Two lectures and a seminar

- Laplacian Growth
- Stochastic Loewner evolution and critical phenomena in 2D (CFT=Conformal Field Theory)
- Aspects of "Bosonization"

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Laplacian Growth

Laplacian Growth

Singular patterns in nonequlibrium regime

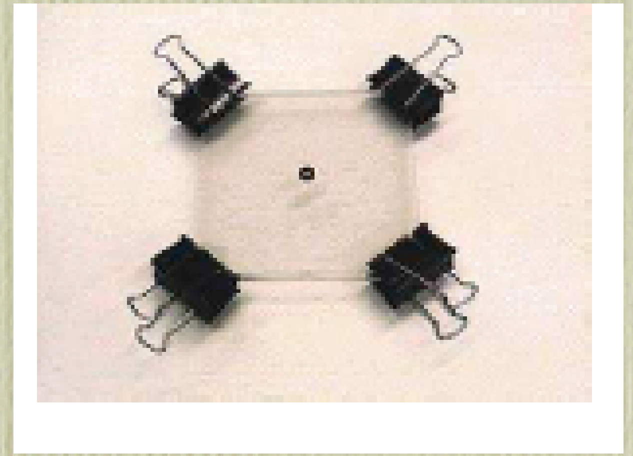
Other names:

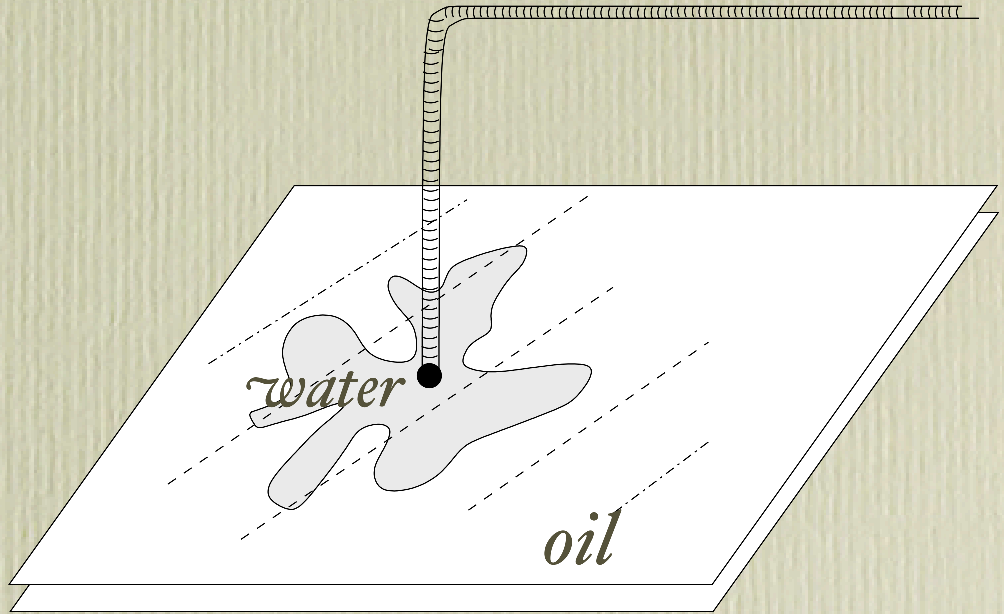
- Hele-Shaw problem;
- Saffman-Taylor problem;
- Fingering instability;
- Growth in the diffusion limit;
- ....(few more, like secondary oil recovery, ...

An interplay between:

- Deterministic Growth
- Stochastic Growth

Hele-Shaw cell





Oil (exterior)-incompressible liquid with high viscosity

Water (interior) - incompressible liquid with low viscosity

Laplacian growth;

diffusion driven patterns.

$$v_n = -\nabla_n P \quad \text{on the interface}$$

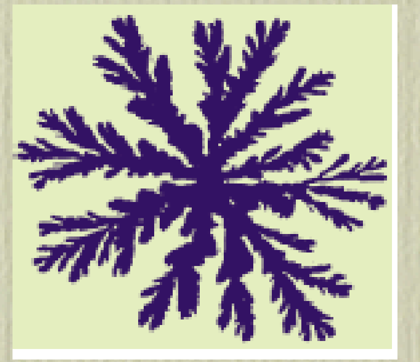
$$\Delta P = 0 \quad \text{in oil,}$$

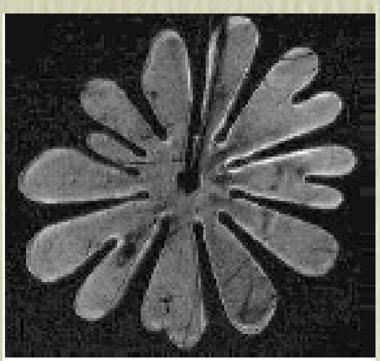
$$P = \sigma \times \text{curvature} \quad \text{in water}$$

σ -surface tension

Velocity= gradient of a harmonic function

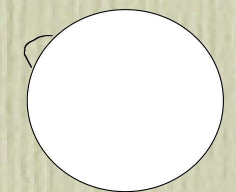
Fingering Instability

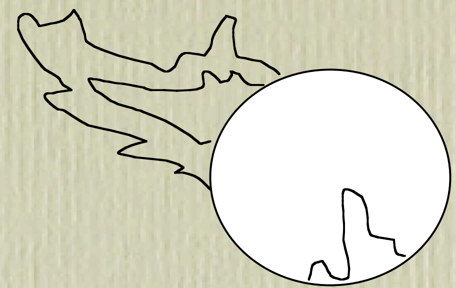




Large flux, small surface tension

Small Flux, large surface tension





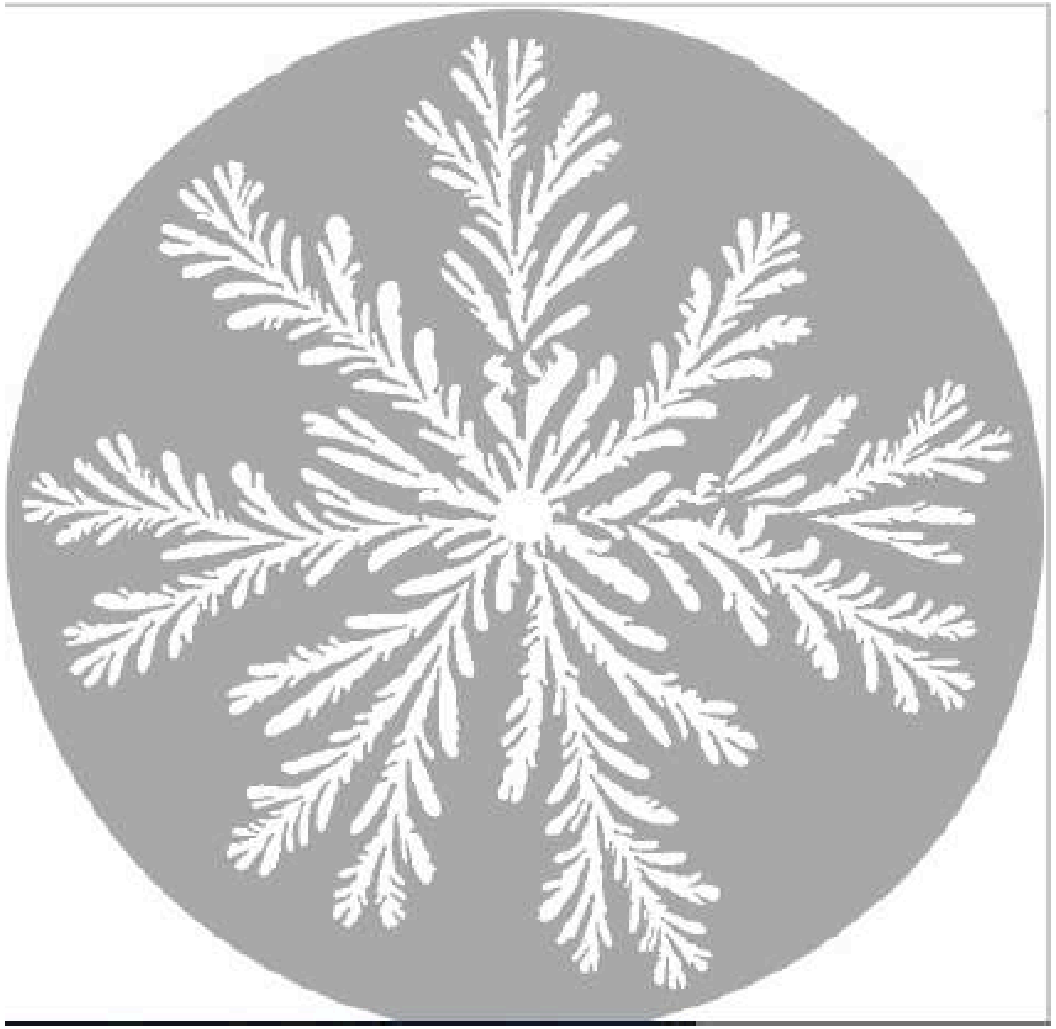
Fingering
instability

An arbitrary small bump gives rise to a cascade of fingers

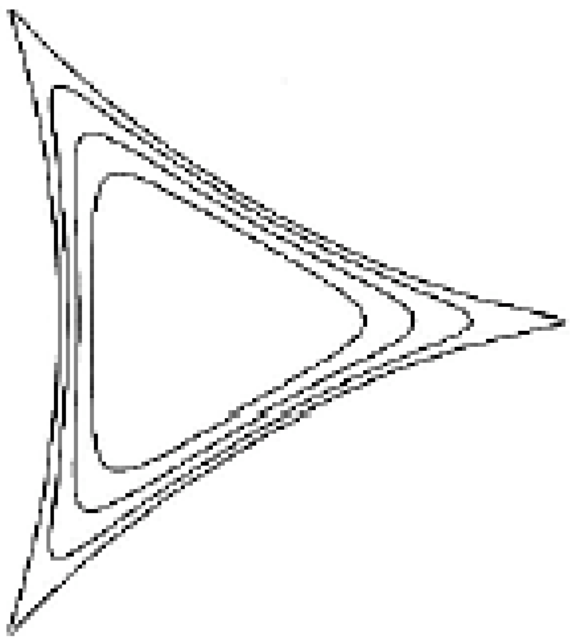
Fingering instability is typical if **convection is suppressed, diffusion drives the** game

Flame with no convection

after Sharon, Moore, McCormick, and Swinney, University of Texas at Austin



Finite time singularities



Hypotrochoid grows until it reaches a critical point.

- Fingering Instability
- Finite time cusp-like singularities
- Necessity of a regularization of short distances

Interpretation:

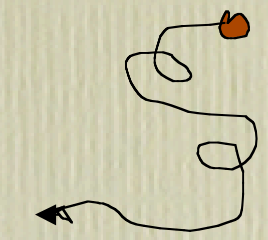
pressure - solution of the Dirichlet problem

$p = \log w(z)$ - *conformal map*

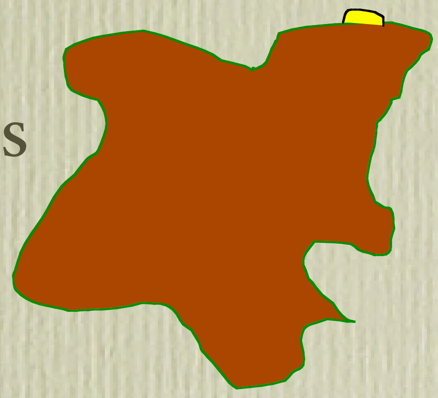
velocity -

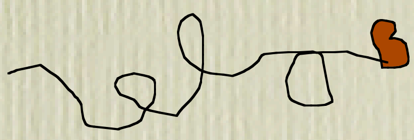
$v_n = \nabla p = |w'(z)|$ - *Conformal measure*

A probability of a Brownian mover to arrive and join the aggregate



the probability happens to be a conformal measure



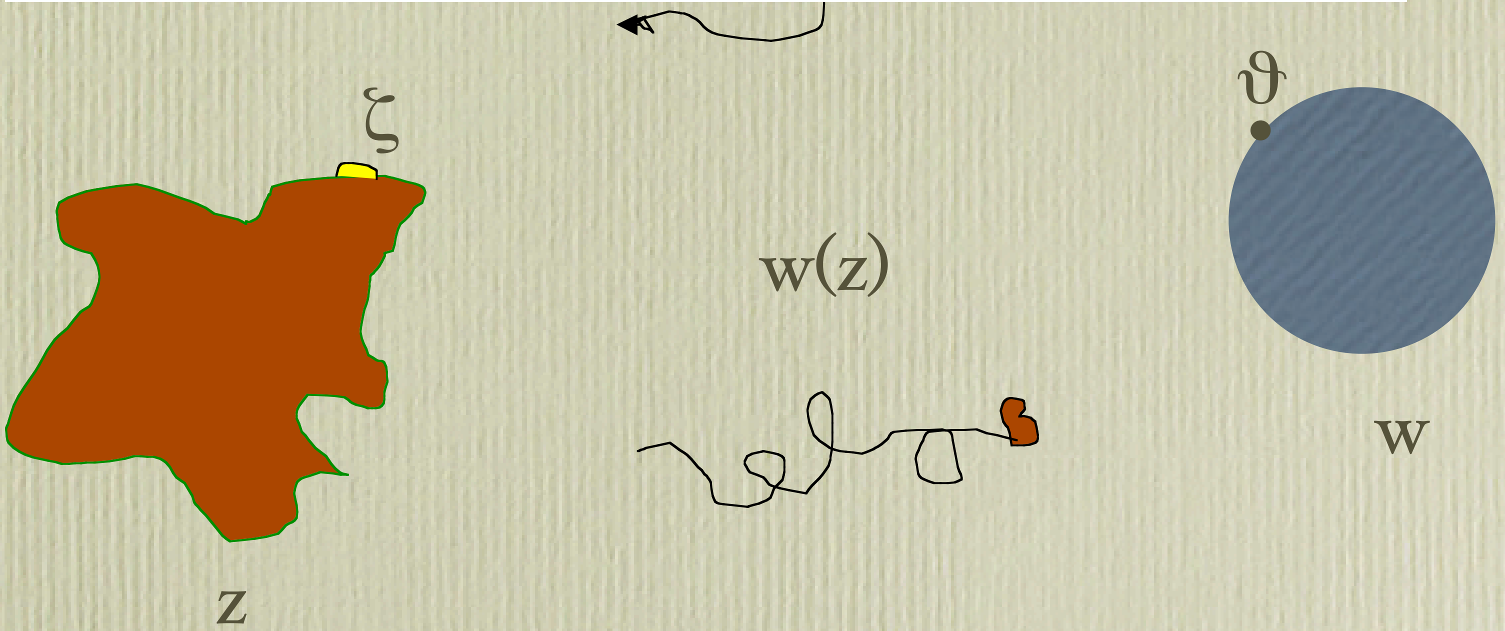


While joining the aggregate a mover becomes a *fermion*

can not sit on top of each other

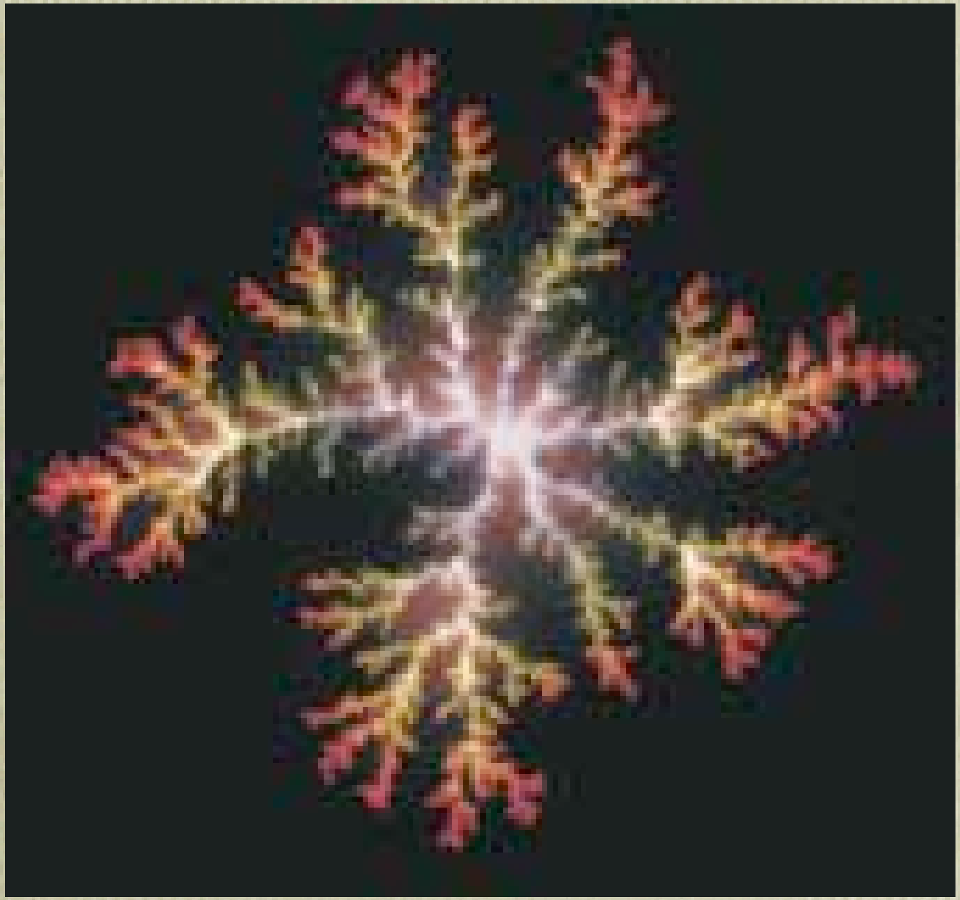
Hadamard formula

$$\frac{\delta w(z)}{w(z)} = \frac{w(z)+e^{i\theta}}{w(z)-e^{i\theta}} \cdot |w'(\zeta)|^2 \mathrm{d(Area)}$$

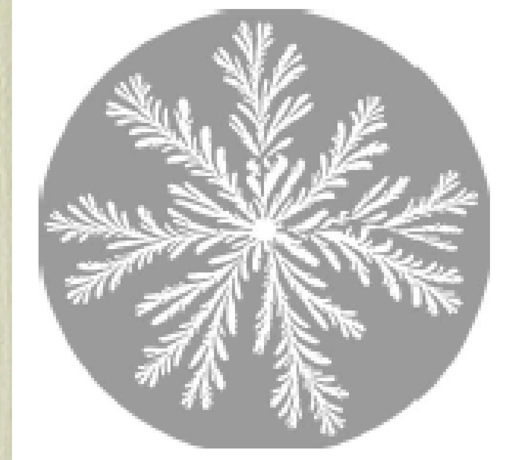


$$\frac{d}{dt} \log w(z) = \frac{w(z)+e^{i\theta}}{w(z)-e^{i\theta}} \cdot |w'(\zeta)|^2$$

Diffusion limited aggregation



turbulent regime -
collection of singularities



Fluid dynamics

Continuos (hydrodynamics problem is ill-defined especially in the at the turbulent regime -many singularities);

Turbulent (singular) regime occurs inevitably;

Cut-off singularities at a small scale is necessary;

DLA - Diffusion limited aggregation provides a regularization: Arriving particles have a finite size;

Noise - a mechanism of stabilization;

Recent developments:

- Link to Random (non-Hermitian) Matrix Ensembles - particles are eigenvalues of a random matrix;

- Link to QHE and Correlation functions in quantum wires: water domain= Quantum droplet on the LLL, or a Fermi surface in the phase space;

- Link to a Boundary Conformal Field Theory;

- Link to Integrable system - Toda lattice hierarchy (singular limit of dispersive waves)

- Link to problems of 2D - Quantum gravity.

Random Matrices

$M_{ij} \quad - N \times N -$ matrix with random elements

$dP(M_{ij}) = e^{-\frac{1}{\hbar}\mathrm{Tr}V(M)} dM_{ij}$ Probability distribution

$V(M) = M^2 -$ Gaussian Potential

Non Gaussian potential

Goal: having a distribution of matrix elements, determine a distribution of eigenvalues

In the limit $\hbar \to 0, \to \infty \quad N\hbar = t =$ fixed

$$M = U^{-1} \mathrm{diag}(x_1, \ldots. x_n) U$$

Integration over U gives the distribution of x's

Hermitian matrices:

$$dP(x) = \prod_{i>j} (x_i - x_j)^2 e^{-\frac{1}{\hbar} \sum_i V(x_i)} dx = |\Psi(x_1, \ldots x_N)|^2$$

Wave function of free fermions in a parabolic potent
Slatter determinant

$$\Psi(x_1, \ldots, x_N) = \det_{i,n} ||\psi_n(x_i)|| = \Delta(x) e^{-\frac{1}{\hbar} \sum_i x_i^2}$$

$$\Delta(x) = \prod_{i>j} (x_i - x_j) - \text{VanderMonde determinant}$$

Hermitian matrix ensemble:

eigenvalues are real, distributed in an interval of the real axis

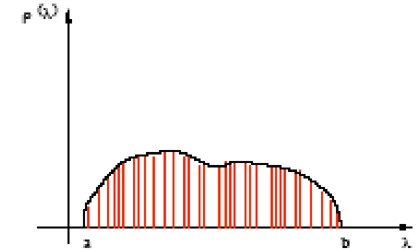
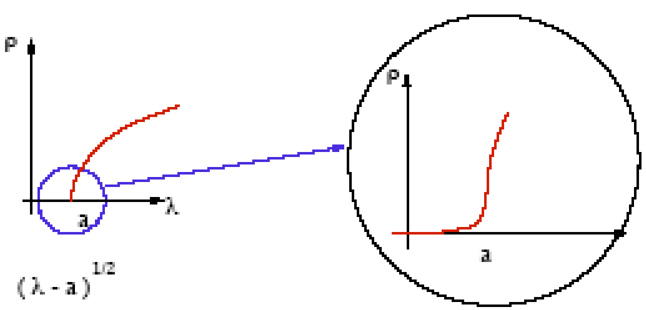


Figure 2.2: Typical spectrum of a matrix. In the large N limit, the spectrum is well approximated by a continuous density of levels $\rho(\lambda)$.

Bulk properties are universal (do not depend on potential V(M))

Edge properties are also universal but do depend on potential



e 2.6: The density of eigenvalues has square-root singularities at the edges. An gement of the edge's vicinity shows a universal tail related to the Airy function.

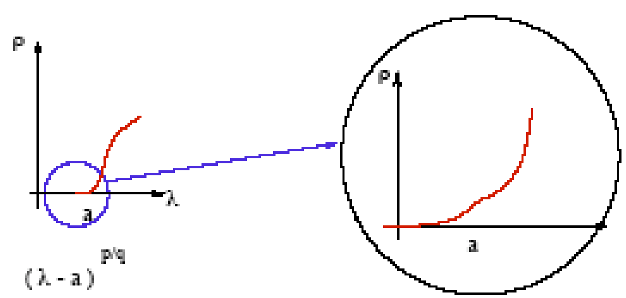


Figure 2.7: At a critical point, the density of eigenvalues has power law singularities at the edges. An enlargement of the edge's vicinity shows a universal tail related to some integrable hierarchy of differential equations.

Edge singularities are characterized by p, q

Normal Matrix ensemble: [M,M*]=0;

Eigenvalues are complex: $z_1, \ldots z_N$

They densely occupies a domain in a complex plane

$$dP = |\Delta(z)|^2 e^{-\frac{1}{\hbar}\sum_i (|z_i|^2 - V(z_i) - \overline{V(z_i)})} = |\Psi(z_1, \ldots, z_N)|^2$$

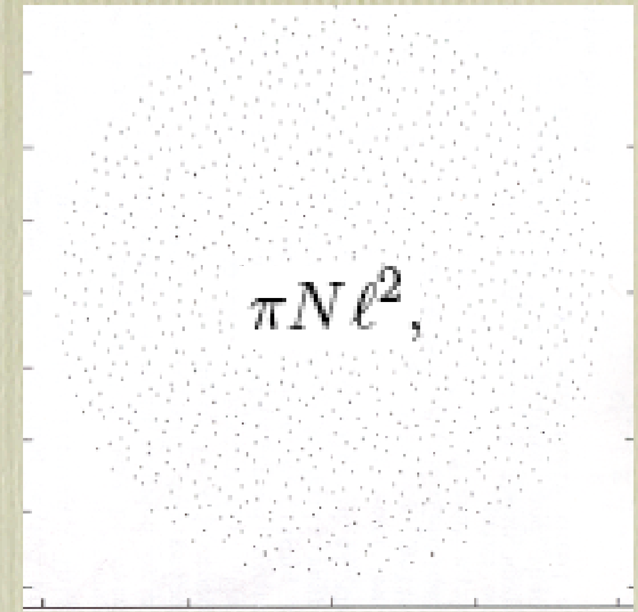
$$\Psi(z_1, z_2, \ldots, z_N) = \prod_{m<n} (z_n - z_m)\, e^{-\sum_n \frac{|z_n|^2}{2\ell^2}}$$

Two interpretations:

1) 1D fermions in coherent state representation z=x+ip

2) 2D Fermions on the LLL

Particles on LLL in a uniform magnetic field form a circular droplet



Every particle occupies a box of in the phase space 2π\hbar=(magnetic length)^2

The area of the droplet is 2π\hbar·N droplet is incompressible

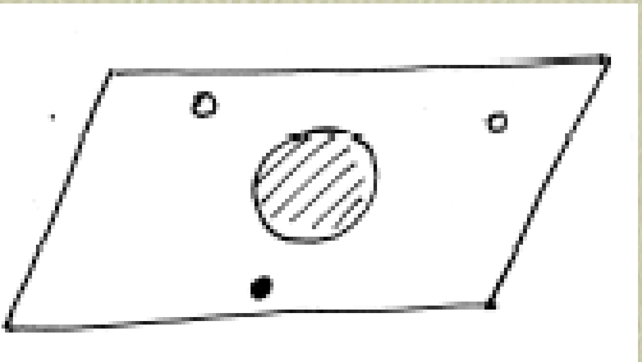
$$\Psi(z_1, z_2, \ldots, z_N) = \prod_{m<n} (z_n - z_m)\, e^{-\sum_n \frac{|z_n|^2}{2\ell^2}}$$

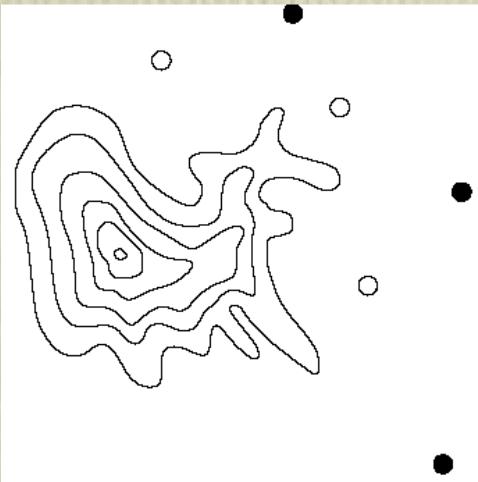
If N⇒N+1 grows, the area also grows, the circle remains to be a circle

Aharonov-Bohm Effect on LLL

The are of the droplet is controled by the number of particles, but who controls the shape?

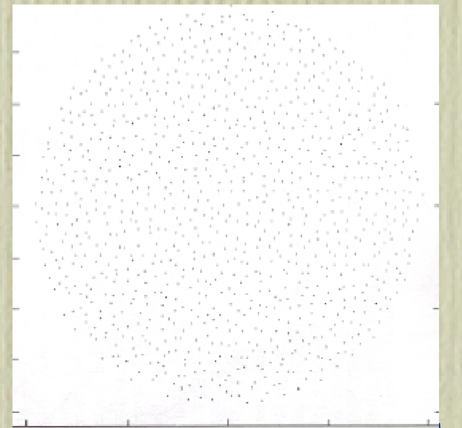
$$\Psi = \Delta(z) e^{-\frac{1}{\hbar}\sum_i(|z_i|^2 - V(z_i) - \overline{V(z_i)})}$$

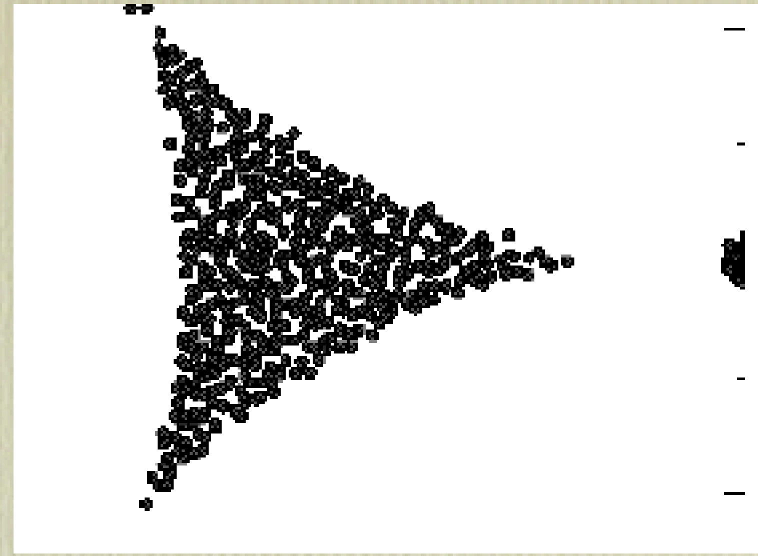




$$\text{Magnetic field} = B = -\partial_z \partial_{\bar{z}}(|z|^2 - V(z) - \overline{V(z)})$$

The circular shape is unstable under gradients of magnetic field





Semiclassical limit of the Growth:
N→ N+1 provides the same hydrodynamics as Laplacian growth.

Quantum limit provides stochastic (dispersive) regularization

If a potential of a nonuniform part of magnetic field is

$$V(z) = Re \sum_{k \geq 1} t_k z^k .$$

Then the droplet's harmonic moments are exactly t_k, the area is determined by the homogeneous part of the magnetic field

- the area of the domain is $\pi N \ell^2$;
- the harmonic moments of the exterior of the domain

$$t_k = -\frac{1}{\pi k} \int z^{-k} d^2 z , \quad k = 1, 2, \ldots$$

$$\Psi(z_1, \cdots, z_N) = \frac{1}{\sqrt{N!\tau_N}} \Delta(z) e^{-(\sum_n \frac{1}{2\ell^2}|z_n|^2 - V(z_n))}.$$

The saddle point equation (4) is transformed accordingly

$$\sum_{m \neq n}^{N} \frac{2\ell^2}{z_n - z_m} = \bar{z}_n - 2\ell^2 \frac{\partial}{\partial z} V(z).$$

Semiclassical approximation

Interpretation: $$\psi_{N+1}(z) = \int \Psi(z, \xi_1, \ldots, \xi_N) \overline{\Psi(\xi_1, \ldots, \xi_N)} \prod_{n \le N} d^2\xi_n.$$

$|\psi_N|^2$ - probability for a new particle to arrive to a point z- an overlap between N and N+1 sates

In semiclassical regime - it is proportional to a conformal measure

$$|\psi_N|^2 \simeq \frac{1}{2\pi} |w'(z)| \delta(z),$$

Quantization can be seen as a noise

Noise is negligible in the semiclassical regime (a smooth droplet)

Important in the turbulent (singular) regime, where semiclassics fails.

Singular semiclassical expansion at the cusps

- Laplacian growth=Witham hierarchy of soliton equations
- Dispersive regularization of nonlinear waves

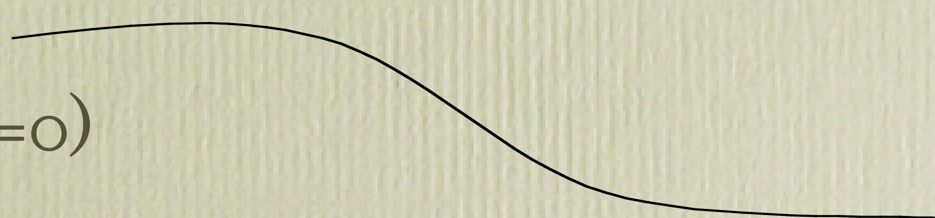
- Witham hierarchy = a semiclassical limit of soliton equations

- Example KdV:

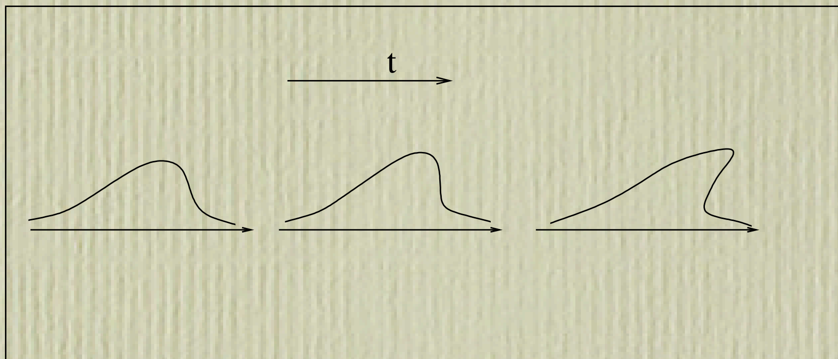
$$\boxed{\mathrm{u}_t - u \cdot u_x + \hbar \cdot u_{xxx} = 0}$$

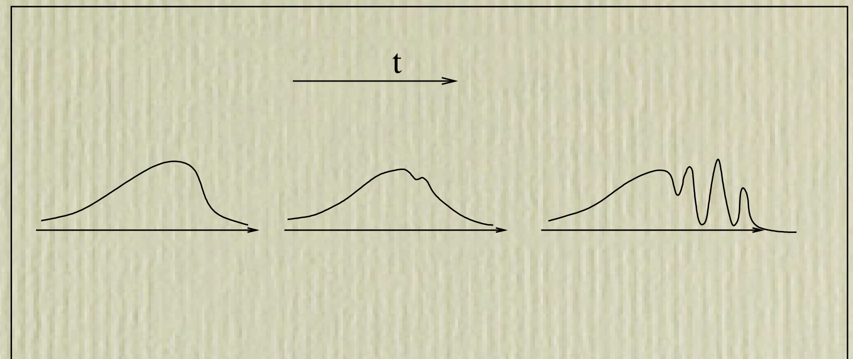
- Periodic (soliton-like) solution: waves in shallow water;

- Non-periodic solutions;



u(x,t=0)





Overhang - a result of
ill- approximation

Oscillations is a
dispersive regularization

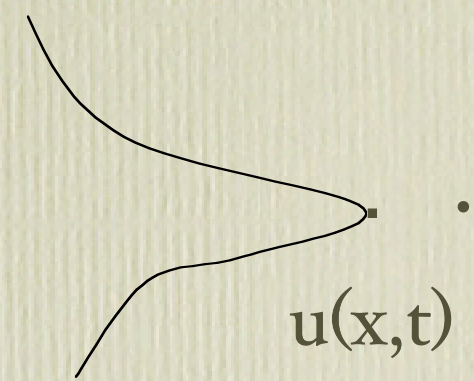
$$u_t - u \cdot u_x + \hbar \cdot u_{xxx} = 0 \quad \Rightarrow \quad u_t - u \cdot u_x = 0$$

Fine structure of a cusp

Painleve I equation

$$\boxed{\hbar \cdot u_{tt} - u^2 = t - t_c}$$

u(t) - a coordinate of a tip



Universal character of the fine structure of singularities

Universal character of of singularities

Asymmetries and tip splitting

Higher order singularities - higher order Painleve like equations classified by two integers (p,q) according to Kac table of CFT

Set of exponents - gravitational descendants

$$\Delta^{m}_{p,q} = \frac{q-p+m}{p+q-1}, \quad m = 1, \ldots, q < p$$

Are fractal exponents of DLA among this set?