Kondo Effect in Quantum Dots

Jan von Delft, August 2004, Windsor

Lecture 1: From Anderson model to Kondo Model

- Experimental results
- AM $\xrightarrow{\text{Schriefer-Wolff}}$ KM
- AM with two leads, conductance: $G = G_0 \sin^2 \delta_1$

Lecture 2: Kondo model – scattering rate, scaling, phase shifts

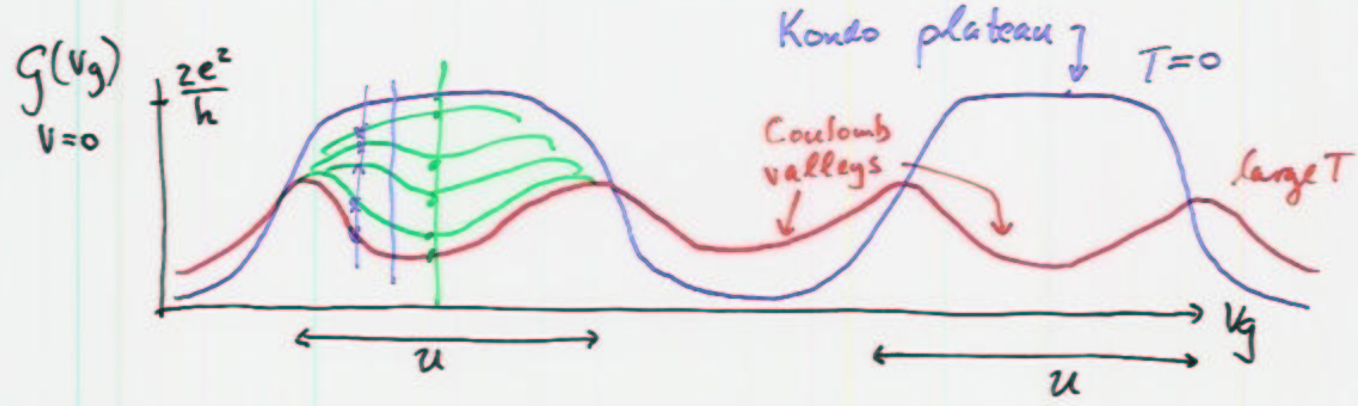
- T-matrix in pert. theory, log T/D divergencies
- Anderson's "poor man's scaling", T_K
- Strong coupling regime. Friedel sum rule
- Kondo resonance

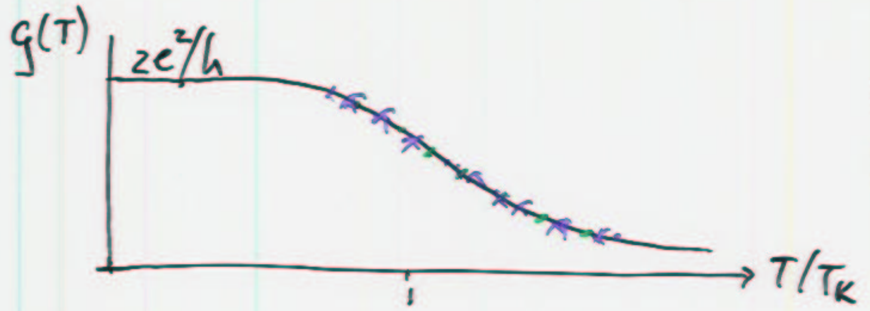
Lecture 3: Flow equation renormalization group

- General idea: diagonalize Hamiltonian by unitary transf.
- Application to AM $\qquad H(B) = U^\dagger(B) H(0) U(B)$
- Application to KM $\qquad H(\infty) = \text{diagonal}.$

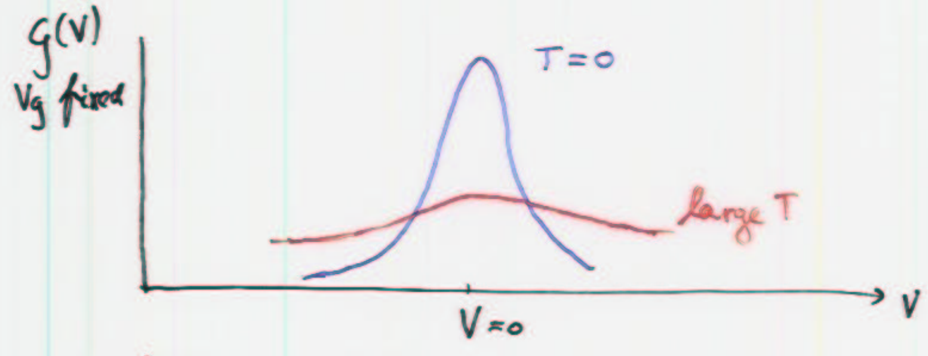
From Anderson model (AM) to Kondo model (KM)

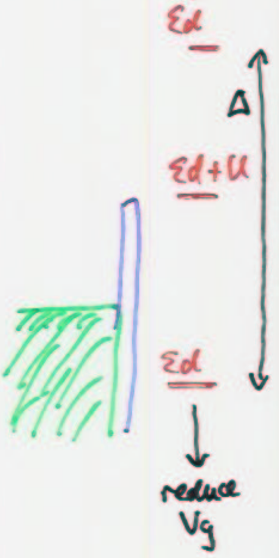
Phenomenology of conductance $G(V_g, V, T)$

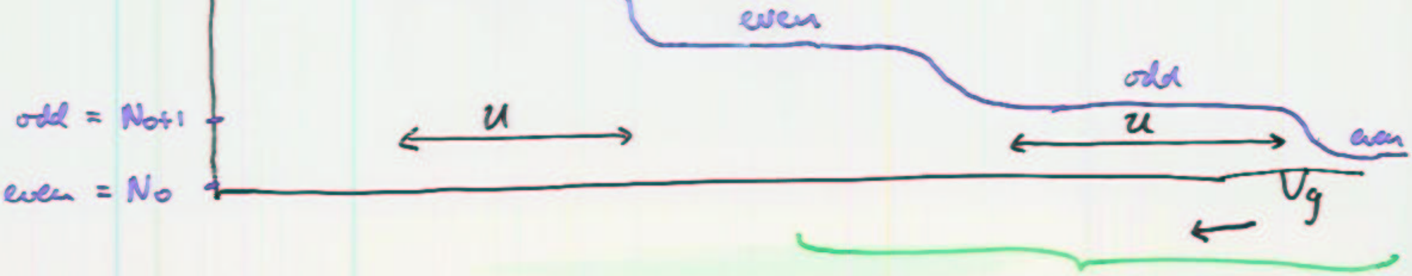




scaling collapse by picking new T_K for each V_g



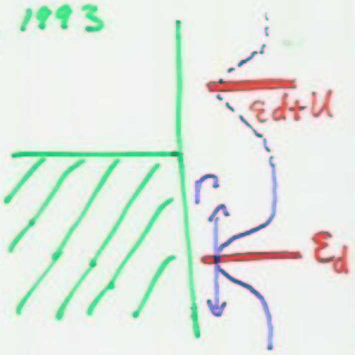




described by single-level AM

Single-level AM

[Anderson, PR 124, 41 (1961)
Hewson, "The Kondo Problem to Heavy Fermions", Cambridge, 1993]



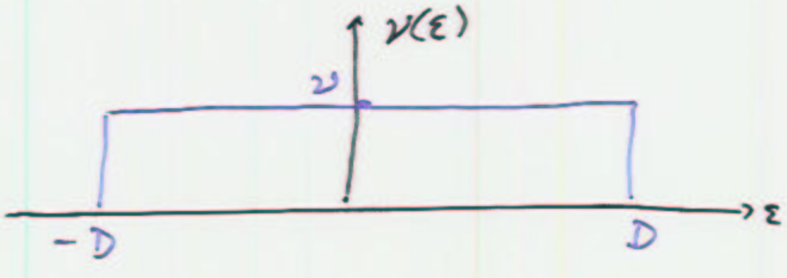
$$H = H_0 + H_1$$

$$H_0 = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \varepsilon_d \underbrace{\sum_\sigma d^\dagger_\sigma d_\sigma}_{\hat{n}_d} + U d^\dagger_\uparrow d_\uparrow d^\dagger_\downarrow d_\downarrow \qquad (1a)$$

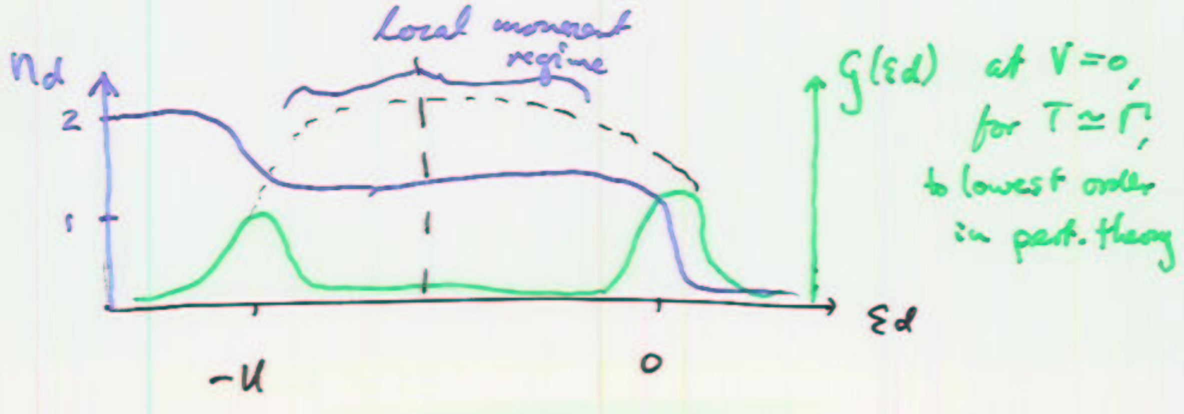
$$H_1 = \sum_k V_k \left[c^\dagger_{k\sigma} d_\sigma + d^\dagger_\sigma c_{k\sigma} \right] \qquad (1b)$$

[take V_k = real, k-independent]

choose flat DOS:



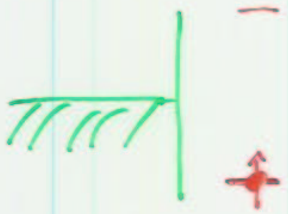
level width : $\Gamma = \pi \nu V^2$ (golden rule)

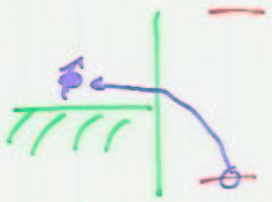


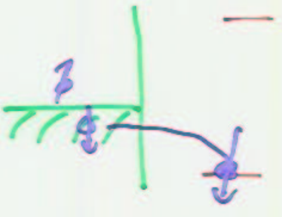
$g(\varepsilon_d)$ at $V=0$, for $T \simeq \Gamma$, to lowest order in pert. theory

What is origin of "Kondo plateaux"

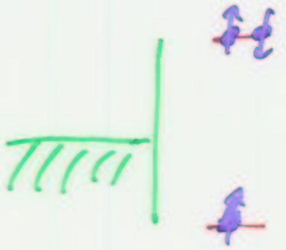
- occurs for $T \to 0$, when $n_d = 1$.
- $n_d = 1 \Rightarrow$ localized spin
- localized spin + conduction band = Kondo model
- $\Rightarrow$ spin-flip scattering !!

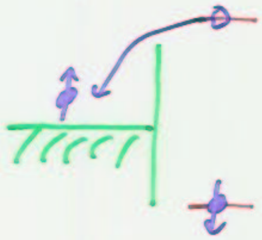






or:





spin-flip processes occur via virtual intermediate states

Schrieffer-Wolff-transformation (SWT) [PR 169, 491 (1966)]

Idea: seek effective $\tilde{H}$ in subspace of $n_d = 1$.

try unitary transformation:
$$\tilde{H} = e^{A} H e^{-A} \, , \quad \text{with } A^{+} = -A \tag{2}$$

A has pert. expansion in V_k:
$$A = 1 + O(V_k) + O(V_k^2) + \dots$$

expand
$$\widehat{H} \overset{(2)}{=} H_0 + \overbrace{H_1 + [A, H_0]}^{=0} + [A, H_1] + \overbrace{\tfrac{1}{2}[A[A, H_0]]}^{-\frac{1}{2}[A, H_1]} + o(V^3) \tag{3}$$

demand: $\tilde{H}$ contains no order $O(V_k^1)$

$$\Rightarrow \quad H_1 = -[A, H_0] \tag{4}$$

then
$$\widehat{H} = H_0 + \tfrac{1}{2}[A, H_1] + O(V^3) \tag{5}$$

(4) is satisfied by [illegible] (check algebra!)

$$A = \sum_{k\sigma} V_k \left[\frac{1}{\varepsilon_k - \varepsilon_d} c^{+}_{k\sigma} c_{d\sigma} + \frac{U}{(\varepsilon_d - \varepsilon_k)(\varepsilon_d + U - \varepsilon_k)} d^{+}_{-\sigma} d_{-\sigma} c^{+}_{k\sigma} d_\sigma \right] - (h.c.) \tag{6}$$

Effective Hamiltonian for nd = 1

(5) yields: $$\tilde{H}\Big|_{nd=1} = \sum_k \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{kk'} V^{(2)}_{kk'}\, \vec{s}_{kk'} \cdot \vec{S} \qquad (7)$$

+ ($c^\dagger_{k\sigma} c_{k'\sigma}$ - term: potential scattering)

local spin operators: $$S^+ = d^\dagger_\uparrow d_\downarrow \quad S^- = d^\dagger_\downarrow d_\uparrow \quad S^z = \tfrac{1}{2}(d^\dagger_\uparrow d_\uparrow - d^\dagger_\downarrow d_\downarrow) \qquad (8a)$$

conduction band spin operators: $$\vec{s}_{kk'} = \sum_{\sigma\sigma'} c^\dagger_{k\sigma} \tfrac{1}{2} \vec{\sigma}_{\sigma\sigma'} c_{k'\sigma'} \qquad (8b)$$

coupling: $$V^{(2)}_{kk'} = -\tfrac{1}{2} V_k V_{k'} \frac{U}{(\varepsilon_d - \varepsilon_k)(\varepsilon_d + U - \varepsilon_k)} + k \to k' \qquad (9)$$

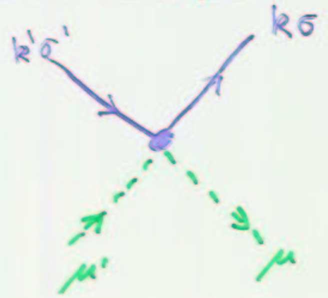
Reduces to Kondo model "at low energies":

for $|\varepsilon_k|, |\varepsilon_{k'}| \ll |\varepsilon_d|, |\varepsilon_d + U|$:

$$\boxed{H_{\text{Kondo}} = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + J \sum_{\substack{kk'\\ \mu\mu'}} \vec{s}_{kk'} \cdot \vec{S}_{\mu\mu'}} \qquad (10)$$

$$J = V^{(2)}_{k_F k_F} \overset{(9)}{=} - \frac{V_{k_F}^2 U}{\varepsilon_d(\varepsilon_d + U)} \qquad (11)$$

$$= \frac{4 V_{k_F}^2}{U} \; (>0) \quad \text{for } \varepsilon_d = -U/2$$



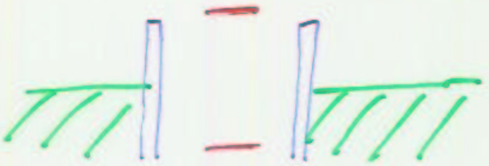
Low-energy properties of AM for nd = 1 are described by KM

Scattering term favors singlet formation

$\frac{1}{\sqrt{2}}(|\Uparrow\downarrow\rangle - |\Downarrow\uparrow\rangle)$

Single-level QD with two leads

[Pustilnik, Glazman, PRL $\underline{87}$, 216601 (2001) ; cond-mat/0302159
0401517]



$\alpha = L, R =$ lead index

$$H = \sum_{\underline{k\alpha\sigma}} \varepsilon_k c^\dagger_{k\alpha\sigma} c_{k\alpha\sigma} + \sum_{\underline{k\alpha\sigma}} V_k^\alpha c^\dagger_{k\alpha\sigma} d_\sigma + h.c. \tag{12}$$

+ usual H_{dot}

SWT as before, with $\sum_k V_k \rightarrow \sum_{k\alpha} V_k^\alpha$

$$H_{\substack{Kondo\\ int.}} = \underbrace{\sum_{\alpha\alpha'} \left[- \frac{V_{k_F}^{\alpha} V_{k_F}^{\alpha'} U}{\varepsilon_d(\varepsilon_d + U)} \right]} \underbrace{\left(\sum_{\substack{kk'\\ \sigma\sigma'}} c^\dagger_{k\alpha\sigma} \tfrac{1}{2} \vec{\sigma}_{\sigma\sigma'} c_{k'\alpha'\sigma'} \right)} \cdot \vec{S} \tag{13a}$$

$$= \sum_{\alpha\alpha'} J_{\alpha\alpha'} \vec{\mathcal{S}}_{\alpha\alpha'} \cdot \vec{S} \tag{13b}$$

Coupling matrix: $J_{\alpha\alpha'} = \tilde{c} \begin{pmatrix} V_L^2 & V_L V_R \\ V_R V_L & V_R^2 \end{pmatrix}$ (14)

$$\boxed{\begin{array}{l} \text{with } V_{k_F}^\alpha \equiv V_\alpha \\ \tilde{c} = - \dfrac{U}{\varepsilon_d(\varepsilon_d + U)} \end{array}}$$

$\det J = V_L^2 V_R^2 - (V_L V_R)^2 = 0$

one eigenvalue $= 0$.

$J_{\alpha\alpha'}$ is diagonalized by

$$W = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad , \quad \tan\theta = -V_R/V_L \qquad (15)$$

$$\text{with} \quad W J W^{+} = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} = \begin{pmatrix} \hat{c}(V_L^2 + V_R^2) & 0 \\ 0 & 0 \end{pmatrix} \qquad (16)$$

[Comment: for <u>multi</u>level AM, J_2 can be nonzero]

Rotate basis: $\psi_{k\gamma\sigma} = C_{k\alpha\sigma} W_{\alpha\gamma}$, $\gamma = 1,2$ (17)

(18)

$$H_{\substack{\text{Kondo}\\\text{int}}} \overset{(13b)}{=} \mathrm{Tr}(J\vec{s}) \cdot \vec{S} = \mathrm{Tr}\left(W^{+} J W \, W^{+} \vec{s} W\right) \vec{S}$$

$$= J_1 \underbrace{\vec{s}_1 \cdot \vec{S}}_{\sum_{kk'\sigma\sigma'} \psi^{+}_{k1\sigma} \frac{1}{2} \vec{\sigma}_{\sigma\sigma'} \psi_{k'1\sigma'}} \qquad (19)$$

$$H = \sum_{\substack{k\sigma\\\gamma}} \varepsilon_k \psi^{+}_{k\gamma\sigma} \psi_{k\gamma\sigma} + J_1 \vec{s}_1 \cdot \vec{S}$$

$$\boxed{\text{One mode decouples completely, other mode makes Kondo}}$$

Conductance through (many-level) QD with 2 leads

[Pustilnik, Glazman]

Consider $T=0$, $B \neq 0$ but $\to 0$ (to ensure non-degenerate ground state)

Then incident electrons experience only potential scattering, described by 2×2 $\mathcal{S}$-matrix:

$$\mathcal{S}_{\sigma,\alpha\alpha'} = W^{\dagger} D_{\sigma} W \underset{\uparrow \text{ same as in (15)}}{} , \qquad D_{\sigma} = \begin{pmatrix} e^{2i\delta_{1\sigma}} & 0 \\ 0 & e^{2i\delta_{2\sigma}} \end{pmatrix} \qquad (20)$$

Phase shifts: $\delta_{\gamma\sigma}$, with $\gamma = 1,2$, $\sigma = \uparrow,\downarrow = +,-$

Landauer-Formula for conductance:

$$G(T{=}0) = \frac{e^2}{h} \sum_{\sigma} \left| \mathcal{S}^{*}_{\sigma,RL} \right|^2 \overset{(20)}{=} \frac{e^2}{h} \sum_{\sigma} \left| (W^{\dagger} D W)^{*}_{RL} \right|^2 \qquad (21)$$

[illegible] $\rightsquigarrow$

$$= G_0 \, \frac{1}{2} \sum_{\sigma} \sin^2 (\delta_{1\sigma} - \delta_{2\delta}) \qquad (22)$$

with
$$G_0 = \frac{2e^2}{h} \sin^2 2\theta = \frac{2e^2}{h} \frac{4(V_L V_R)^2}{(V_L^2 + V_R^2)^2} = \frac{2e^2}{h} \text{ if } V_L = V_R \qquad (23)$$

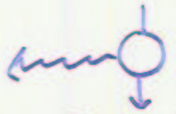
$$\boxed{T=0 \text{ conductance is determined purely by phase shifts}}$$

Comments about phase shifts

- phaseshifts defined only mod (π)
- choose $\delta_{\gamma\sigma} = 0$ for $J_\gamma = 0$, then $|\delta_{\gamma\sigma}| \le \pi/2$
- H_{Kondo} is invariant under particle-hole transf,

$$\psi_{k\alpha\sigma} \to \sigma\, \psi^\dagger_{-k\alpha,-\sigma}$$

- thus particle $(k\sigma)$ scatters same way as hole $(-k,-\sigma)$



$$\text{diag}\left(e^{i2\delta_{\gamma\sigma}}\right) = S_\sigma = S^\dagger_{-\sigma} = \text{diag}\left\{e^{-2i\delta_{\gamma,-\sigma}}\right\} \tag{24}$$

$$\delta_{\gamma\uparrow} = -\delta_{\gamma\downarrow} \equiv \delta_\gamma \tag{25}$$

Final very useful formula for conductance at $T=0$:

$$\frac{G(T=0)}{G_0} \overset{(23),(25)}{=} \sin^2(\delta_1 - \delta_2) \tag{26}$$

(for many-level AM).

Conductance through 1-level QD with 2 leads

For 1-level AM, $J_2 \overset{(16)}{=} 0 \Rightarrow \delta_2 = 0$

We'll see (next lecture) : at $T=0$, $\boxed{\delta_1 = \pi/2}$ (27)

next time.

$$\Rightarrow \quad G(T=0) \overset{(26)}{=} G_0$$

$$= \frac{2e^2}{h} \quad \text{if } V_L = V_R$$

= "unitarity limit", maximal possible value, as though channel were completely open!