Non-Parametric Estimation for Intermittent Demands

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Agenda

- Introduction

- Motivation of this Research

- Properties of Parametric Methods

- Properties of Overlapping and Non-Overlapping Blocks

- Bootstrapping Approaches: Bias and Variance

- Numerical Investigation

- Conclusions

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Introduction: Parametric Approaches

Parametric distributions often recommended for inventory models

- ❑ Normal

- ❑ Gamma

- ❑ Bernoulli (and Compound Bernoulli)

- ❑ Poisson (and Compound Poisson)

- ❑ Negative Binomial

Empirical evidence

- ❑ Good support for Bernoulli / Poisson models of demand incidence for intermittent demand items.

- ❑ Quite good support for Negative Binomial Distribution model of demand

- ❑ BUT: many demands not well modelled by *any* parametric distribution.

Example (History = 10 periods)

P1	P2	P3	P4	P5	P6	P7	P8	P9	P10

Suppose LT = 3 and Block-Size (m) = 3   [Natural application in inventory context]

1. Non-Overlapping Blocks

Block 1 = {P2, P3, P4}, Block 2 = {P5, P6, P7}, Block 3 = {P8, P9, P10}
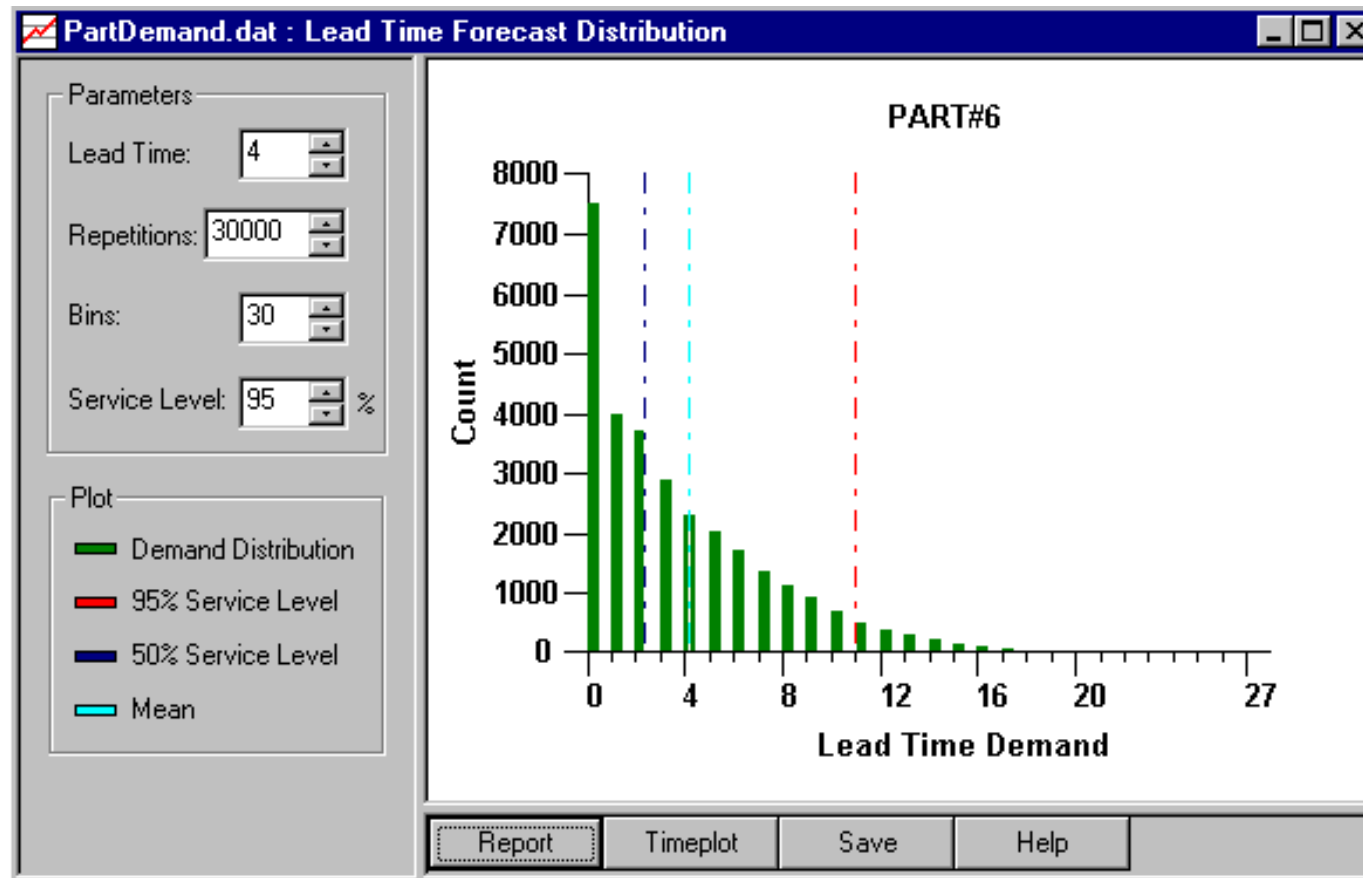
2. Overlapping Blocks

Block 1 = {P1, P2, P3}, Block 2 = {P2, P3, P4} , ... , Block 8 = {P8, P9, P10}

3. Bootstrapping

Randomly drawn from all combinations (with replacement) – extensions not considered

eg: Block 1 = {P7, P2, P9}, Block 2 = {P2, P3, P3}, ...

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Introduction: Bootstrapping Software

- Smart Software incorporates Markov Chain switching (between 'demand ' and 'non-demand' states), and "jittering".

- These extensions not addressed in this research so far.

Motivation of this Research

Bootstrapping approaches for intermittent demand have some empirical evidence in their support (Willemain, 2004).

Theoretical properties not investigated in an inventory context in which cumulative distributions need to be estimated.

This study aims to investigate the bias and variance properties of bootstrapping methods:

- With replacement

- Without replacement

and compare the performance of these methods with other non-parametric methods.

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Properties of Parametric Methods

Previous Research

- Normal distribution: biased estimate of CSL (Strijbosch et al, 1997; Janssen et al, 2009)

- Gamma distribution: biased estimate of CSL (Janssen et al, 2007)

Poisson Demand

$$E[P^{Poiss}(y)] = \sum_{a=0}^{y} \frac{E[e^{-\hat{\lambda}}\hat{\lambda}^a]}{a!} = 1 - \frac{1}{(y+1)!}E(\hat{\lambda}^{y+1}) + \frac{y+1}{(y+2)!}E(\hat{\lambda}^{y+2}) - \frac{(y+1)(y+2)}{(y+3)!2!}E(\hat{\lambda}^{y+3}) + \dots$$

- $\hat{\lambda}$ is the estimate of the mean parameter

- Suppose $\hat{\lambda}$ is unbiased

- Then the Poisson estimate of CSL is not necessarily unbiased because $E(\hat{\lambda}^c) \neq \lambda^c$ for c $\geq$ 2.

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Properties of Non-Overlapping Blocks

Estimation Problem

- Suppose History Length = **n** and we are given an (integer-valued) quantity **y** and the population cumulative distribution function is $P_m(y)$ for total demand over **m** periods

- We produce an estimate, $\hat{P}_m(y)$, based on **k=n/m** Non-Overlapping Blocks

- Statistical properties are well established for stationary i.i.d. demand:

Unbiased Estimate

$$E[\hat{P}_m(y)] = P_m(y)$$

Variance of the Estimate

$$Var[\hat{P}_m(y)] = \frac{P_m(y)}{k} - \frac{P_m(y)^2}{k}$$

- Variance depends only on **number of blocks** (*k*) and **CSL** ($P_m(y)$)

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Number of Blocks

❑ This increases from n/m to $n-m+1$
(eg from 12 to 34 for History Length=n=36 and Block Size=m=3).

Unbiased Estimate .

$$E[\hat{P}_m(y)] = P_m(y)$$

Variance of the Estimate

$$Var[\hat{P}_m(y)] = \frac{1}{k}P_m(y) + \left[\frac{(k-m)(k-m+1)}{k^2} - 1\right]P_m(y)^2$$

$$+ \frac{2}{k^2}\sum_{s=1}^{m-1}(k-m+s)\sum_{Y_1=0}^{y}\cdots\sum_{Y_m=0}^{y-Y_1-\ldots-Y_{m-1}}\sum_{Y_{m+1}=0}^{Y-Y_{m-s+1}-\ldots Y_m}\cdots\sum_{Y_{2m-s}=0}^{y-Y_{m-s+1}-\ldots-Y_{2m-s-1}}\prod_{j=1}^{2m-s}p(Y_j)$$

❑ s : number of overlapping observations between two blocks

❑ p : probability mass function (pmf) of the demand in one period

❑ Performance also depends on block size (*m*) and the pmf (*p*)

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Comparison of Variances

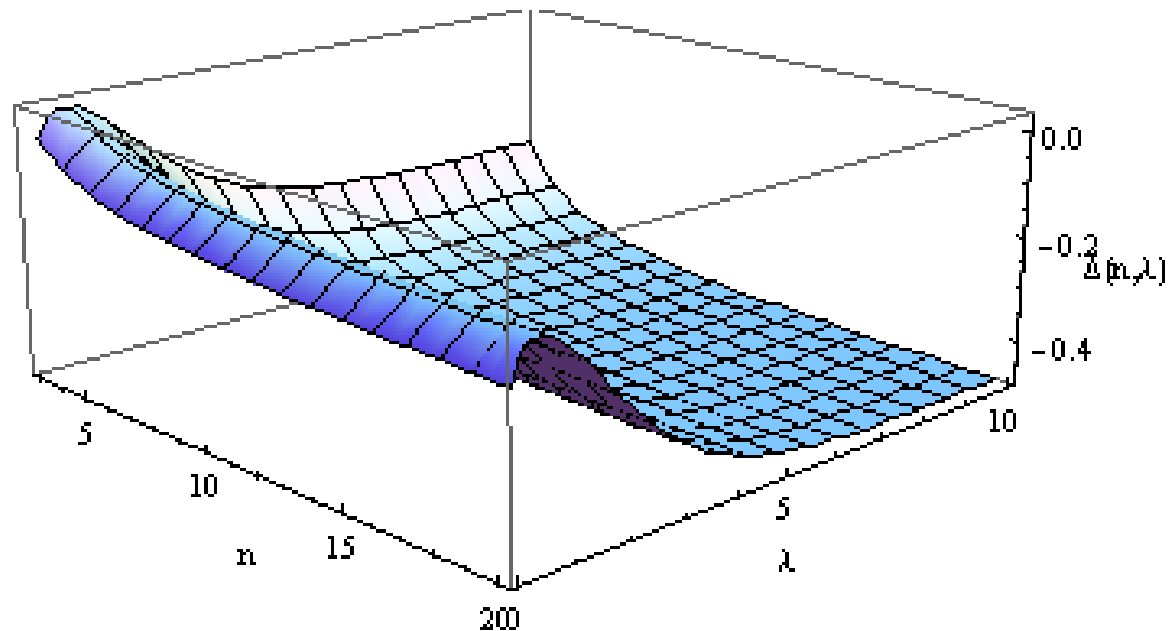❑ Let $\Delta(\Theta, n)$ denote **the variance reduction ratio** from using OB instead of NOB (for fixed LT, *m*)

$$\Delta(\Theta, n) = \frac{Var\left[\hat{P}_m^{OB}(y)\right] - Var\left[\hat{P}_m^{NOB}(y)\right]}{Var\left[\hat{P}_m^{NOB}(y)\right]}$$

❑ The NOB approach has lower variance than the OB approach if and only if:

$$\sum_{s=1}^{m-1} \frac{2(n-2m+s+1)}{(n-m+1)^2} \Theta_s \geq$$

$$\left[\frac{m}{n} - \frac{1}{(n-m+1)}\right] P_m(y) + \left[1 - \frac{m}{n} - \frac{(n-2m+1)(n-2m+2)}{(n-m+1)^2}\right] P_m(y)^2$$

Variance Reduction using OB instead of NOB

❑ Poisson distributed demand with $\lambda \in [0,10]$; $n \in [3,20]$ and $m = 2$

❑ In almost all cases, the OB approach leads to a variance reduction.

❑ For very low values of λ (i.e. slow moving demand), the benefit from using OB instead of NOB is very low when n is low.

❑ There are very few values where NOB outperforms OB; these cases occur when both λ and n are very low.

Bootstrapping without Replacement: Properties

Unbiased Estimate

$$E[\hat{P}_m(y)] = P_m(y)$$

Variance of the Estimate

\# combinations
of i from m

\# permutations
of i from m

$$Var[\hat{P}_m^{Boot(NR)}(y)] = \frac{1}{\prod\limits_{i=0}^{m-1}(n-i)^2}\left[\sum\limits_{i=0}^{m}\Theta_i C(m,i)P(m,i)\prod\limits_{j=0}^{2m-j-1}(n-j)\right] - P_m(y)^2$$

$$\Theta_0 = P_m(y)^2$$

$$\Theta_m = P_m(y)$$

Special Case (m=2)

$$Var[\hat{P}_2^{Boot(NR)}(y)] = \frac{1}{n(n-1)}[(n-2)(n-3)P_2(y)^2 + 4(n-2)\Theta_1 + 2P_2(y)] - P_2(y)^2$$

- ❑ Same Θ factors as in the Overlapping Blocks approach.
- ❑ Other special cases are obtained straightforwardly.

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Bootstrapping with Replacement: Bias Properties

Biased Estimate

$$E[\hat{P}_m^{Boot(R)}(y)] \neq P_m(y)$$

Special Case (m=2)

$$E[\hat{P}_2^{Boot}(y)] = \frac{1}{n} P_1(\lfloor y/2 \rfloor) + \left(\frac{n^2 - n}{n}\right) P_2(y)$$

$$Bias = E[\hat{P}_2^{Boot}(y)] - P_2(y) = \frac{1}{n}[P_1(\lfloor y/2 \rfloor) - P_2(y)]$$

Special Case (m=3)

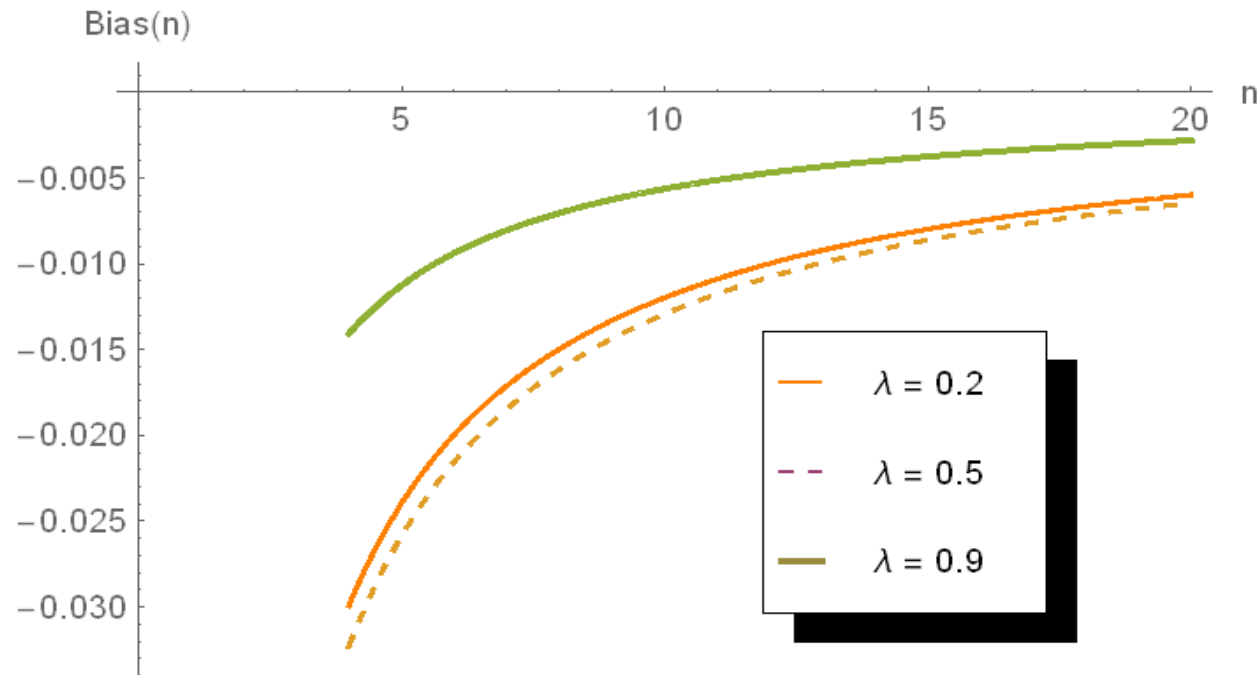$$Bias = \frac{1}{n^2} P_1(\lfloor y/3 \rfloor) + 3\left(1 - \frac{1}{n}\right)\Lambda_{21}(y) - \left(\frac{3}{n} - \frac{2}{n^2}\right)P_3(y) \qquad \Lambda_{21}(y) = \sum_{Y_1=0}^{\lfloor y/2 \rfloor} \sum_{Y_2=0}^{y-Y_1} p(Y_1)p(Y_2)$$

- ❑ Different factor (Λ) than in the Overlapping Blocks approach.

- ❑ New factor takes into account repeated selection of same time index.

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Quantifying Bias for Bootstrap with Replacement
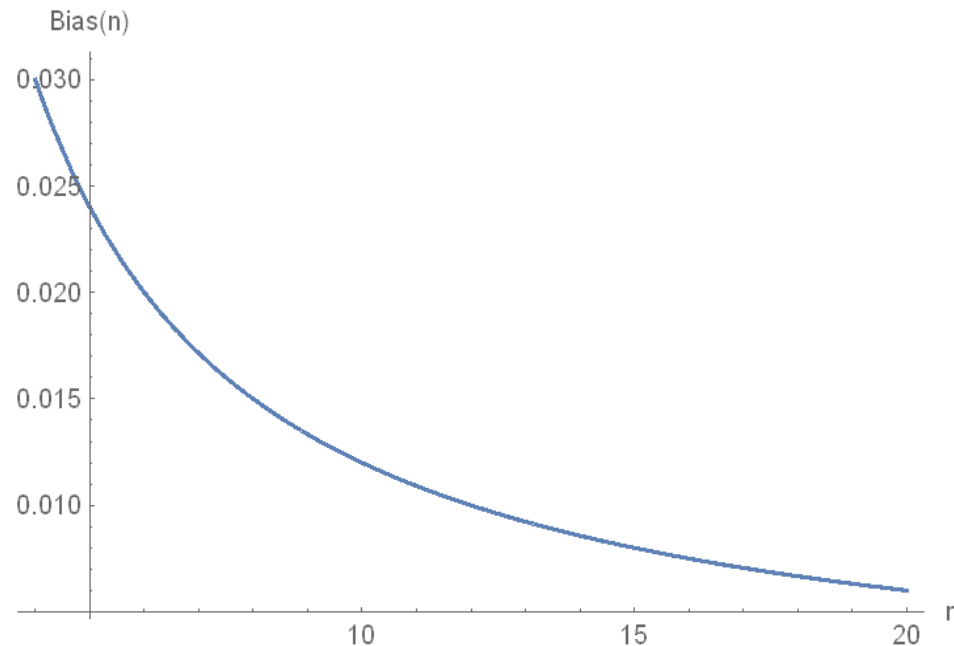
Poisson Demand (non-lumpy)

y=1

❑ Underestimates CSL

❑ Bias in CSL can be significant for short histories.

❑ The stronger the intermittence, the greater the bias. .

Quantifying Bias for Bootstrap with Replacement (2)

Lumpy Demand [p(0)=0.6, p(1)=0.1, p(10)=0.3]

y=1

- ❑ Overestimates CSL

- ❑ Bias in CSL can be significant for short histories.

- ❑ So, bias can be positive or negative depending on the distribution of demand.

Bootstrapping with Replacement: Variance Properties

$$Var(\hat{P}_m(y)) = \frac{1}{k}P_m(y) + \frac{1}{k^2}\sum_{|Si\cap S_j=l|}\sum_{l=1}^{m-1}E[I_{[0,y]}(D_i)I_{[0,y]}(D_j)]P_m(y)^2 + \frac{1}{k^2}\sum_{|S_i\cap Sj|=m}P_m(y)$$

- ❑ First term – selection of same indices in same sequence

- ❑ Second term – some indices different

- ❑ Third term – same indices in different sequence

Special Case (m=2)

$$n^4E(\hat{P}_2^{Boot}(y)^2) = n(n-1)[P_1(\lfloor y/2\rfloor)^2 + 2(n-2)P_1(\lfloor y/2\rfloor)P_2(y) + (n-2)(n-3)P_2(y)^2]$$
$$+ 4n(n-1)\Lambda_1(y) + 4n(n-1)(n-2)\Theta_1(y) + nP_1(\lfloor y/2\rfloor) + 2(n^2-n)P_2(y)$$

$$Var[\hat{P}_2^{Boot(R)}(y)] = E[P_2^{Boot(R)}(y)^2] - E[P_2^{Boot(R)}(y)]^2$$

- ❑ Special case for m=3 has been derived – very messy!

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Variance Reduction Ratio of bootstrapping with replacement compared to OB (y=1)
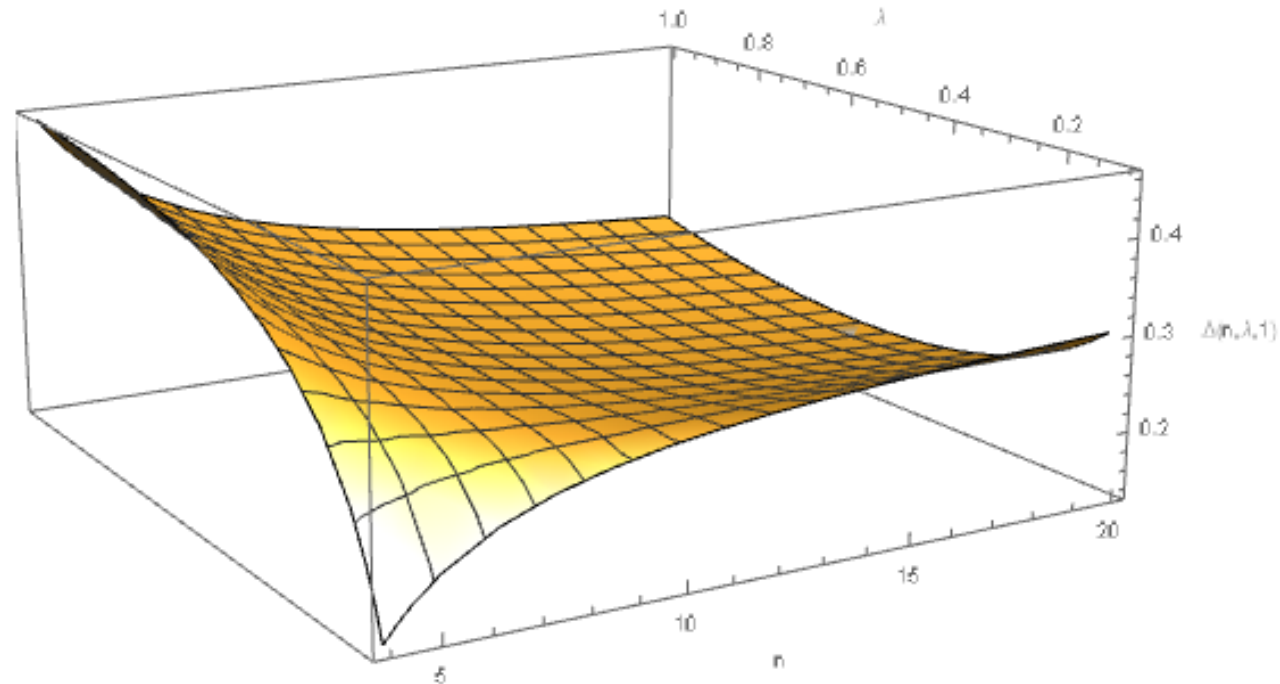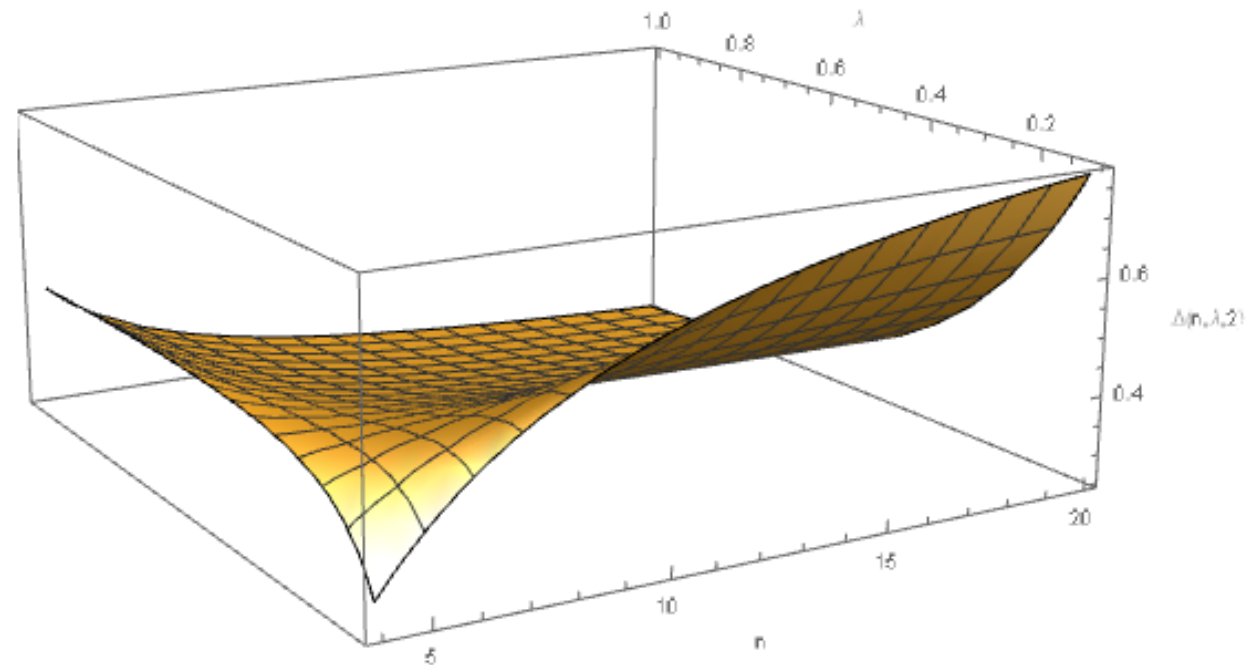

- ❑ Variance always reduced by using Bootstrapping with Replacement

- ❑ Strong reduction across most parameters
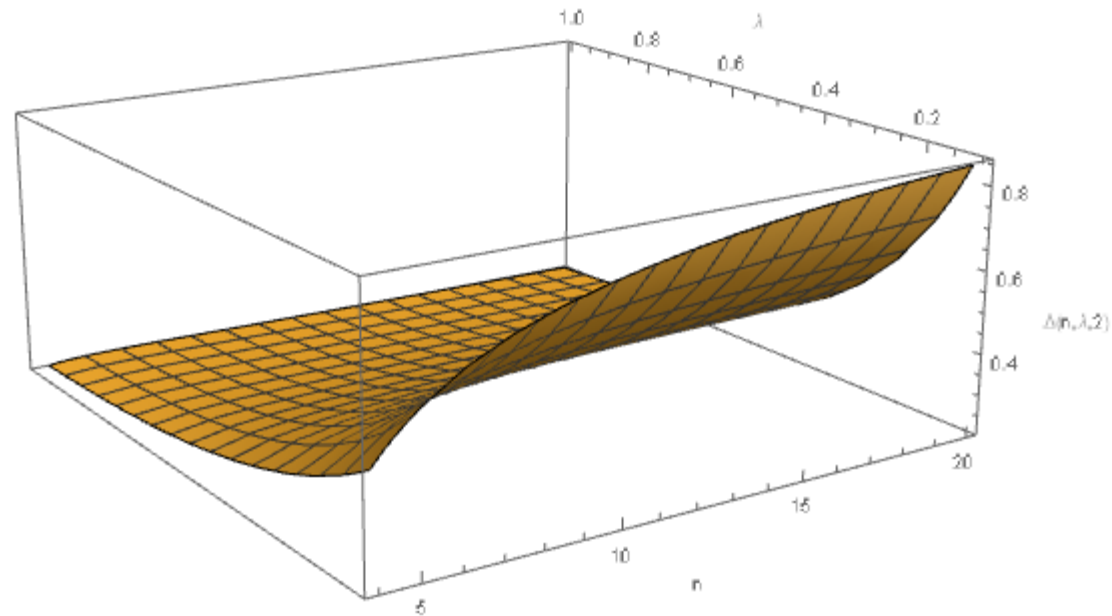
- ❑ Exception is for short demand histories.

. .

Relative performance of bootstrapping **with** replacement compared to OB (y=2)

- ❑ Variance always reduced by using Bootstrapping with Replacement
- ❑ Stronger reductions for y=2 compared with y=1
- ❑ Stronger effect of the length of demand histories

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Variance Reduction Ratio of bootstrapping **without** replacement compared to OB (y=2)

- ❑ Variance always reduced by using Bootstrapping without Replacement

- ❑ Demand history effect as before

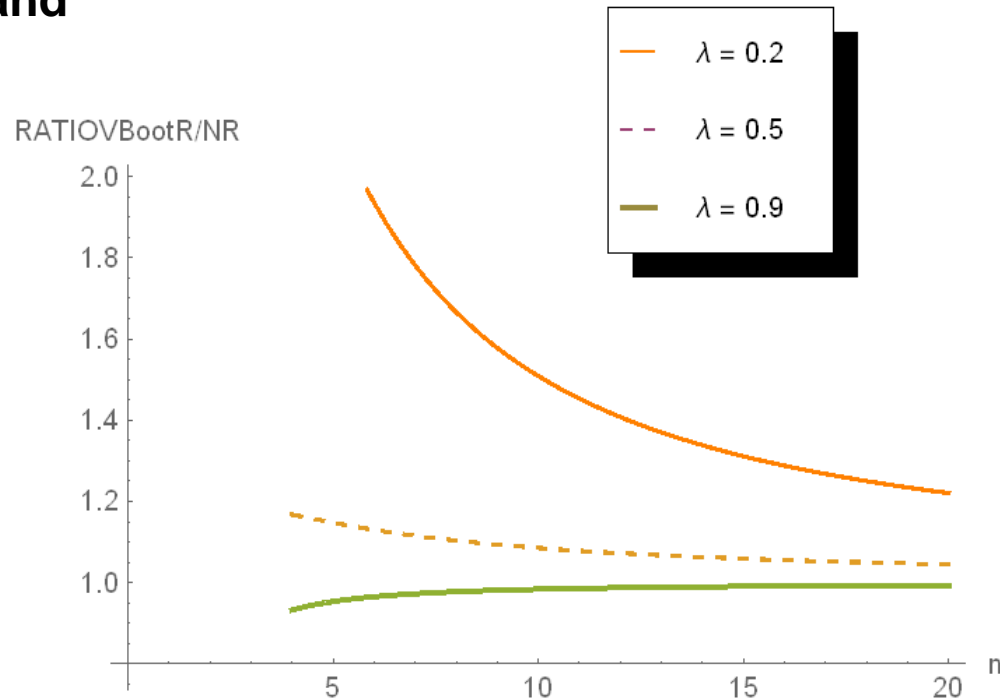- ❑ For fast-moving, very modest reduction.

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Poisson Demand
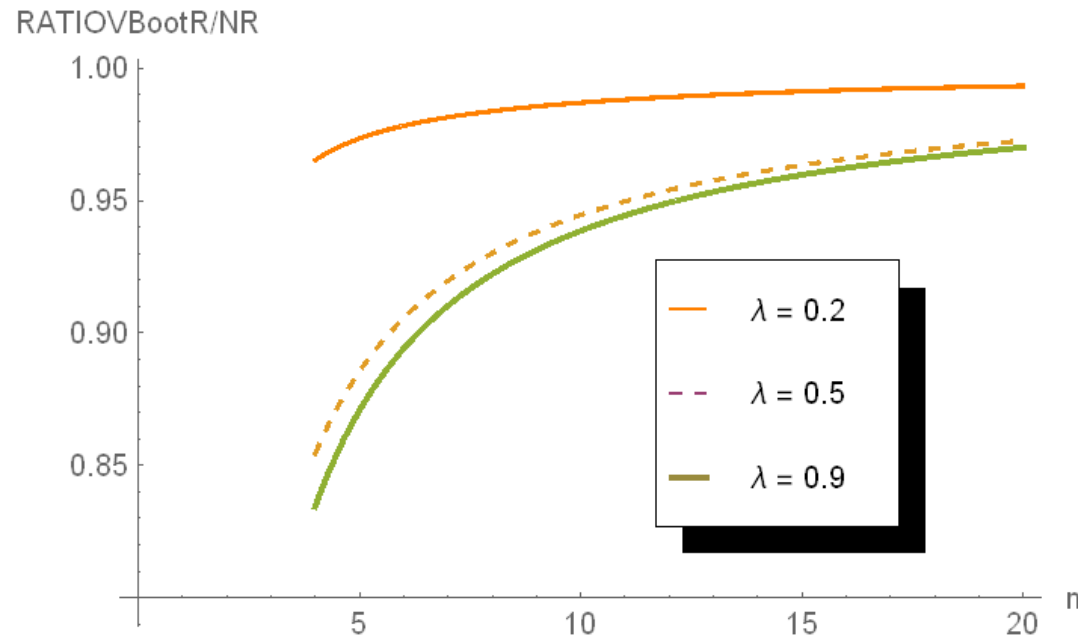
- ❑ With Replacement always gives lower variance

- ❑ For highly intermittent demand, little difference

- ❑ For shorter histories, difference may become significant for less intermittent demand .

Poisson Demand

- ❑ With Replacement does not always give lower variance

- ❑ For highly intermittent demand, Without Replacement can give lower variance

Conclusions

- Bias and variance properties of Bootstrapping (with and without replacement) have been derived for CDF estimates.

- Bootstrapping with Replacement is generally biased in its estimate of the CDF.

- Direction of bias depends on the nature of the distribution.

- Bootstrapping (with and without replacement) has lower variance than Overlapping Blocks approach.

- Comparison of variance of Bootstrapping With and Without Replacement estimates depends on the value for which the CDF is being estimated.

- Further research needed to examine this further.

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Thank you for your attention!

Q&A?

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