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On the stability of the CRRA utility under high degrees of uncertainty

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Abstract

Economic growth models under uncertainty and rational agents with CRRA utility have been shown to provide quite fragile explanations of consumers' choice as equlibrium comsumption paths (expected utility) are drastically dependant on distributional assumptions. We show that assuming a SNP distribution for random consumption provides stability to general equilibrium models as expected utility exists for any value of the marginal rate of substitution over time.

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1 Introduction

King *et al.* (1990) and Geweke (2001) highlight a number of difficulties in the application of rational expectations models to choice under uncertainty when the distribution of micro and macro aggregates is heavy-tailed. They show that when a Student's t distribution rather than log normality is assumed, the constant relative risk aversion (CRRA) expected utility model (EU) does not have a solution leading to operational problems in the theory of choice. Further Geweke notes that even when log normality is assumed EU does not have a solution regardless of the priors from which Bayesian learning begins. More generally, Geweke (2001) shows that general equilibrium models which embody rational expectations are quite fragile with respect to different distributional assumptions (which are often non distinguishable on econometric grounds) and infinitesimal changes in over-time marginal rates of substitution which can lead to different equilibrium paths with quite different properties (see also Pesaran *et al.* 2007). Yoon (2004) also shows that similar fragility applies to a standard asset pricing model when the endowment follows a stochastic unit root process. In this letter we propose one solution to this problem. We assume that the logarithm of macroeconomic variables (e.g. log-consumption) follow a semi-nonparametric (SNP) density . The assumption of a SNP distribution for random log-consumption provides stability to general equilibrium models as the expected CRRA utility exists under Bayesian learning for any value of the over-time marginal rate of substitution. The rest of the letter is structured as follows. In Section one we describe properties of the SNP probability distribution. In Section two we extend the results on expected utility under CRRA assuming SNP densities, and also show that EU is well-defined under Bayesian updating for this density. The final section is a brief conclusion.

2 The SNP distributions

This section describes properties of the SNP pdf which will be useful throughout the paper.

Proposition *1* (Cramér 1925)

Let x be a continuous random variable distributed according to a certain pdf $f(x)$ which has a continuous derivative such that

$$\int_{-\infty}^{\infty}\left(\frac{df(x)}{dx}\right)^{2} e^{\frac{1}{2}x^{2}}dx < \infty \quad \text{and} \quad f(x) \underset{|x|\to\infty}{\to} 0, \tag{1}$$

then, $f(x)$ can be expanded formally on a (infinite) series of derivatives of the standard Normal density, denoted as $\phi(x)$, as follows

$$f(x) = \sum_{s=0}^{\infty} \kappa_s H_s(x)\phi(x), \tag{2}$$

where $\kappa_s = \frac{1}{s!}\int_{-\infty}^{\infty} H_s(x)\phi(x)dx$, *and* $H_s(x)$ *is the* s^{th} *order Chebyshev-Hermite polynomial, which can be defined by the identity in equation* (3),[1]

$$\frac{d^s\phi(x)}{dx^s} = (-1)^s H_s(x)\phi(x),\ \forall s \geq 1. \tag{3}$$

Proposition 1 allows to define a general family of SNP distributions, $g\left(x;\boldsymbol{\delta}\right)$, $\boldsymbol{\delta} \in \mathbb{R}^n$, $\boldsymbol{\delta} = (\delta_1, ..., \delta_n)$, which can approximate any probability density function (pdf hereafter) to any degree of accuracy depending on the truncation order n.[2]

$$g\left(x;\boldsymbol{\delta}\right) = \left[1 + \sum_{s=1}^{n} \delta_s H_s(x)\right]\phi(x) \simeq f\left(x\right). \tag{4}$$

The rationale behind this density expansion lies in the properties of the Chebyshev-Hermite polynomials, which form an orthonormal basis with respect to the weight function $\phi(x)$ (see Abramowitz and Stegun, 1972, or Kendall and Stuart, 1977, for further details).

$$\int_{-\infty}^{\infty} H_s(x)H_r(x)\phi(x)dx = \begin{cases} 0, & s \neq r \\ s!, & s = r. \end{cases} \tag{5}$$

$$\frac{dH_s(x)}{dx} = sH_{s-1}(x), \forall s \geq 1. \tag{6}$$

Based on these properties other characterizations of the SNP distributions can be obtained straightforwardly, for example equation (7) gives the cumulative distribution function (cdf hereafter), and equation (8) is the moment generating function (mgf hereafter) (see Proofs 1 and 2 in the Appendix),

$$G_x(a) = \int_{-\infty}^{a} g(x;\boldsymbol{\delta})dx = \int_{-\infty}^{a} \phi(x)dx - \phi(a)\sum_{s=1}^{n} \delta_s H_{s-1}(a), \tag{7}$$

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} g(x;\boldsymbol{\delta})dx = e^{t^2/2}\left[1 + \sum_{s=1}^{n} \delta_s t^s\right]. \tag{8}$$

[1] This is the so-called Gram-Charlier series of Type A. By convention it is usually assumed $H_0(x) = 1$.

[2] The truncation of the expansion involves positivity issues, which can be addressed through either pdf reformulations (Gallant and Nychka, 1987) or parametric constraints (Jondeau and Rockinger, 2001). In most cases the use of maximum likelihood estimation techniques for empirical purposes does not require any restriction to obtain well-defined densities at the optimal δ_s parameters. Also note that without loss of generality we consider $\delta_0 = 1$.

In addition, all order SNP distribution moments exist and are functions of the density parameters. For example, the first four central moments of the SNP pdf are: $E[x] = \delta_1$, $E[x^2] = 1 + 2\delta_2$, $E[x^3] = 3\delta_1 + 6\delta_3$ and $E[x^4] = 3 + 12\delta_2 + 24\delta_4$, and the distribution has zero mean and unit variance if $\delta_1 = \delta_2 = 0$ with δ_3 and δ_4 capturing the skewness and excess kurtosis of the distribution, respectively (see Proof 3 in the Appendix).

3 The CRRA utility under SNP distributions

This section extends the EU results under CRRA by assuming SNP distributed micro and macroeconomic variables. We show that in contrast with the Student's t distribution, a log-SNP pdf is valid for a wider range of possibilities thus providing consistency to rational expectations models in regards to heavy-tailed distributional assumptions.

Definition *1*

Let $x = \log(z)$ *be a random variable with pdf* $f(x)$, *and* α *a strictly positive parameter. We define the EU of* $z > 0$ *under CRRA as*

$$EU(z) = \begin{cases} \log(z) & \text{if} \quad \alpha = 1, \\ (1-\alpha)^{-1} E\left[z^{1-\alpha}\right] & \text{if} \quad \alpha \neq 1. \end{cases} \tag{9}$$

The EU function defined above is valid for any strictly nonnegative random variables, e.g. consumption, provided that:

1. The function $f(x)$ is known;

2. The mgf, $M_x(t) = E\left[e^{tx}\right]$, exists for all $t = 1 - \alpha$ and $\alpha \neq 1$.

Then, under assumptions 1 and 2 it is clear to show that the $EU(z)$ exists for all $\alpha \neq 1$ and it is given by,

$$EU(z) = (1-\alpha)^{-1} E\left[e^{(1-\alpha)x}\right] = (1-\alpha)^{-1} M_x(1-\alpha). \tag{10}$$

The most common distributional assumption in the literature on choice theory for the pdf of z is the lognormal, i.e. $x \sim N(\mu, \sigma^2)$, as it delivers estimates with almost surely known properties. From a pragmatic viewpoint, improvements in model reliability and theory predictions can be achieved assuming alternative densities to the lognormal which may better capture features of the data such as heavy tails. A straightforward way of assuming a heavy-tailed distribution for x is by means of the Student's t distribution with ν degrees of freedom, i.e. $x \sim t(\mu, \sigma^2; \nu)$. But under this assumption the CRRA utility is shown not to be well-defined since the mgf of the Student's t distribution fails to exist for any $t \neq 0$. This case is examined in Geweke (2001) and points out the limitations of the CRRA utility when the lognormal assumption is relaxed to incorporate a more realistic heavy-tailed pdf, (see also Kendall and Stuart (1977) p. 60). If period utility is bounded and expected utility guaranteed to exist, Geweke's result would not apply. This is the case for some classes of HARA utility function (e.g. Cogley, 2009). It is also possible to obtain existence of expected utility even though period utility is unbounded. For instance, in the case of a 'finite-state' economy (see Cogley and Sargent, 2008).

Definition *2*

We say that variable $z > 0$ is log-SNP distributed if the pdf of the variable $x = \log(z)$ is that in equation (4) above, $x \sim SNP(\boldsymbol{\delta})$. For that variable, and provided that $\alpha > 0$, the *EU consistent with the CRRA hypothesis is defined as,*

$$EU(z) = \begin{cases} \log(z) & \text{if} \quad \alpha = 1, \\ (1-\alpha)^{-1} e^{(1-\alpha)^2/2} \left[1 + \sum_{s=1}^{n} \delta_s (1-\alpha)^s \right] & \text{if} \quad \alpha \neq 1. \end{cases} \tag{11}$$

The EU in equation (11) is well-defined as all order moments of the SNP distribution and the mgf (equation (8)) exist. Consequently the log-SNP model appears useful as a method of generating solutions in the EU model when disdtributions are assumed to exhibit heavy tails.

We also note that assumption 1 does not seem to be consistent under CRRA if the EU is accomplished by Bayesian updating. This fact makes rational expectations models dependent not only on the distributional assumptions but also on the subjective distribution of its priors. In this case Geweke (2001) argues that the EU fails to exist even in the lognormal case and regardless of the priors from which Bayesian learning begins.[3] Nevertheless, conditions can be

[3]See Examples 4 and 5 in Geweke (2001). It is worth mentioning that recent contributions in the area of asset pricing have circumvented the problems with Bayesian learning highlighted by Geweke in different

found for which the EU is well-defined under Bayesian updating. In particular, Definition 3 below describes the EU for log-SNP nonnegative random variables in a Bayesian framework.

Definition *3*

Let $x = \log(z)$, $z > 0$, *be a random variable with pdf* $g(x;\boldsymbol{\delta})$ *(equation (4)) where the parameter vector* $\boldsymbol{\delta}$ *is unknown and has subjective pdf* $\varphi(\boldsymbol{\delta})$ *with support* $\mathbb{R}^n$. *If* $E(x) = \int \cdots \int_{\boldsymbol{\Delta}} E\left[x\,|\boldsymbol{\delta}\right]\varphi(\boldsymbol{\delta})d\boldsymbol{\delta}_1 \cdots d\boldsymbol{\delta}_n$ *exists, then*

$$EU(z) = \begin{cases} E(x) & \text{if} \quad \alpha = 1, \\ (1-\alpha)^{-1}E_{\boldsymbol{\delta}}\left[M_x(1-\alpha;\boldsymbol{\delta})\right] & \text{if} \quad \alpha \neq 1, \end{cases} \tag{12}$$

since $M_x(1-\alpha;\boldsymbol{\delta})$ *exists for all* $\boldsymbol{\delta} \in \mathbb{R}^n$ *and is finitely integrable with respect to* $\boldsymbol{\delta}$.

For example, if $x\,|(\boldsymbol{\delta}) \sim SNP(\boldsymbol{\delta})$ and $d_i \sim N(\overline{\delta_i}, \overline{q})$ $\forall i = 1, ..., n$,[4] then, $x \sim SNP(\overline{\boldsymbol{\delta}})$. In this case, it follows that

$$EU(z) = \begin{cases} \overline{\delta_1}, & \text{if} \quad \alpha = 1, \\ (1-\alpha)^{-1}e^{(1-\alpha)^2/2}\left[1 + \sum\limits_{s=1}^{n} \overline{\delta_s}(1-\alpha)^s,\right] & \text{if} \quad \alpha \neq 1. \end{cases} \tag{13}$$

The example above illustrates the fact that the log-SNP pdf may be used to overcome the fragility of rational expectations models under CRRA utility and reasonable assumptions about the subjective distribution of SNP pdf parameters.

4 Conclusion

Recent evidence in the literature (see the seminal paper of Geweke (2001)) shows that traditional equilibrium models of growth under the common assumptions of CRRA utility and rational expectations may not be well-defined when macroeconomic variables exhibit

ways. Bidarkota et al. (2009) use a sub-family of $\alpha-$stable distributions (the ones with maximal negative skewness of -1) to provide an operational theory under uncertainty for that particular case. Bakshi and Skoulakis (2010) develop further the model by Weitzman (2007) and obtain a model that (with subjective expectations) yields well-defined expected utility and finite moment generating function of the predictive distribution of consumption growth.

[4]E.g., given a prior $d_i \sim N(\underline{d_i}, \underline{q})$ and T i.i.d. observations $x_1, ..., x_T$, $\overline{q}^{-1} = \underline{q}^{-1} + T$ and $\overline{d_i} = \overline{q}\left(\underline{q}^{-1}\underline{d_i} + T\overline{x}\right)$. See Example 3 in Geweke (2001).

heavy tails and/or learning is accomplished by Bayesian updating. In these cases the existence of expected utility becomes very fragile with respect to the distributional assumptions leading to a non-operational theory of choice, particularly when the model tries to embody high levels of economic uncertainty through heavy-tailed distributions.

In this paper we follow an alternative approach to that of Geweke (2001) to recover an operational theory of choice. We propose the replacement of the traditional assumption of lognormality by (possibly heavy-tailed and skewed) log-semiparametrically distributed random macroeconomic variables (e.g. log-consumption). The advantages of this approach are twofold: *a)* As in Geweke (2001), it provides stability to the existence of the agents' expected utility under CRRA and Bayesian learning, and *b)* it provides the agents with the possibility to better identify the probabilities of extreme events which may occur under economic situations of high uncertainty. Thus, while Geweke (2001) provides a solution to the stability of expected utility through modifications of the agents' Bayesian learning process still under lognormality, we focus on the whole distributional assumption for log-consumption. We argue that our approach allows to recover an even more operational theory of choice with CRRA utility, Bayesian learning and, in particular, heavy-tailed distributed random consumption.

Appendix

Proof *1. The cdf of the SNP distribution is given by,*

$$\begin{aligned} G_x(a) &= \int_{-\infty}^{a} g(x;\boldsymbol{\delta})dx = \int_{-\infty}^{a} \phi(x)dx + \sum_{s=1}^{n} \delta_s \int_{-\infty}^{a} H_s(x)\phi(x)dx = \\ &= \int_{-\infty}^{a} \phi(x)dx - \sum_{s=1}^{n} \delta_s H_{s-1}(x)\phi(x) \,\Big|_{-\infty}^{a} \\ &= \int_{-\infty}^{a} \phi(x)dx - \phi(a)\sum_{s=1}^{n} \delta_s H_{s-1}(a), \end{aligned} \tag{A1}$$

since $\lim_{x\to\pm\infty} H_s(x)\phi(x) = 0 \ \forall s \geq 1$, *we obtain*

$$\begin{aligned} \int H_s(x)\phi(x)dx &= \int (-1)^s \frac{d^s\phi(x)}{dx^s} dx_t = (-1)^s \frac{d^{s-1}\phi(x)}{dx^{s-1}} \\ &= (-1)^s(-1)^{s-1} H_{s-1}(x)\phi(x) = -H_{s-1}(x)\phi(x). \end{aligned} \tag{A2}$$

Proof *2*. *The mgf of the SNP distribution is given by,*

$$
\begin{aligned}
M_x(t) &= \int_{-\infty}^{\infty} e^{tx} g(x;\boldsymbol{\delta})dx = \int_{-\infty}^{\infty} e^{tx}\phi(x)dx + \sum_{s=1}^{n}\delta_s \int_{-\infty}^{\infty} e^{tx} H_s(x)\phi(x)dx \\
&= e^{t^2/2} + \sum_{s=1}^{n}\delta_s \left[-e^{tx}H_{s-1}(x_t)\phi(x_t)\,|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} te^{tx}H_{s-1}(x_t)\phi(x_t)dx\right] \\
&= e^{t^2/2} + \sum_{s=1}^{n}\delta_s \int_{-\infty}^{\infty} t^s e^{tx}\phi(x)dx = e^{t^2/2}\left[1+\sum_{s=1}^{n}\delta_s t^s\right]. \qquad \text{(A3)}
\end{aligned}
$$

Integrating by parts and taking into account the property in (6) and $e^{tx}H_s(x)\phi(x) \underset{x\to\pm\infty}{\longrightarrow} 0$ $\forall s \geq 1$, *we have,*

$$
\begin{aligned}
u &= e^{tx} \Longrightarrow du = te^{tx}dx \qquad \text{(A4)}\\
dv &= H_s(x)\phi(x)dx \Longrightarrow v = -H_{s-1}(x)\phi(x). \qquad \text{(A5)}
\end{aligned}
$$

Proof *3*. *The four first central moments of the SNP distributions are:*

$$
\begin{aligned}
\frac{dM_x(t)}{dt}\,|_{t=0} &= \left[e^{t^2/2}\left\{t+\sum_{s=1}^{n}\delta_s\left(t^{s+1}+st^{s-1}\right)\right\}\right]_{t=0} = \delta_{1.} \qquad \text{(A6)}\\
\frac{d^2M_x(t)}{dt^2}\,|_{t=0} &= \left[e^{t^2/2}\left\{1+t^2+\sum_{s=1}^{n}\delta_s\left(t^{s+2}+(2s+1)t^s+s(s-1)t^{s-2}\right)\right\}\right]_{t=0} = 1+2\delta_{2.} \qquad \text{(A7)}\\
\frac{d^3M_x(t)}{dt^3}\,|_{t=0} &= \left[e^{t^2/2}\left\{3t+t^3+\sum_{s=1}^{n}\delta_s\left(t^{s+3}+3(s+1)t^{s+1}+3s^2t^{s-1}+s(s-1)(s-2)t^{s-3}\right)\right\}\right]_{t=0}\\
&= 3\delta_1+6\delta_{3.} \qquad \text{(A8)}\\
\frac{d^4M_x(t)}{dt^4}\,|_{t=0} &= \left[e^{t^2/2}\left\{3+6t^2+t^4+\sum_{s=1}^{n}\delta_s\left[t^{s+4}+3(s+1)t^{s+2}\right.\right.\right.\\
&\quad \left.\left.\left.+(6s^2+6s+3)t^s+2s(s-1)(2s-1)t^{s-2}+s(s-1)(s-2)(s-3)t^{s-4}\right]+\right\}\right]_{t=0}\\
&= 3+12\delta_2+24\delta_{4.} \qquad \text{(A9)}
\end{aligned}
$$

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