



# Nonlinear bosonization

Hydrodynamic description of 1D fermions

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Goal:

to express fermionic objects

$$H = c^\dagger \left( -\frac{\nabla^2}{2m} - \mu \right) c + \text{interaction}$$

through hydrodynamics modes:

$$\rho(x) = \sum_i \delta(x - x_i) = c^\dagger(x)c(x)$$

$$j(x) = \sum_i \dot{x}_i \delta(x - x_i) = c^\dagger(x)i\nabla c(x)$$

Same is applied to bosons

$$H = b^\dagger \left( -\frac{\nabla^2}{2m} - \mu \right) b + \text{interaction}$$

- Why it is possible?
- When it is effective?

Math: appearance of complex structure

Physics: If there is no, or just few channels of scattering (energy dissipation)

Two major examples:

- 1) 1D- electrons - forward-backward scattering;
- 2) Superconductor

Linear bosonization:

Linearisation of the spectrum:

$$\frac{p^2}{2} - \epsilon_F \sim \pm v_F(p \pm k_F)$$

$$c(x) \sim \psi_R(x)e^{ik_Fx} + \psi_L(x)e^{-ik_Fx}$$

Rules:  $\rho(x) = c^\dagger c \rightarrow -\nabla\varphi$

$$j(x) = i c^\dagger \nabla c \rightarrow -\dot{\varphi}$$

$$\psi_{R,L} \sim e^{2\pi i \varphi_{RL}}$$

$$K = \psi_R^\dagger \nabla \psi_R + \psi_L^\dagger \nabla \psi_L \sim (\nabla \varphi_R)^2 + (\nabla \varphi_L)^2$$

# Free Fermions=Free bosons

$$c^\dagger \left( i\partial_t - \frac{\nabla^2}{2} - \varepsilon_F \right) c \sim (\dot{\varphi})^2 - (\nabla\varphi)^2$$

**Accuracy?**  $O(1/N) = O(1/k_F)$

example

$$<<\rho(1)\rho(2)\rho(3)>> \sim \partial_1\partial_2\partial_3 <<\varphi(1)\varphi(2)\varphi(3)>= 0$$

$$<<\rho(1)\rho(2)\rho(3)>> \sim \partial_1\partial_2\partial_3 <<\varphi(1)\varphi(2)\varphi(3)>=0$$

$$<<\rho(x_1)\dots\rho(x_n)>> = (k_F)^{-n+2} f_n(x_1, \dots, x_n)$$

Linear “bosonization” fails when spectrum asymmetry is important, and when one goes beyond a linear response.

Sakita 1974

recognized in string theory as collective field theory

## First quantization

$$H_F = - \sum_i \frac{\partial^2}{\partial x_i^2} - \text{fermions: antisymmetric w.f.}$$

$$H_B = - \sum_i \frac{\partial^2}{\partial x_i^2} - \text{bosons: symmetric w.f.}$$

$$H = c^\dagger \left( -\frac{\nabla^2}{2m} - \mu \right) c + \text{interaction}$$

## Collective variables (modes)

$$\phi_k = \sum_{i=1}^N x_i^k$$

$$\pi_k = \frac{\partial}{\partial \phi_k}$$

$$\nu_k = -ik\pi_k$$

$$[v_k \phi_n] = -ik\delta_{kn}.$$

## Density

$$u_-(z) = \sum_i \frac{1}{z - x_i} = \sum_i z^{-k-1} \phi_k$$

$$u_-(x + i0) - u_-(x - i0) = 2\pi i \sum_i \delta(x - x_i) = 2\pi i \rho(x)$$

## Current, velocity

$$u_+(z) = -iv(z) \qquad v(z) = \sum_{k \geq 0} z^{k-1} v_k,$$

$$j(x) = \sum_i \dot{x}_i \delta(x - x_i) = \rho(x) v(x)$$

Continuity equation

$$\dot{\rho} + \nabla \cdot j = 0$$

# Bose field

$$u(z) = u_+(z) + u_-(z) = \sum_{k>0} v_k z^{k-1} + \sum_i \phi_i z^{-k-1}$$

bosonic notations

$$a_n = i v_n, \quad a_{-n} = \phi_n, \quad , n > 0$$

$$u(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad [a_n, a_k] = n \delta_{n+k}$$

$$u(z) = \partial_z \varphi(z)$$

# Fermions (Bosons) in collective variables

$$H_F = \frac{\rho v^2}{2} + \frac{\pi^2}{3} \rho^3$$

$$H_B = \frac{\rho v^2}{2} - \frac{1}{8} \frac{(\nabla \rho)^2}{\rho}$$

Accuracy?  $O(e^{-N})$

$$H_F = \frac{\rho v^2}{2} + \frac{\pi^2}{3} \rho^3 \quad [v(x), \rho(y)] = -i\delta'(x - y)$$

I) Kinetic energy: Galileian inv.

2) Fermi energy (pressure)

$$\rho \varepsilon(\rho) = \frac{\pi^2}{6} \rho^3, \quad P = \rho^2 \frac{\partial \varepsilon}{\partial \rho} = \frac{\pi^2}{3} \rho^4$$

$$E_F = \int_{-k_F}^{k_F} \frac{k^2}{2} \frac{dk}{2\pi} = \frac{k_F^3}{6\pi} = \frac{\pi^2}{6} \rho^3 \quad \rho = \int_{-k_F}^{k_F} \frac{dk}{2\pi} = \frac{k_F}{\pi}$$

# Small waves - linear bosonization (linear hydrodynamics)

$$\rho = \rho_0 + \delta\rho$$

density

$$K = \frac{\rho v^2}{2} \sim \rho_0 \frac{v^2}{2}$$

Kinetic energy

$$\Pi = \frac{2}{6}\rho^3 \sim \Pi_0 + \pi^2 \rho_0^2 (\delta\rho)^2$$

Potential energy (Fermi pressure)

$$H = K + \Pi \sim \frac{1}{2\rho_0}(j^2 + v_0^2(\delta\rho)^2) = \frac{\kappa}{2}[(\dot{\varphi})^2 + v_0^2(\nabla\varphi)^2]$$

$$j = \dot{\varphi}, \quad \delta\rho = -\nabla\varphi$$

# Equations: Euler's Hydrodynamics

$$H_F = \frac{\rho v^2}{2} + \frac{\pi^2}{3} \rho^3$$

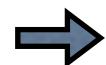
$$\dot{\rho}_t = i[H, \rho] \quad \partial_t \rho + \partial_x (\rho v) = 0$$

$$\dot{v} = i[H, v] \quad \partial_t v + v \partial_x v = \partial_x \left[ \epsilon + \rho \frac{\partial \epsilon}{\partial \rho} \right] = \frac{1}{\rho} \partial_x P,$$

$$\epsilon(\rho) = \frac{\pi^2}{6} \rho^2, \quad P = \frac{\pi^2}{3} \rho^3$$

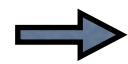
Hopf Equation:  $u(x + i0) = v + i\rho,$

$$\partial_t \rho + \partial_x(\rho v) = 0$$

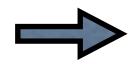


$$\dot{v} + v \cdot \nabla v = \pi^2 \rho \nabla \rho$$

$$\dot{u} + u \cdot u_x = 0$$



$$[u(x), u(y)] = i\delta'(x - y)$$



$\dot{u} + u_x = 0$  – linearized version

## Solution of Hopf equation

$$\dot{u} + u \cdot u_x = 0$$

$$u(x, t) = u_0(x - u(x, t) \cdot t), \quad u_0(x) = u(x, 0)$$

## Solution of wave equation

$$\dot{u} + u_x = 0 \text{ -- linearized version}$$

$$u(x, t) = u_0(x - t), \quad u_0(x) = u(x, 0)$$

**Hamiltonian in terms of Bose field:**  $H_F = \frac{\rho v^2}{2} + \frac{\pi^2}{3} \rho^3$

$$u(z) = u_+(z) + u_-(z) = \sum_{k>0} v_k z^{k-1} + \sum_i \phi_k z^{-k-1}$$

$$\phi_k = \sum_{i=1}^N x_i^k$$

$$u(z) = \partial_z \varphi(z)$$

$$H_F=\frac{1}{12}\oint_C u^3(z)\frac{dz}{2\pi i}=\int [\frac{\rho v^2}{2}+\pi^2 \frac{\rho^3}{6}]dx$$

$$\nabla_{\!X} Y = [X,Y]$$

$$u(z)=\partial_z\varphi(z)$$

$$H_F=\frac{1}{12}\oint_C u^3(z)\frac{dz}{2\pi i}$$

$$(\mathcal{L}_\theta \mathcal{L}_{\theta'} - \mathcal{L}_{\theta'} \mathcal{L}_\theta) \mathcal{L}_\theta$$

$$H_F=\frac{1}{12}\oint_C (\partial\varphi)^3\frac{dz}{2\pi i}$$

$$\varphi=2k_Fx+\phi\qquad\qquad H_F\rightarrow k_F\int(\partial\phi)^2\frac{dx}{2\pi}$$

$$(\mathcal{L}_\theta \mathcal{L}_{\theta'} - \mathcal{L}_{\theta'} \mathcal{L}_\theta) \mathcal{L}_\theta$$

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## Calogero-model

$$H = - \sum_i \partial_i^2 + \sum_{i \neq j} \frac{\beta/2(\beta/2 - 1)}{(x_i - x_j)^2}$$

Ground state

$$\Psi_0(x_1, \dots, x_N) = \Delta^\beta(x), \quad \Delta(x) = \prod_{i>j} (x_i - x_j)$$

Excited states

$$\Psi(x_1, \dots, x_N) = \Delta^\beta(x) J(x_1, \dots, x_N)$$

J(x) - Jack -polynomial - symmetric function

$\beta=0$  - bosons,     $\beta=2$  - fermions

$$H=\frac{1}{2}\sum_i(-\partial_i+A(x_i))(\partial_i+A(x_i)):$$

$$A(x)=-\frac{\beta}{2}\sum_i\frac{1}{x-x_i}$$

$$(\partial_i+A(x_i))\Psi_0(x)=0,\quad \Psi_0(x)=\Delta^{\beta/2}(x)$$

**Goal: rewrite everything in terms of collective modes**

$$\tilde{H} = -\Delta^{-\beta/2} H \Delta^{\beta/2} = \sum_i \left( -\partial_i + \beta \sum_j \frac{1}{x_i - x_j} \right) \partial_i$$

$$\tilde{H} J_E(x) = E J_E(x)$$

**Both, the Hamiltonian and the w.f's are symmetric functions**

$$\dot{\phi}_k = [\tilde{H}, \phi_k] \quad \dot{v}_k = [\tilde{H}, v_k]$$

$$-\frac{1}{l}\dot{v}_l = \sum_k v_k v_{l-k+2} + 2v_k \phi_{k-l-2} + \alpha(l+1)v_{l+2},$$

$$\alpha = 1 - \beta/2$$

$$\frac{1}{l}\dot{\phi}_l = \sum_k \phi_k \phi_{l-k-2} + 2v_k \phi_{l+k-2} + \alpha(l-1)\phi_{l-2},$$

**Goal:** to rewrite these eqs. in terms of the Bose field

$$u(z) = u_+(z) + u_-(z) = \sum_{k>0} v_k z^{k-1} + \sum_i \phi_k z^{-k-1}$$

$$\dot{u} + \frac{\beta}{2} u \partial_z u + i \left( \frac{\beta}{2} - 1 \right) \partial_z^2 (u_+ - u_-) = 0$$

## Benjamin-Ono equation

$\beta=0$  – bosons,     $\beta=2$  – fermions

$$\dot{\varphi} + \beta (\partial \varphi)^2 + i \left( \frac{\beta}{2} - 1 \right) \partial_z^2 (\varphi_+ - \varphi_-) = 0$$

$$u = \partial_z \varphi$$