \left\langle\!\left\langle \mathrm{e}^{-S_{\mathrm{imp}}} \right\rangle\!\right\rangle$$ is to be calculated

Calculating Z_λ

Symbolically:

$$Z_\lambda = \sum_{n=0}^{\infty} \frac{{v_\mathrm{F}}^{2n}}{(n!)^2} \left\{ \left\langle \left(\int_\xi \psi_+^* \psi_- \right)^{2n} \right\rangle_f \left\langle \int_{\xi\xi'} \mathrm{e}^{n(\theta-\theta')} \right\rangle_f \right\}$$

The fermionic average cancels an unpleasant denominator of the bosonic one, both resulting in a formal expression in terms of

$$z = \pi T(\tau + ixg/v_\mathrm{F})$$

Formal Results

$$Z_\lambda = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{T}{2\alpha^{1-g}} \right)^{2n} \qquad (\alpha \sim T/\varepsilon_{\mathrm{F}} \ll 1)$$

$$\times \prod_{k=1}^{n} \int \mathrm{d}\xi_k \mathrm{d}\xi'_k \, \lambda(x_k)\lambda^*(x'_k) \, |P_n(z)|^{2g}$$

$$P_n(z) = \frac{\prod_{i<j}^{n} \sin(z_i - z_j) \sin(z'_i - z'_j)}{\prod_{i,j=1}^{n} \sin(z_i - z'_j)}$$

Re-Bosonization

Z_λ in terms of a new impurity functional above is given by

$$Z_\lambda = \langle e^{-S_\lambda[\Theta]} \rangle_0, \quad S_\lambda[\Theta] = -\frac{T}{\alpha} \int \mathrm{d}\xi \, \lambda(x) \cos\Theta(\xi)$$

The functional average $\langle\ldots\rangle_0$ is performed with the weight

$$S_0[\Theta] = \frac{1}{8\pi g v} \int \mathrm{d}\xi \left[(\partial_\tau \Theta)^2 + v^2 (\partial_x \Theta)^2 \right]$$

The Green's function (with $\chi(z_1) = \arg \frac{\sin(z_1 - z)}{\sin(z_1 - z')}$)

$$\widetilde{\mathcal{G}}_{\eta\eta'}(\xi,\xi') = Z_\lambda^{-1} \left\langle \mathrm{e}^{\frac{i\eta}{2}\Theta(\xi) - \frac{i\eta'}{2}\Theta(\xi')} \mathrm{e}^{-S_\lambda[\Theta - \chi]} \right\rangle_0$$

Self-consistent harmonic approximation

Assumption: $\alpha \ll \lambda \ll 1 \quad (\alpha \sim T/\varepsilon_F)$

the impurity potential $\lambda(x)=\lambda\delta(x)$ is weak but non-perturbative

SCHA: the deviation of Θ from χ is prohibitive so that cos potential is substituted by the quadratic approximation:

$$S_\Lambda[\Theta - \chi] = \frac{\Lambda T}{2\alpha} \int \mathrm{d}\tau_1 \left[\Theta(0,\tau_1) - \chi(0,\tau_1)\right]^2$$

What is left is a (rather gory) calculation of Gaussian integrals

Green's Function

$$\frac{2\pi}{p_{\rm F}}\mathcal{G}(x,x;\tau) =$$

$$\begin{cases} \dfrac{\alpha^{\frac{1}{g}}}{(\sin\alpha\tilde{\tau})^{\frac{1}{g}}}\left[\max(\Lambda^{-1},\tilde{x})\right]^{\frac{1}{2}\left(\frac{1}{g}-g\right)}\left[1-\cos(2p_{\rm F}x+\Phi)\right], & \max(\tilde{x}\,,\Lambda^{-1})\ll\tilde{\tau} \\ \dfrac{\alpha^{\frac{1}{2}\left(\frac{1}{g}+g\right)}}{(\sin\alpha\tilde{\tau})^{\frac{1}{2}\left(\frac{1}{g}+g\right)}}\left[1-\left(\dfrac{\sin\alpha\tilde{\tau}}{\sinh\alpha\tilde{x}}\right)^{g}\cos(2p_{\rm F}x+\Phi)\right], & \min(\tilde{x}\,,\Lambda^{-1})\gg\tilde{\tau} \end{cases}$$

x is the distance from the impurity, $\Lambda = \lambda^{\frac{1}{1-g}}$, $\tilde{x} = gp_{\rm F}|x|$, $\tilde{\tau} = \varepsilon_{\rm F}|\tau|$

Friedel oscillations can be easily extracted

Well known LDoS

Suppression of the LDoS at the impurity point

$$\nu(\varepsilon) \propto \begin{cases} \varepsilon^{\frac{1}{g}-1}, & \text{at the impurity} \\ \varepsilon^{\frac{1}{2}\left(\frac{1}{g}+g\right)-1}, & \text{in the bulk} \end{cases}$$

Corresponds to "end" and bulk LDoS in Leonid Glazman's lecture this morning

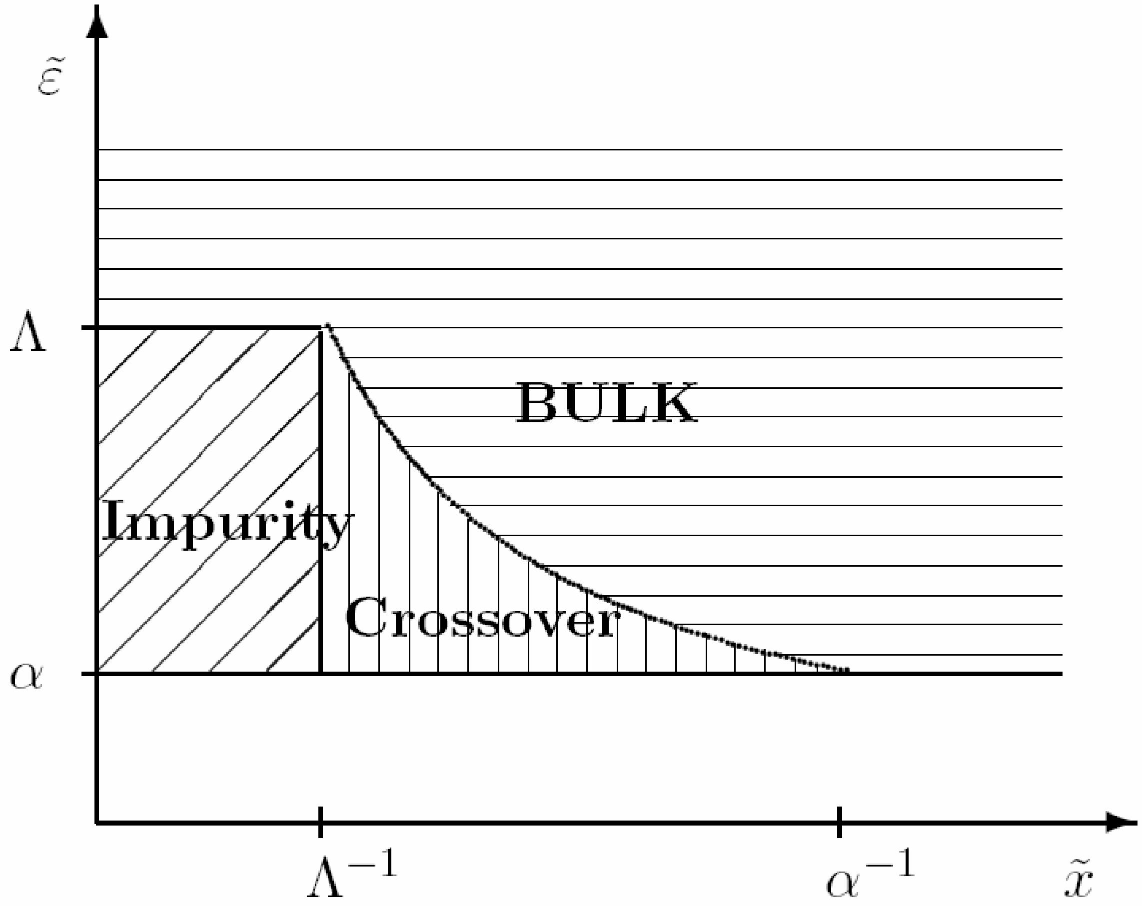
Local Tunnelling DoS

is obtained at an arbitrary distance from the impurity

$$\nu(x,\varepsilon) \sim \begin{cases} \varepsilon^{\frac{1}{g}-1}\Lambda^{-\frac{1}{2}\left(\frac{1}{g}-g\right)}, & \tilde{x} \ll \Lambda^{-1} \ll \varepsilon^{-1} \\ \varepsilon^{\frac{1}{g}-1}\tilde{x}^{\frac{1}{2}\left(\frac{1}{g}-g\right)}, & \Lambda^{-1} \ll \tilde{x} \ll \varepsilon^{-1} \\ \varepsilon^{\frac{1}{2}\left(\frac{1}{g}+g\right)-1}, & \tilde{x} \gg \Lambda^{-1}, \varepsilon^{-1} \end{cases}$$

Local Tunnelling DoS

different regions at a distance x from the impurity



$\alpha \sim \mathrm{T}/\varepsilon_{\mathrm{F}}$

$$\Lambda = \lambda^{\frac{1}{1-g}}$$

$$\tilde{x} = g p_{\mathrm{F}} |x|,$$

$$\tilde{\tau} = \varepsilon_{\mathrm{F}} |\tau|$$

Summary

- Functional Bosonization works: it reproduces known results for Friedel Oscillations & LDoS and allows one to find LDoS at an arbitrary distance from the impurity

- In many way, it is analogue of field-theoretical treatment of higher-dimensional models

- The operator and functional approaches are equivalent for exact results; however, ease of use for approximations may be different

Appendix : Jacobian

$$\ln\ J[\phi] = \sum_{a=\pm} \mathrm{Tr}\ln\left|\frac{\partial_a - i\phi}{\partial_a}\right| = -\sum_{a=\pm}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\,(i\phi g_a)^n$$

where $g_-(\xi,\xi') = g_+^*(\xi) = \frac{T}{2v_{\mathrm{F}}}\frac{1}{\sin(z_{\mathrm{F}} - z'_{\mathrm{F}})}$

The n[th] term is

$$\mathrm{Tr}\,(g_a\,\phi)^n = \int \prod_{k=1}^{n} \mathrm{d}x_k \mathrm{d}\tau_k\ \Gamma_n^{(a)}(z_{\mathrm{F}_1};\ldots;z_{\mathrm{F}\,n}) \prod_{i=1}^{n} \phi(x_i,\tau_i),$$

with Γ

$$\Gamma_n^{(a)}(z_{\mathrm{F}_1};\ldots;z_{\mathrm{F}\,n}) = \prod_{i=1}^{n} g_a(z_{\mathrm{F}\,i} - z_{\mathrm{F}\,i+1}) \propto \prod_{i=1}^{n}\frac{s_i}{s_i - s_{i+1}}, \qquad s_i = e^{2iz_{\mathrm{F}\,i}}$$

2: Jacobian

Only symmetric part of the vertex contributes to the integral:

$$\text{Sym}\left[\Gamma_n^+(z_{F_1};\ldots;z_{F_n})\right] \propto \frac{\mathcal{A}_n(s_1,\ldots s_n)}{\prod_{i<j}^n (s_i - s_j)} \prod_{k=1}^n s_k$$

where $\mathcal{A}_n$ is an absolutely anti-symmetric polynomial

By power counting, its order is n(n-3)/2>n(n+1)/2 – only possible for n=1 and n=2 loops, whose calculation is straightforward (after dealing with inevitable divergences)