Prediction of Non-Abelian Statistics for Quasiparticles in the $\nu = 5/2$ Fractional Quantized Hall State

Windsor Summer School, August 10, 2007

Quasiparticles with "non-abelian statistics":

Moore and Read in 1991, proposed a new kind of trial wave function, involving a Pfaffian, to explaion the recently discovered fractional quantized Hall state at filling fraction $\nu = 5/2$, which would have quasiparticles with non-abelian statistics. Not yet confirmed by experiments.

Recent advances in materials should make experiments possible--for the 5/2 state--at least show whether the state is indeed of the Moore-Read type, and hopefully confirm the existence of non-abelian quasiparticles.

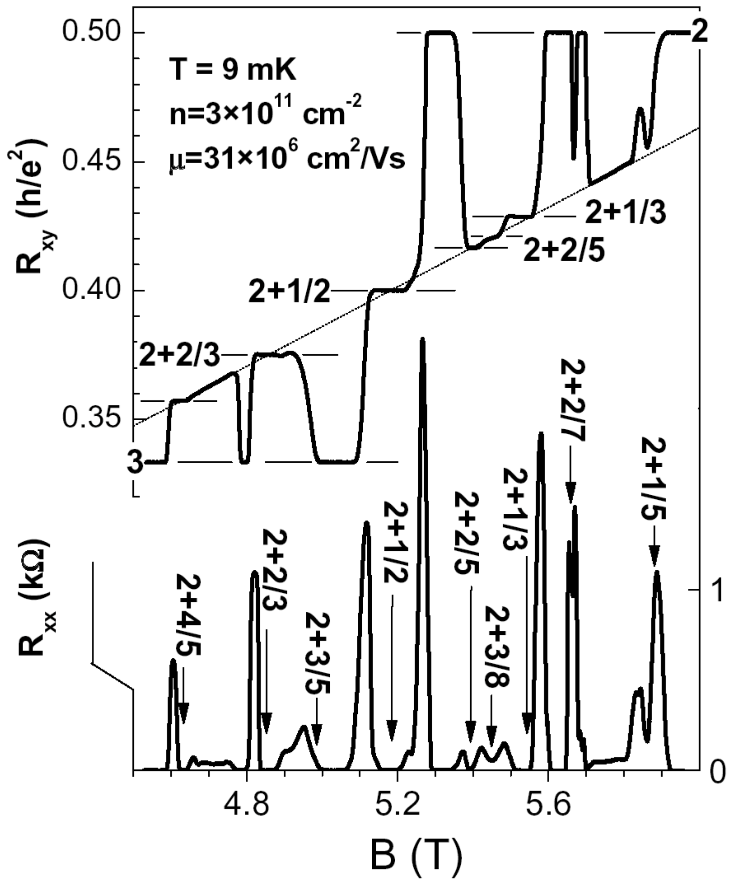
Outline

What do we mean by Non-Abelian statistics? How do they work in the Moore-Read "Pfaffian" state? Evidence for the Pfaffian state at $\nu=5/2$.

Description of the Pfaffian state as a p-wave BCS superconductor of "composite fermions". $\Delta \sim (p_x + ip_y)$

Some consequences of the non-abelian statististics.

Existence of a quantized Hall state at $\nu=5/2$



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Non-abelian statistics for Moore-Read 5/2 state

Consider a system containing 2N localized quasiparticles, far from each other and far from boundaries. Then there exist $M=2^{N-1}$ orthogonal degenerate ground states, which cannot be distinguished from each other by any local measurement.

Moving various quasiparticles around each other and returning them to their original positions, or interchanging quasiparticles, can lead to a nontrivial unitary transformation of the ground states, which depends on the order in which the winding is performed. (Unitary matrix depends on the topology of the braiding of the world lines of the quasiparticles. Matrices form a representation of the braid group).

If two quasiparticles come close together, degeneracy is broken; but energy splittings fall off exponentially with separation.

Topological quantum computation

Non-abelian quasiparticles may be useful for "topological quantum computation".

[Kitaev, quant-ph/9707021; Freedman, Larson, Wang, Commun. Math Phys (2002); Bonesteel, et al PRL (2005).]

Manipulation of qubits would be carried out by moving quasiparticles around each other, not bringing them close together. Advantage: exponentially long decoherence times.

Caveat: Moore-Read state is not rich enough for general topological quantum computation).

More complicated non-abelian states have been proposed, which would allow universal quantum computation. (E.g., Read-Rezayi k=3 state; may be realized at $\nu = 12/5$.)

What is the evidence that the ν =5/2 Quantized Hall State is indeed of the Moore-Read type ?

Evidence comes primarily from numerical calculations on finite systems.(Morf & collaborators, 2002, 2003; Das Sarma et al. 2004).

Using electron-electron interactions appropriate for electrons in the second Landau level, with parameters appropriate to GaAs samples, find a spin-polarized ground state, which seems to have an energy gap, and which has good overlap with Pfaffian wave function. Relatively small changes in parameters can lead to other ground states, which are not quantized Hall states. (As is found experimentally for samples in large in-plane magnetic field.)

Proposed experiments

Moving one quasiparticle around another can be done in principle by means of gates which couple electrostatically to the charge of the quasiparticles; but we are far from being able to accomplish this technologically.

We seek other experiments to examine the $\nu=5/2$ state to see if it is of the Moore-Read type.

*Measurements of the quasiparticle charge. (e.g. using SETs, as in studies of $\nu = 1/3$ by Yacoby et al.) Moore-Read quasiparticles have charge e/4 .

*Measurements of spin polarization. Moore-Read has complete polarization in second Landau Level.

***Interference-type experiments related to non-abelian statistics.**

Moore-Read state and the Fermion-Chern-Simons description at $\nu=1/2$

If the true filling fraction is $\nu=1/2$, then for the transformed particles:

(2 flux quanta per electron): $\Delta B \equiv B - 4\pi n_e = 0$

Mean-field approximation $\Rightarrow$ FREE FERMIONS in ZERO MAGNETIC FIELD

Ground state = filled Fermi Sea $k_F = (4\pi n_e)^{1/2}$

If this is correct, then there is **no energy gap**, **no QHE**.

Depending on the short-distance interactions between fermions the Fermi surface may be unstable, e.g., to **formation of p-wave superconductivity**

If a superconducting energy gap forms at the Fermi surface, then state is stabilized at precisely $\nu = 1/2$. Deviations in filling fraction => $B_{eff} \neq 0$ => require vortices, cost finite energy.

Get plateau in Hall conductance at $\nu = 1/2$: fractional quantized Hall state.

Apparently: Superconductivity does not occur for electrons in the lowest Landau level ($\nu = 1/2$) but interactions are different for electrons in the second Landau level, and it looks like pairing does occur occur for electrons at $\nu = 2 + 1/2$.

Moore-Read quasiparticle <=> vortex in superconductor

By Meissner effect, vortex must bind 1/2 quantum of magnetic flux to have finite energy. With a Chern-Simons gauge field, the source of magnetic flux is charge, rather than current.

1/2 quantum of Chern-Simons flux requires 1/4 electric charge.

Analogy between Moore-Read 5/2 state (p_x+ip_y) superconductor .

Analogy elucidated particularly by Read and Green (PRB 2000) -- emphasized important difference between a BCS p_x+ip_y superconductor and a Bose condensate of tightly bound p-wave pairs.

Tightly bound pairs could also be a mechanism for producing even-denominator fractional quantized Hall states (Halperin, 1983), but this would not give rise to non-abelian statistics.

Quasiparticles in the quantized Hall state correspond to vortices in the superconductor.

For a BCS superconductor with pairing function $\Delta \propto (p_x + ip_y)$: Energy gap in the bulk, but vortices have zero energy states. If there are 2N vortices present, there are $M=2^{N-1}$ degenerate ground states.

Moving vortices around each other generates a unitary transformation on these states similar to that in the Moore-Read state.

*Effects of vortex motion on zero energy states elucidated by Ivanov (PRL 2001); and Stern, von Oppen and Mariani (PRB 2004), using Bogoliubov-de Gennes equations for superconductor.

Zero-energy modes

Specifically, in a p_x+ip_y superconductor, an isolated vortex, at point $\mathbf{R}_i$, has a zero energy mode, with Majorana fermion operator γ_i :

$$\gamma_i = \gamma_i^\dagger \ , \quad \gamma_i^2 = 1 \ , \quad \{\gamma_i , \gamma_j\} = 2\delta_{ij}$$

To form ordinary fermion creation or annihilation operator: need pair of vortices: e.g.

$$c_{12} = (\gamma_1 + i\,\gamma_2) / 2 \ , \qquad c_{12}^\dagger = (\gamma_1 - i\,\gamma_2) / 2 \ ,$$

obey usual commutations rules

$N_{12} = c_{12}^\dagger c_{12}$ has eigenvalues = 0, 1. $[N_{12}, N_{34}] = 0$, etc.

Constraint : Number of occupied pairs = $N_{electrons}$ (mod 2) .

-> 2N vortices gives 2^{N-1} independent states

Bogoliubov-de Gennes Equations for an Inhomogeneous Superconductor

Generalization of BCS, in presence of vortices, boundaries, or other inhomogeneities. Here, p-wave superconductor for spinless electrons. Effective Hamiltonian has the form

$$H = \int dr\, \psi^\dagger(r)\, [\, - c\, \nabla^{2} + V(r) - \mu]\, \psi\, (r)$$

$$+ \int dr\, dr'\, [\, \Delta\, (r,r')\, \psi\, (r)\;\; \psi\, (r') + \text{h. c.}\,]$$

where V and Δ must be determined self-consistently, $c = \hbar^2/2m$.

Solution of the BDG equations

Construct quasiparticle operators γ_i , related by unitary transformation to electron creation and annihilation operators,

$$\gamma_i = \int d\mathbf{r}\, [\, u_i(\mathbf{r})\, \psi(\mathbf{r}) + v_i(\mathbf{r})\, \psi^\dagger(\mathbf{r})\,]$$

which diagonalize the BdG Hamiltonian

$$H = \sum_i E_i\, \gamma_i^\dagger\, \gamma_i \text{ - constant.}$$

Generally, E_i occur in pairs: $E_i = - E_j$, $\gamma_i = \gamma_j^\dagger$, $u_i = v_j^*$, $u_j = v_i^*$. Restrict sum to positive energy states to avoid double counting.

Zero energy modes: Explicit relation between Majorana operator and electron operators

For zero-energy modes: $E_i = -E_i$, can have just a single solution: $i = j$

$$\gamma_i = \int d\mathbf{r} \, [\, u_i(\mathbf{r}) \, \psi(\mathbf{r}) + v_i(\mathbf{r}) \, \psi^\dagger(\mathbf{r}) \,] \, , \quad \text{with } v_i(\mathbf{r}) = u_i^*(\mathbf{r}) \, ,$$

localized near each vortex.

$$\gamma_i = \gamma_i^\dagger$$

Phase of u_i depends on positions of other vortices, changes by -1 if a vortex j is moved around vortex i.

Braiding properties

Vortices at points $\mathbf{R}_1$ $\mathbf{R}_2$ $\mathbf{R}_3$ $\mathbf{R}_4$

Move vortex 2 around vortex 3:

Changes N_{12} -> $(1-N_{12})$, N_{34} -> $(1-N_{34})$.

Changing the sign of γ_2 interchanges

$c_{12} = (\gamma_1 + i\,\gamma_2) / 2$ <==> $c_{12}^{\dagger} = (\gamma_1 - i\,\gamma_2) / 2$,

Gives unitary transformation $\sim \gamma_2\,\gamma_3$.

Braiding properties

Vortices at points $\mathbf{R}_1$ $\boxed{\mathbf{R}_2}$ $\mathbf{R}_3$ $\mathbf{R}_4$

Move vortex 2 around 3 and 4. Gives unitary transformation $\sim \gamma_2\gamma_4\,\gamma_2\gamma_3 = \gamma_3\,\gamma_4$: leaves N_{12} and N_{34} unchanged

Boundary states

Finite sample with an odd number of elementary vortices will have a zero energy state at the boundary. Form pair between boundary state and one of the vortices. Again have constraint : Number of occupied pairs = $N_{electrons}$ (mod 2) .

Generally, edge of superconductor has a series of low-energy fermion modes, with energies $E_m = m\,(\pi\, v / L)$,

$m = 0, \pm 1, \pm 2, \ldots$ if number of vortices is odd,

$m = \pm 1/2, \pm 3/2, \ldots$ if number of vortices is even.

Get even-odd alternation in energy to add an electron to the system, if number of vortices is even, not if number of vortices is odd. Alternation energy $\sim v / L$; goes to 0, for $L \rightarrow \infty$

Contrast to s-wave superconductor

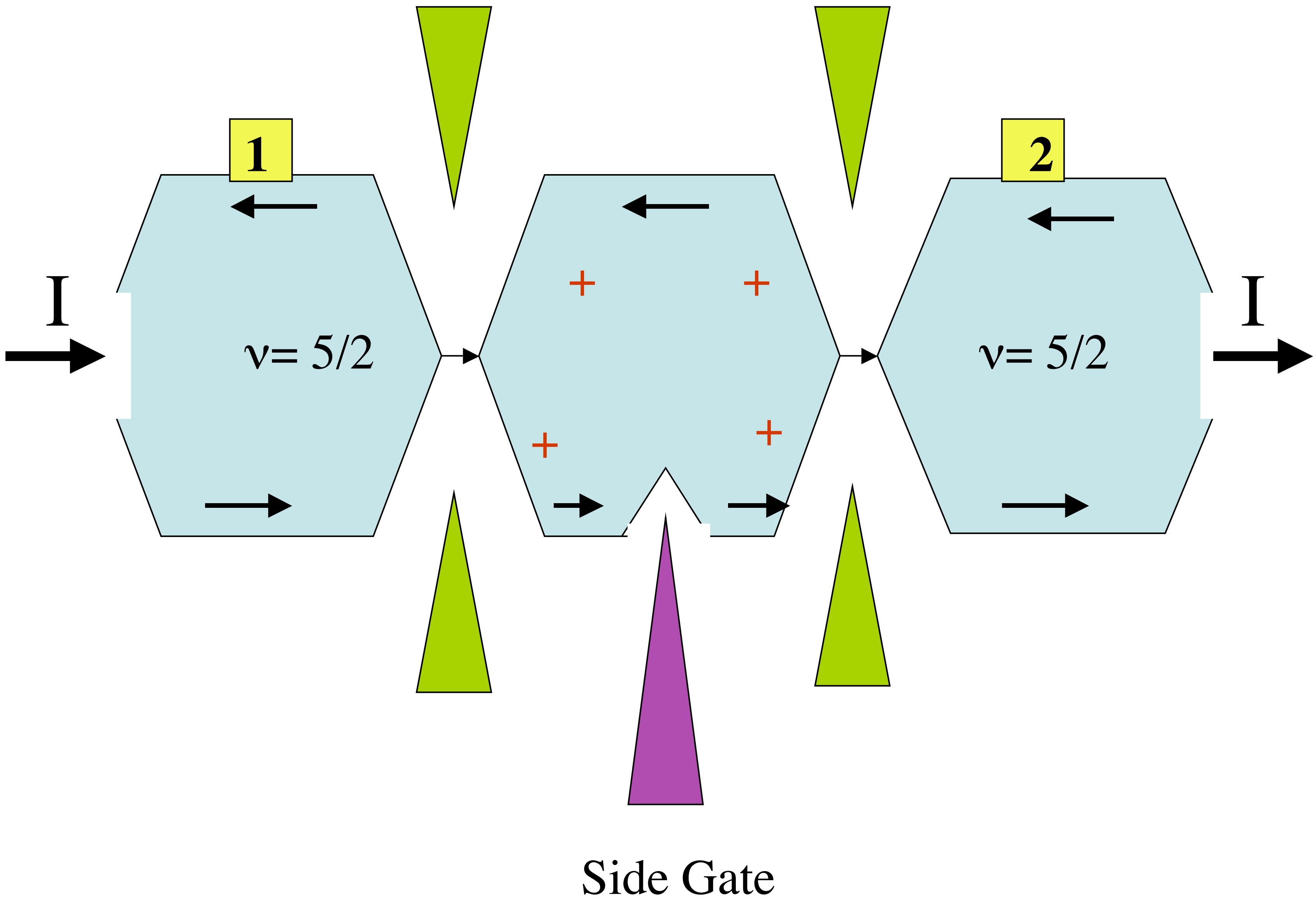
s-wave superconductor has no low-energy fermion states at boundary.

Energy to add an electron has an even-odd alternation independent of whether there are an even or odd number of vortices present, and independent of the perimeter of the sample.

True also for gapless d-wave superconductor, $\Delta \propto (p_x+ip_y)^{2,}$ or for a Bose condensate of tightly bound p-wave pairs .

It may be possible to detect even-odd effect experimentally in an FQHE system at $\nu = 1/2$.

Coulomb blockade regime, R_{12} is large, except on resonance, when $E_N = E_{N+1}$



If the number of quasiholes is even, E_N has an even-odd alternation in the electron number N.

The period for oscillations in R_{12} is then $\Delta A = 4\Phi_0/B$, corresponding to the addition of two electrons to the partially-filled landau level.

If the number of quasiholes is odd, there is no even-odd alternation in E_N. The period oscillations is then $\Delta A = 2\Phi_0/B$, there is no oscillation with period $\Delta A = 4\Phi_0/B$.

Similar effects are predicted for weak back-scattering.

Different behavior for even and odd quasihole number is a consequence of the particular non-abelian statistics of the Moore-Read state.

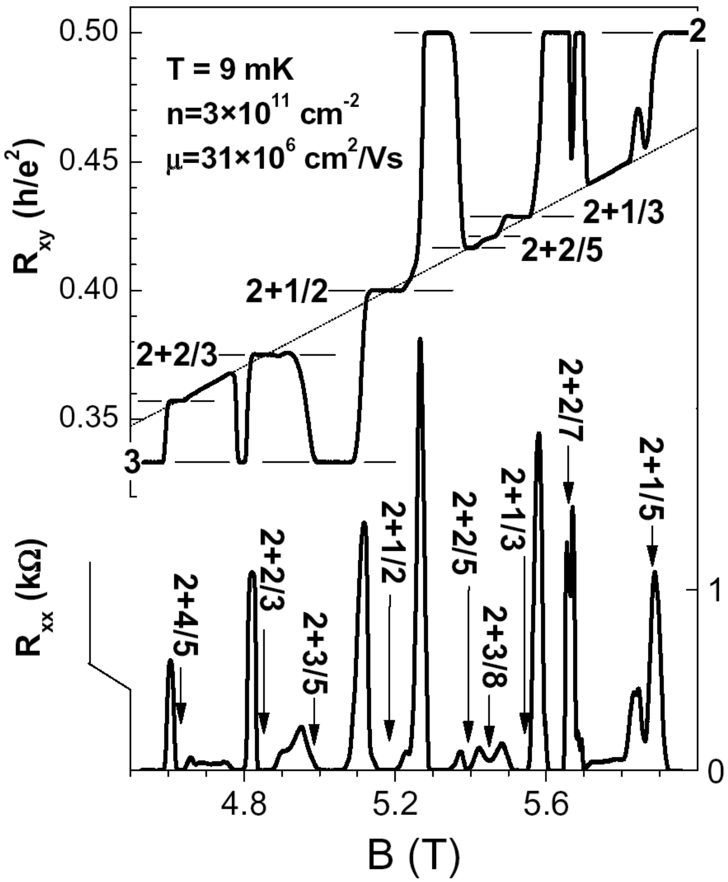
Other Phenomena in the Second and Higher Landau Levels

Re-entrant integer QHE

And Resistance Anisotropies

(Stripe and Bubble Phases)

R_{xy} and R_{xx} in the Second Landau Level



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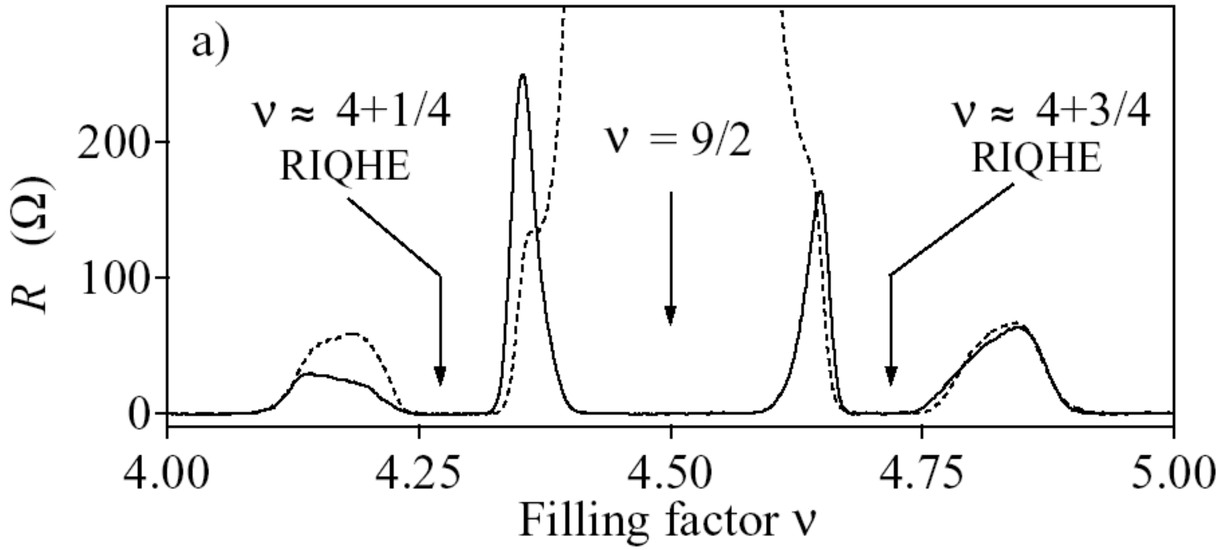
Third Landau Level



Hall Resistance (not shown)

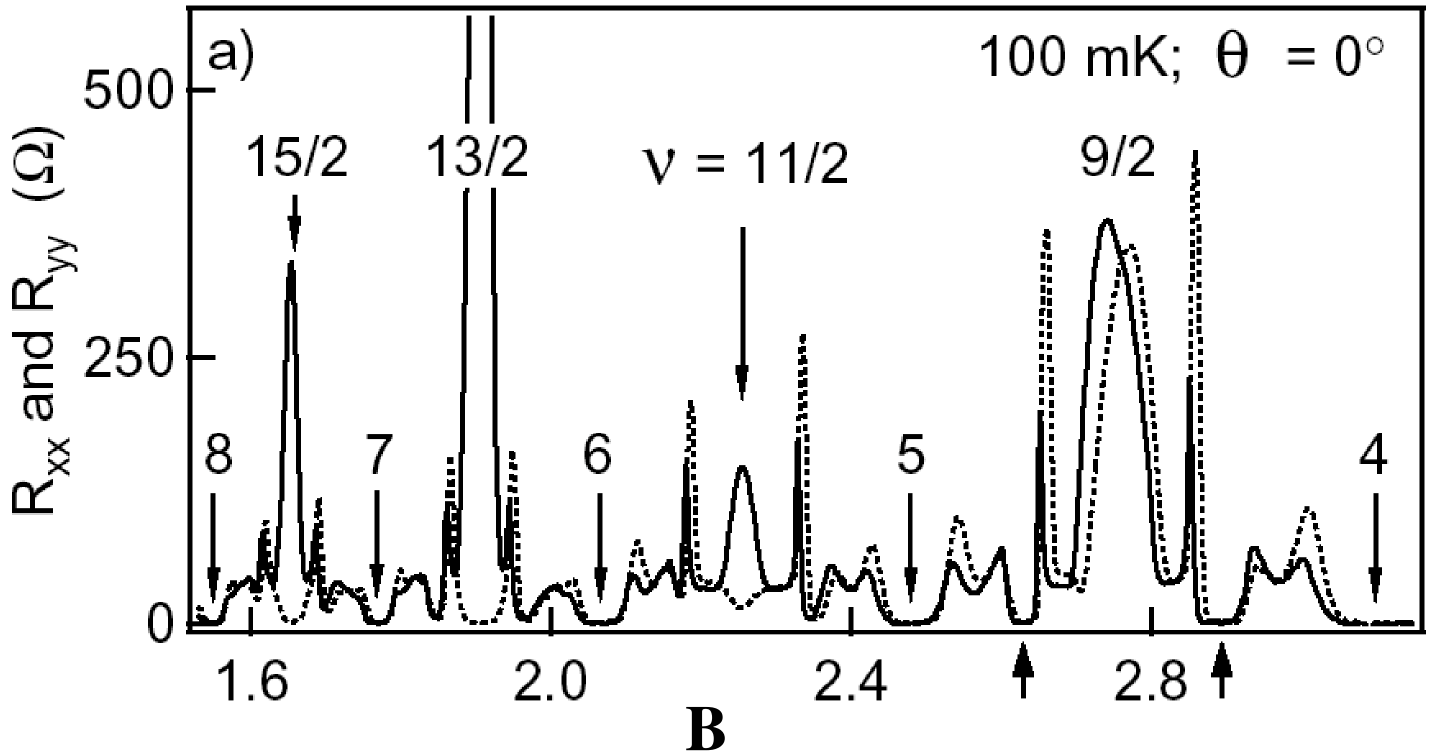
T = 50 mK

$= h/4e^2$ at $\nu \approx 4.25$

$= h/5e^2$ at $\nu \approx 4.75$

From Cooper, Eisenstein, Pfeiffer and West, PRL 90, 226803 (2003)

Higher Landau Levels: R_{xx} and R_{yy}



From Cooper, Eisenstein, Pfeiffer and West, PRL 2004

Stripe and Bubble Phases

Predicted by

Fogler, Koulakov, & Shklovskii (1996)

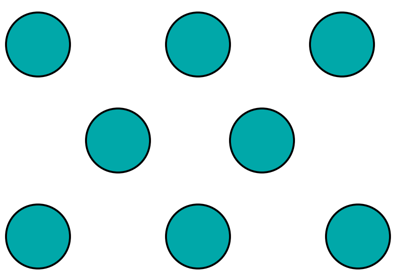
Moessner & Chalker (1996)

Fukuyama, Platzman, & Anderson (1979): Partially full Landau level should be unstable to formation of charge density waves, due to exchange interaction. (Hartree-Fock approximation).

Bubble Phases

A regular array of “bubbles”, each one containing n=2 or more electrons, embedded in the otherwise empty Landau Level,

or containing n=2 or more holes in the otherwise full Landau Level. (n=1 would be a Wigner crystal of electrons or holes)



Localized electrons or holes do not contribute to transport.

Get integer Quantized Hall conductance, $R_{xx} = R_{yy} = 0$,

Stripe Phases

Alternating stripes of full and empty Landau Level.

Current can flow easily in y-direction (along edges of stripes) R_{yy} -> 0.

Difficult for current to flow in x-direction: R_{xx} can be very large.

Interesting issues: effects of dislocations, and other disorder? What determines orientation of stripes?

Concluding Reminder

These lectures have covered only a fraction of the wide range of phenomena included in the subject of quantum Hall effects, and have touched on only a few of the theoretical ideas used to explain these phenomena.

There still are many open questions in the field, and a number of experimental results which are poorly understood.