Introduction in Bosonization I

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Non-interacting or Interacting?

In every physical system there are correlations between constituent particles due to various interactions

- "Non-interacting systems" - systems whose physics can be understood using the single particle picture and where correlations are an annoying ingredient reducing the predictive power of the theory
- "Interacting systems" - systems whose essential physical properties would not exist without interactions

Systems Where Interactions are Essential

Here are some textbook examples:

- Ferro and antiferromagnets
- Superconductors
- Fractional quantum Hall systems
- Kondo impurities
- Systems at Coulomb Blockade
- Luttinger liquids

The Aim of this Lecture Course

To give an elementary introduction into the physics of 1D Luttinger liquids, the basics of Bosonization technique and the scaling theory.

Outline

Part I

This is a formal part, where we will discuss the fermion-boson correspondence, the Luttinger model and the Bosonization.

Part II

In this part a contact to reality will be made. The concept of Luttinger Liquid will be introduced and physical examples will be given.

Part III

In this part applications of Luttinger Liquid theory to quantum impurity problems will be discussed.

The Fermion-Boson Correspondence

The Harmonic Oscillator

Bosonic Oscillator

The Bosonic algebra:

$$[b, b^\dagger] = 1$$

$$[b, b] = [b^\dagger, b^\dagger] = 0$$

The Fock space:

$$b|0\rangle = 0, \quad |n\rangle = (b^\dagger)^n|0\rangle$$

The Hamiltonian

$$H_B = \hbar\omega N_B, \quad N_B = b^\dagger b$$

Fermionic Oscillator

The Fermionic algebra:

$$\psi\psi^\dagger + \psi^\dagger\psi = 1$$

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0$$

The Fock space:

$$\psi|0\rangle = 0, \quad |1\rangle = \psi^\dagger|0\rangle$$

The Hamiltonian

$$H_F = \hbar\omega N_F, \quad N_F = \psi^\dagger\psi$$

Occupation Numbers

Bosons

The extended Bosonic algebra:

$$b_n,\ b_n^\dagger, \quad n = 1, \ldots, N$$

$$[b_n, b_m^\dagger] = \delta_{nm}$$

The Fock space is spanned by states of given occupation numbers $0 \leq n_k \leq \infty$

$$|n_1 \ldots n_N\rangle = (b_1^\dagger)^{n_1} \ldots (b_N^\dagger)^{n_N} |0\rangle$$

$$N_B = b_1^\dagger b_1 + \cdots + b_N^\dagger b_N$$

Fermions

The extended Fermionic algebra:

$$\psi_n,\ \psi_n^\dagger, \quad n = 1, \ldots, M$$

$$\{\psi_n, \psi_m^\dagger\} = \delta_{nm}$$

The Fock space is spanned by states of given occupation numbers $0 \leq n_k \leq 1$

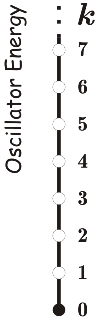
$$|n_1 \ldots n_N\rangle = (\psi_1^\dagger)^{n_1} \ldots (\psi_M^\dagger)^{n_M} |0\rangle$$

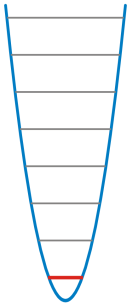
$$N_F = \psi_1^\dagger \psi_1 + \cdots + \psi_M^\dagger \psi_M$$

Fermion-Boson Correspondence: $N_F = 1$

$$H_F = \sum_{k=0}^{\infty} k\psi_k^\dagger \psi_k$$

$$H_B = b^\dagger b$$





For extended fermionic oscillator algebra ψ_k, $\psi_k^\dagger$, $0 \le k \le \infty$ and

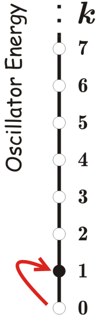
$$H_F = \sum_{k=0}^{\infty} kN_k, \qquad N_k = \psi_k^\dagger \psi_k$$

The Correspondence of States:

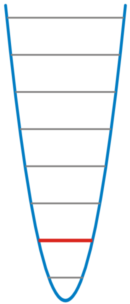
The eigenstates of H_F for $N_F = 1$ are in one to one correspondence to the eigenstates of $H_B = b^\dagger b$.

Fermion-Boson Correspondence: $N_F = 1$

$$H_F = \sum_{k=0}^{\infty} k \psi_k^{\dagger} \psi_k$$



$$H_B = b^{\dagger} b$$



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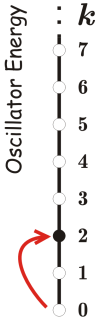

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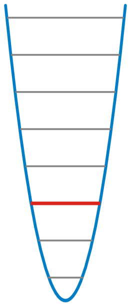

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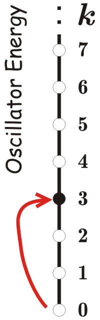

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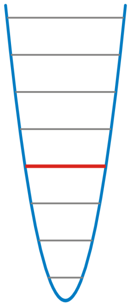

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The Correspondence of States:

The eigenstates of H_F for $N_F = 1$ are in one to one correspondence to the eigenstates of $H_B = b^\dagger b$.

Fermion-Boson Correspondence: $N_F = 2$

Fermion Hamiltonian

$$H_F = -1 + \sum_{k=0}^{\infty} k \psi_k^\dagger \psi_k,$$

for $N_F = 2$.

Boson Hamiltonian

$$H_B = b_1^\dagger b_1 + 2 b_2^\dagger b_2$$

Eigenstates of H_F and H_B

Fermions	Bosons	E
$\|110000\ldots\rangle$	$\|00\rangle$	0
$\|101000\ldots\rangle$	$\|10\rangle$	1
$\|011000\ldots\rangle$ $\|100100\ldots\rangle$	$\|01\rangle$ $\|20\rangle$	2
$\|010100\ldots\rangle$ $\|100010\ldots\rangle$	$\|11\rangle$ $\|30\rangle$	3

Energy levels and their degeneracies coincide for H_F and H_B

Fermion-Boson Correspondence: general N_F

Consider a pair of Hamiltonians

$$H_F = -E_N + \hbar\omega \sum_{k=0}^{\infty} k\psi_k^\dagger \psi_k \quad \text{and} \quad H_B = \hbar\omega \sum_{m=1}^{N} m b_m^\dagger b_m.$$

where H_F constrained onto the subspace of fixed particle number $N_F = N$. Here $E_N = \hbar\omega N(N-1)/2$.

Theorem

The eigenvalues of H_B and H_F coincide and are given by $E_n = \hbar\omega n$, where n is a non-negative integer. Each level E_n has the same degeneracy $D(n)$ for H_B and H_F.

Fermion-Boson Correspondence. Exercises.

Problem 1

Prove the theorem in the previous slide.

Problem 2

For the eigenvalue E_n of H_B calculate the degeneracy $D(n)$ of the corresponding eigenspace. Calculate large n asymptotics of $D(n)$ for $n \ll N$ and for $n \gg N$.

Problem 3

Calculate the quantum partition function $Z(\beta) = \sum_n D(n) e^{-\beta E_n}$, free energy and the specific heat of the system described by the Hamiltonian H_B (H_F) in the large N limit.

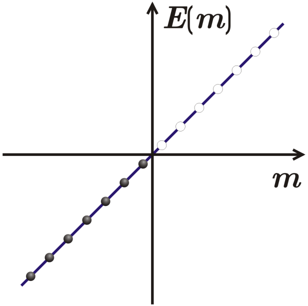
Weyl Fermion in 1+1 Dimensions I

Consider the Hamiltonian:

$$H_F = \frac{v}{r} \sum_{m=-\infty}^{\infty} \left(m - \frac{1}{2} \right) : \psi_m^\dagger \psi_m :$$

The ground state is the Dirac vacuum:

$$\psi^\dagger_{-m+1}|0\rangle = \psi_m|0\rangle = 0, \quad m \geq 1$$



Weyl Fermion in 1+1 Dimensions II

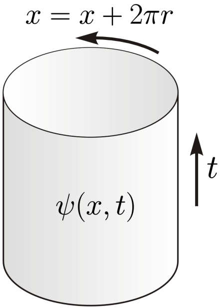
Introduce a local field:

$$\psi(x) = \frac{1}{\sqrt{2\pi r}} \sum_{m=-\infty}^{\infty} e^{i\frac{mx}{r}} \psi_m$$

$$\{\psi(x), \psi^\dagger(x')\} = \delta(x - x')$$

Time evolution is generated by Hamiltonian

$$H_F = v \int_0^{2\pi r} dx : \psi^\dagger(x)(-i\partial_x)\psi(x) :$$



Singularities and Normal Ordering

Consider a point-split product $\psi^\dagger(x)\psi(x+\epsilon)$. It is singular in the following sense:

$$\langle \psi^\dagger(x)\psi(x+\epsilon)\rangle = \frac{1}{2\pi i\epsilon} + O(1), \qquad \epsilon \longrightarrow 0$$

This can be used for an alternative definition of normal ordering:

$$: \psi^\dagger(x)\psi(x) := \lim_{\epsilon\longrightarrow 0}\left[\psi^\dagger(x)\psi(x+\epsilon) - \frac{1}{2\pi i\epsilon}\right]$$

Weyl Fermion: The Current Anomaly

The density operator is $\rho(x) = \psi^\dagger(x)\psi(x)$.

- Normally, the density operator satisfies $[\rho(x), \rho(x')] = 0$
- For the Weyl fermion $\psi^\dagger(x)\psi(x)$ is singular
- The non-singular (normal ordered) expression is $\rho(x) =: \psi^\dagger(x)\psi(x) :$

The Current Anomaly:

The normal ordered density satisfies $[\rho(x), \rho(x')] = \frac{i}{2\pi}\delta'(x - x')$

Weyl Fermion. Exercises.

Problem 1

Derive the formula for the current anomaly. (Hint: get rid of the singular operators in the result of the commutation before taking the $\epsilon \longrightarrow 0$ limit)

Problem 2

Derive the equation of motion for the Weyl fermion $\psi(x, t)$. Use the equation of motion to show that all local operators of the theory satisfy

$$\mathcal{O}(x, t) = \mathcal{O}(x - vt)$$

Chiral Boson in 1+1 Dimensions I

Consider a Hamiltonian

$$H_B = \frac{v}{r} \sum_{m \geq 1} m b_m^\dagger b_m + \frac{v}{2r} N^2$$

where N is the "angular momentum" operator of a rotator algebra

$$[\varphi_0, N] = i, \qquad \varphi_0 + 2\pi = \varphi_0.$$

The Fermion-Boson Correspondence in 1+1 D

The energy levels and their degeneracies for the Weyl fermion and the chiral boson on a cylinder coincide. N for the chiral boson corresponds to the particle number of fermions.

Chiral Boson in 1+1 Dimensions II

Introduce a local field:

$$\varphi(x) = \varphi_0 + \frac{x}{r}N + i\sum_{m\geq 0}\frac{1}{\sqrt{|m|}}\left(b_m^\dagger e^{-i\frac{mx}{r}} - b_m e^{i\frac{mx}{r}}\right)$$

this field satisfies $\varphi(x,t) = \varphi(x,t) + 2\pi$ and can be considered as a map from a cylinder onto a unit circle. The commutation relations are

$$[\varphi(x), \varphi(x')] = -i\pi\mathrm{sgn}(x - x')$$

The Hamiltonian becomes

$$H_B = \frac{v}{4\pi}\int_0^{2\pi r} dx(\partial_x\varphi)^2$$

Local Fermion-Boson Correspondence

By comparison of local operator algebras one can establish a local Fermion-Boson correspondence called Bosonization

Example: Bosonization of density operator

For the Weyl fermion	For the chiral boson
$[\rho(x), \rho(x')] = \frac{i}{2\pi}\delta'(x - x')$	$[\varphi(x), \varphi(x')] = -i\pi \mathrm{sgn}(x - x')$

$$\Downarrow$$

$$\rho(x) = \frac{1}{2\pi}\partial_x \varphi$$

Bosonization of Fermion Field

Local Fermion Algebra

$$\psi(x)\psi(x') + \psi(x')\psi(x) = 0, \quad \psi(x)\psi^\dagger(x') + \psi^\dagger(x')\psi(x) = \delta(x-y)$$

$$[\rho(x), \psi(x')] = -\psi(x)\delta(x-x')$$

Bosonization of the local fermion algebra

$$\rho(x) = \frac{1}{2\pi}\partial_x\varphi, \qquad \psi(x) = c : e^{i\varphi(x)} :$$

Here $: e^{i\varphi} :\equiv e^{i\varphi_+}e^{i\varphi_-}$, that is all annihilation operators in φ are put to the right of creation operators. The constant c depends on the ultraviolet regularization.

Bosonization Exercies

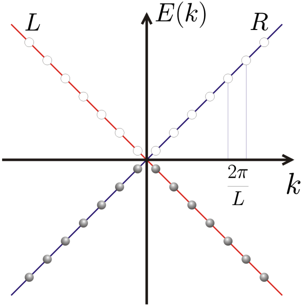
Problem 1

Prove that the operators $\frac{1}{2\pi}\partial_x\varphi, \; : e^{i\varphi(x)} :$ satisfy the same algebra as $\rho(x), \; \psi(x)$. Hint: use the Campbell- Hausdorff formula for two operators whose commutator is a c-number:

$$e^A e^B = e^B e^A e^{[A,B]}$$

Free Massless Dirac Fermions in 1+1 Dimensions

Free Massless Dirac Fermion = 2 Weyl Fermions



$$\psi_{L,R}(x) = \frac{1}{\sqrt{2\pi r}} \sum_k e^{ikx} \psi_{L,R}(k)$$

$$H_F = v \int dx \left[\psi_R^\dagger(-i\partial_x)\psi_R - \psi_L^\dagger(-i\partial_x)\psi_L \right]$$

Bosonization of Free Massless Dirac Field

Each Weyl fermion is Bosonized by its chiral boson $\varphi_{L(R)}$.

$$[\varphi_L(x), \varphi_L(x')] = i\pi \mathrm{sgn}(x - x') \qquad [\varphi_R(x), \varphi_R(x')] = -i\pi \mathrm{sgn}(x - x')$$

$$H_B = \frac{v}{4\pi} \int dx \left[(\partial_x \varphi_R)^2 + (\partial_x \varphi_L)^2 \right]$$

Bosonization Rules

$$\rho_R(x) = \frac{1}{2\pi} \partial_x \varphi_R, \qquad \psi_R(x) = e^{i\varphi_R(x)}$$

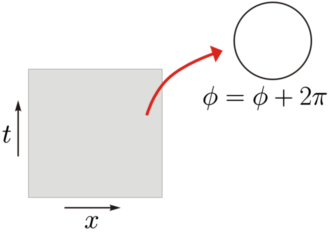
$$\rho_L(x) = \frac{1}{2\pi} \partial_x \varphi_L, \qquad \psi_L(x) = e^{-i\varphi_L(x)}$$

Free Compact Boson Field in 1+1 Dimensions

Change of Variables

$$\phi = \varphi_R + \varphi_L \qquad \text{and} \qquad \theta = \varphi_R - \varphi_L$$

$$H_B = \frac{v}{2\pi}\int dx\,\left[(\partial_x\theta)^2 + (\partial_x\phi)^2\right]$$



Fields ϕ and $\Pi = \partial_x\theta/\pi$ are canonically conjugate. In Lagrangian formulation:

$$S_B = \frac{1}{2\pi v}\int dxdt\,\left[(\partial_t\phi)^2 - v^2(\partial_x\phi)^2\right]$$

Summary.

Weyl=Chiral Boson

$$v\int dx\psi^\dagger(-i\partial_x)\psi = \frac{v}{4\pi}\int dx(\partial_x\varphi)^2, \quad \psi = e^{i\varphi}, \quad \rho(x) = \frac{1}{2\pi}\partial_x\varphi$$

Massless Dirac=Free Boson

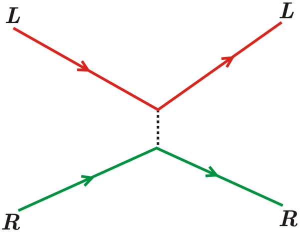
$$v\int dx\left[\psi_R^\dagger(-i\partial_x)\psi_R - \psi_L^\dagger(-i\partial_x)\psi_L\right] = \frac{v}{2\pi}\int dx\left[(\partial_x\theta)^2 + (\partial_x\phi)^2\right]$$

$$\psi_L = e^{i(\theta-\phi)}, \qquad \psi_R = e^{i(\theta+\phi)}, \qquad [\phi(x'), \partial_x\theta(x)] = i\pi\delta(x-x')$$

The Luttinger Model

The Luttinger Model

The Luttinger model is probably the simplest model describing interacting relativistic fermions. The interaction is characterized by a dimensionless coupling γ.



The Luttinger Hamiltonian

$$H_{\mathrm{LUT}} = v \int dx : \left[\psi_L^\dagger i\partial_x \psi_L - \psi_R^\dagger i\partial_x \psi_R + \gamma \rho_L(x) \rho_R(x) \right] :$$

Note, that all terms are normal ordered!

Perturbation Theory to Luttinger Model

The Schwinger-Dyson perturbation theory:

The perturbation theory contains divergent terms which need to be renormalized in the spirit of Gell-Mann and Low RG. The beta function of this theory vanishes to all orders in parameter γ.

Exact resummation of the perturbation series using the axial Word identities I. E. Dzyaloshinsky and A. I. Larkin, Sov. Phys. JETP 38, 202 (1974)

Bosonization of the Luttinger Model

The kinetic term of the Luttinger Hamiltonian is the massless Dirac Hamiltonian. Its Bosonization is a Boson Hamiltonian. The bosonization of the interaction term is

$$: \rho_R(x)\rho_L(x) :=: \left[\frac{1}{2\pi}(\partial_x\phi + \partial_x\theta)\right]\left[\frac{1}{2\pi}(\partial_x\phi - \partial_x\theta)\right] :$$

therefore

$$H_{\mathrm{LUT}} = v\int dx\left[\psi_L^\dagger i\partial_x\psi_L - v\psi_R^\dagger i\partial_x\psi_R + \gamma\rho_L(x)\rho_R(x)\right]$$

$$\Downarrow$$

$$H_{\mathrm{LUT}}^B = \frac{v}{2\pi}\int dx\left\{(\partial_x\theta)^2 + (\partial_x\phi)^2 + \frac{\gamma}{2\pi}[(\partial_x\phi)^2 - (\partial_x\theta)^2]\right\}$$

Bosonization of the Luttinger Model

Luttinger Hamiltonian. Standard Notations.

$$H_{\mathrm{LUT}} = \frac{v_c}{2\pi}\int dx\left[\frac{1}{K}(\partial_x\phi)^2 + K(\partial_x\theta)^2\right]$$

where

$$[\partial_x\theta(x), \phi(x')] = -i\pi\delta(x-x')$$

$$v_c = v\sqrt{1-\left(\frac{\gamma}{2\pi}\right)^2}$$ is the sound velocity

$$K = \sqrt{\frac{1-\gamma/2\pi}{1+\gamma/2\pi}}$$ is the Luttinger parameter

Luttinger Model. Exercises.

Problem 1

Using the Heisenberg equation $i\partial_t A = [A, H]$ show that field ϕ satisfies the wave equation

$$\partial_t^2 \phi - v_c^2 \partial_x^2 \phi = 0$$

Problem 2

By performing the Legendre transform of the bosonized Luttinger Hamiltonian find the Lagrange density of the Bose field ϕ (in this case $\pi \partial_x \theta$ should be treated as the momentum density).

Luttinger Model. Lagrangian Formulation.

Imaginary time Luttinger Action

Action for the ϕ field:

$$S_i = \frac{1}{2\pi K}\int_0^\beta d\tau \int dx \left[\frac{1}{v_c}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + v_c\left(\frac{\partial\phi}{\partial x}\right)^2\right]$$

Action for the θ field:

$$S_i = \frac{K}{2\pi}\int_0^\beta d\tau \int dx \left[\frac{1}{v_c}\left(\frac{\partial\theta}{\partial\tau}\right)^2 + v_c\left(\frac{\partial\theta}{\partial x}\right)^2\right]$$

Note the duality $\theta \to \phi, \quad K \to K^{-1}$.

Correlation Functions: Bosons I

We start with the imaginary time correlator

$$\mathcal{G}(x,\tau) = \langle T\phi(x,\tau)\phi(x')\rangle$$

It is a Fourier transform of

$$\mathcal{G}(x,\tau) = \beta^{-1}\sum_{\omega_n}\int \frac{dk}{2\pi} e^{ikx - i\omega_n\tau}\mathcal{G}(i\omega_n,k)$$

where $G(i\omega_n, k)$ in a free Boson theory it is given by

$$\mathcal{G}(i\omega_n,k) = \frac{\pi v_c K}{\omega_n^2 + v_c^2 k^2}, \qquad \omega_n = \frac{2\pi n}{\beta}$$

Correlation Functions: Bosons II

Temperature Correlator of Bose fields

$$\mathcal{G}(x,\tau) = -\frac{K}{4}\ln\left[1 - e^{-\frac{2\pi}{\beta v_c}(|x|+iv_c\tau)}\right] - \frac{K}{4}\ln\left[1 - e^{-\frac{2\pi}{\beta v_c}(|x|-iv_c\tau)}\right]$$

In the limit of zero temperature $T \to 0$ this becomes

$$\mathcal{G}(x,\tau) = -\frac{K}{4}\ln(x^2 + v_c^2\tau^2) + c$$

In real time $t = -i\tau$ there are light cone singularities at

$$x = \pm v_c t$$

Gaussian Integration

Write the quadratic action of boson in a symbolic form

$$S_i = \frac{1}{2}\phi\mathcal{G}^{-1}\phi$$

Then for a source field η there is a

Gaussian Integration Formula

$$\langle Te^{i\eta\phi}\rangle = \frac{1}{Z}\int \mathcal{D}\phi e^{-S_i} e^{i\eta\phi} = e^{-\frac{1}{2}\eta\mathcal{G}\eta}$$

For example,

$$\left\langle Te^{i\phi(x,\tau)} e^{-i\phi(x',\tau')}\right\rangle = c\, e^{\mathcal{G}(x-x',\tau-\tau')}$$

here $c = e^{-\langle\phi(0)^2\rangle}$ is an (infinite) constant.

Correlation Function of Fermions. $T = 0$

The right-moving Fermion is given by

$$\psi_R(x) = e^{i\phi_R(x)} \equiv e^{i\theta(x)+i\phi(x)}$$

Applying Gaussian Integration Formula to this expression we find

Gaussian Integration Formula

$$\langle T\psi_R(x,\tau)\psi_R^\dagger(x')\rangle = \frac{c}{(x+iv_c\tau)^{\Delta}(x-iv_c\tau)^{\bar{\Delta}}}$$

where

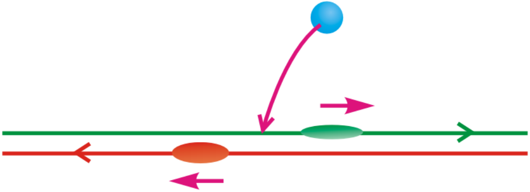
$$\Delta = \frac{(1+K)^2}{4K} \qquad \text{and} \qquad \bar{\Delta} = \frac{(1-K)^2}{4K}$$

Correlation Function of Fermions. $T = 0$

The structure of correlation function

$$\langle T\psi_R(x,\tau)\psi_R^\dagger(x')\rangle = \frac{c}{(x+iv_c\tau)^{\Delta}(x-iv_c\tau)^{\bar{\Delta}}}$$

suggests that in interacting system the "right" electron is no more a pure right-mover. It rather splits into two counterpropagating wave-packets. This is called charge fractionalization.



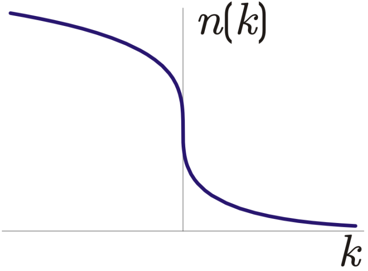
Particle occupation numbers.

The particle occupation numbers are found as

$$n_R(k) = \int dx e^{-ikx} \langle T \psi_R^\dagger(x) \psi_R(x') \rangle = n_0 + c\,\mathrm{sgn}(k) |k|^{\Delta + \bar{\Delta} - 1}$$

$$\Delta + \bar{\Delta} - 1 = \frac{(K-1)^2}{2K} > 0$$

Instead of the sharp Fermi step there is a continuous distribution with a power-law singularity at $k = 0$.



Summary.

- Using the Fermion-Boson correspondence we solved exactly a non-trivial interacting system of fermions.
- We found that the spectrum of the system is described by bosons (phonons) whose velocity is renormalized by interactions.
- The interaction effects are encoded in the Luttinger parameter K.
- For $K \neq 1$ charge fractionalization and the disappearance of Fermi step are observed.