Intermittent state-space model for demand forecasting

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Motivation

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He also assumes that probability is constant between occurrences.
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He also assumes that probability is constant between occurrences.

Syntetos and Boylan (2001, 2005) show that the conditional expectation of Croston’s method is biased.

They propose an approximation, that corrects the error.
Snyder (2002) looks at Croston’s method in details, claiming that the underlying model is: $y_t = x_t \cdot z_t + \epsilon_t$. 

This model produces both positive and negative data. This is a drawback, so Snyder (2002) proposes a modification, taking $\exp$ of non-zero demands.
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They argue that any model underlying Croston’s method must be:

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- defined on continuous space.
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They argue that any model underlying Croston’s method must be:

- non-stationary,
- defined on continuous space.

They conclude that the implied model has non-realistic properties.

They support Snyder (2002) approach with \( \exp \).
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Teunter et al. (2011) propose a model taking inventory obsolescence into account.
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They use MSE calculated as a difference between the estimate and the actual demand.
Motivation

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Introduction

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And proposes two new ones, which improves estimation of methods.

He finds that optimisation of initial states increases forecasting accuracy.
Motivation, overall

There is no concise model, underlying all the methods.
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Intermittent demand methods are disconnected from fast-moving data methods.
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Because of Shenstone and Hyndman (2005) we believe that it doesn’t exist.

Intermittent demand methods are disconnected from fast-moving data methods.

And we still need to make good decisions about replenishment levels.
Universal model
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Very general model:

\[ y_t = o_t \tilde{y}_t, \]  

(1)
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\[ y_t = o_t \tilde{y}_t, \] (1)

where \( o_t \sim \text{Bernoulli}(p_t) \) and \( \tilde{y}_t \) is a statistical model of our choice.

This corresponds to Croston’s original idea.
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Very general model:

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where \( o_t \sim \text{Bernoulli}(p_t) \) and \( \tilde{y}_t \) is a statistical model of our choice.

This corresponds to Croston’s original idea.

If \( o_t = 1 \), for any \( t \), then this is fast-moving data model.
Additive state-space model (Snyder, 1985)

State-space model:

\[
\begin{align*}
y_t &= o_t (w' v_{t-1} + \epsilon_t) \\
v_t &= F v_{t-1} + g \epsilon_t
\end{align*}
\]  

(2)
Additive state-space model (Snyder, 1985)

State-space model:

\[ y_t = o_t(w'v_{t-1} + \epsilon_t) \]
\[ v_t = Fv_{t-1} + g\epsilon_t \]  

\(v_{t-1}\) vector of states, \(w\) is measurement vector, 
\(F\) is transition matrix, \(g\) is persistence vector, 
\(\epsilon_t \sim N(0, \sigma^2)\).
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\(v_{t-1}\) vector of states, \(w\) is measurement vector, \(F\) is transition matrix, \(g\) is persistence vector, \(\epsilon_t \sim N(0, \sigma^2)\).

Example. iETS(A,N,N) with constant probability:

\[
y_t = o_t(l_{t-1} + \epsilon_t) \\
l_t = l_{t-1} + \alpha\epsilon_t
\]

(3)

where \(o_t \sim\) Bernoulli\(p\).
General state-space (based on Hyndman et al. (2008))

State-space model for any ETS:

\[ y_t = o_t \left( w(v_{t-1}) + r(v_{t-1}, \epsilon_t) \right) \]
\[ v_t = F(v_{t-1}) + g(v_{t-1}, \epsilon_t) \quad (4) \]
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\]  
\hspace{1cm} (4)

Example. iETS(M,Ad,N) with constant probability:

\[
y_t = o_t(l_{t-1} + \phi b_{t-1})(1 + \epsilon_t) \\
l_t = (l_{t-1} + \phi b_{t-1})(1 + \alpha \epsilon_t) \\
b_t = \phi b_{t-1}(1 + \beta \epsilon_t)
\]  
\hspace{1cm} (5)

where \( o_t \sim \text{Bernoulli}(p) \), \((1 + \epsilon_t) \sim \log N(0, \sigma^2)\).
Advantages

What are the advantages of such a model?

- Statistical rationale for intermittent demand;
- Connection between conventional and intermittent models;
Advantages

What are the advantages of such a model?

- Statistical rationale for intermittent demand;
- Connection between conventional and intermittent models;
- Correct estimation of mean;
- Simpler variance estimation;
- Prediction intervals;
Advantages

What else?

- Both additive and multiplicative ETS models;
- Any statistical model;
Advantages

What else?

- Both additive and multiplicative ETS models;
- Any statistical model;
- Likelihood function;
- Solution to initialisation and optimisation problems;
Advantages

What else?

- Both additive and multiplicative ETS models;
- Any statistical model;
- Likelihood function;
- Solution to initialisation and optimisation problems;
- Model selection.
Disadvantages

What are the disadvantages of such a model?

- May need more observations...
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- ...Especially for trend and seasonal models;
Disadvantages

What are the disadvantages of such a model?

- May need more observations...
- ...Especially for trend and seasonal models;
- Derivations in some cases may be messy.
$iETS(M,N,N)$,

constant probability
iETS(M,N,N), constant probability

iETS(M,N,N) model has the form:

\[ y_t = o_t l_{t-1} (1 + \epsilon_t) \]
\[ l_t = l_{t-1} (1 + \alpha \epsilon_t) \]

where \( o_t \sim \text{Bernoulli}(p) \).
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where \(o_t \sim \text{Bernoulli}(p)\).

iETS(M,N,N) underlies SES (Hyndman et al., 2008).
iETS(M,N,N), constant probability

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Conditional expectation:

\[ E(y_{t+h}|t) = pE(\tilde{y}_{t+h}|t) = pw'F^{h-1}v_t = pl_t. \]
iETS(M,N,N), constant probability

Conditional variance:

\[
V(y_{t+h}|t) = p(1-p)l_t^2 + pl_t^2 \sigma^2 \left( 1 + \alpha^2 (1 + \sigma^2) \sum_{j=1}^{h-1} (1 + \alpha^2 \sigma^2) \right).
\]
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Messy because of the multiplicative error.
iETS(M,N,N), constant probability

Likelihood can be derived taking probabilities:

\[ P(y_t|o_t = 1, \theta, \sigma^2) = p \frac{1}{y_t} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(1+\epsilon_t)^2}{2\sigma^2}} \]

\[ P(y_t|o_t = 0, \theta, \sigma^2) = 1 - p. \]
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\[ P(y_t | o_t = 0, \theta, \sigma^2) = 1 - p. \]

Product of all the zero and non-zero cases is then:

\[ L(\theta, \sigma^2 | y_t) = \prod_{o_t=1} p \frac{1}{y_t} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(1+\epsilon_t)^2}{2\sigma^2}} \prod_{o_t=0} (1 - p). \quad (7) \]
iETS(M,N,N), constant probability

The concentrated log-likelihood is simple:

$$\ell(\theta, \hat{\sigma}^2|y_t) = -\frac{T_1}{2} \left( \log(2\pi e) + \log(\hat{\sigma}^2) \right) - \sum_{o_t=1} \log(y_t) + T_0 \log(1 - p) + T_1 \log p,$$

(8)
iETS(M,N,N), constant probability

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+ T_0 \log(1 - p) + T_1 \log p,
\]

(8)

where \( T \) is number of all observations, \( T_0 \) is number of zeroes, \( T_1 \) number of non-zero demands.
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\[ + T_0 \log(1 - p) + T_1 \log p, \]

where \( T \) is number of all observations, \( T_0 \) is number of zeroes, \( T_1 \) number of non-zero demands.

The variance of the error estimated using likelihood (8) is:

\[ \hat{\sigma}^2 = \frac{1}{T_1} \sum_{o_t=1} (1 + \epsilon_t). \]
iETS(M,N,N), constant probability

The concentrated log-likelihood is simple:

$$\ell(\theta, \hat{\sigma}^2 | y_t) = - \frac{T_1}{2} \left( \log(2\pi\varepsilon) + \log(\hat{\sigma}^2) \right) - \sum_{o_t=1} \log(y_t)$$

$$+ T_0 \log(1 - p) + T_1 \log p,$$

where $T$ is number of all observations, $T_0$ is number of zeroes, $T_1$ number of non-zero demands.

The variance of the error estimated using likelihood (8) is:

$$\hat{\sigma}^2 = \frac{1}{T_1} \sum_{o_t=1} (1 + \epsilon_t).$$

The probability can also be derived from (8): $p = \frac{T_1}{T}$. 

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Intermittent state-space model for demand forecasting
Example. Intermittent demand

ETS(MNN)

Series
Fitted values
Point forecast
95% prediction interval
Forecast origin

Intermittent state-space model for demand forecasting
Example. Probabilities

Intermittent state-space model for demand forecasting
Simple iETS. Sub-conclusion

- Pretty easy statistical model;
- Multiplicative ETS is possible and makes more sense than additive;
Simple iETS. Sub-conclusion

- Pretty easy statistical model;
- Multiplicative ETS is possible and makes more sense than additive;
- But probability is currently constant;
iETS(M,N,N),
time varying probability,
Croston style
Croston’s iETS(M,N,N)

ETS(M,N,N) + compound Bernoulli distribution:
\[ o_t \sim \text{Bernoulli}(p_t), \text{ where } p_t = \frac{1}{1+q_t}, \]
\[ q_t \text{ are intervals between demands. If } q_t = 0, \text{ then } p_t = 1. \]
Croston’s iETS(M,N,N)

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**Assumption:** Probability changes only when demand occurs.
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**Assumption**: Probability changes only when demand occurs.

State-space model for probabilities:

\[
q_t = l_{q,t-1}(1 + \varepsilon_t) \\
l_{q,t} = l_{q,t-1}(1 + \delta \varepsilon_t)
\]

(9)

where \((1 + \varepsilon_t) \sim \text{log N}(0, \sigma_q^2)\)
Croston’s iETS(M,N,N)

Overall iETS(M,N,N) Croston style is:

\[ y_t = o_t l_{t-1}(1 + \epsilon_t) \]
\[ l_t = l_{t-1}(1 + \alpha \epsilon_t) \]
\[ q_t = l_{q,t-1}(1 + \epsilon_t) \]
\[ l_{q,t} = l_{q,t-1}(1 + \delta \epsilon_t) \] (10)

\[ (1 + \epsilon_t) \sim \log N(0, \sigma^2) \]
\[ o_t \sim \text{Bernoulli}(\frac{1}{1+q_t}) \]
\[ (1 + \epsilon_t) \sim \log N(0, \sigma_q^2). \]
Croston’s iETS(M,N,N)

Overall iETS(M,N,N) Croston style is:

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\begin{align*}
y_t &= o_t l_{t-1}(1 + \epsilon_t) \\
l_t &= l_{t-1}(1 + \alpha \epsilon_t) \\
q_t &= l_{q,t-1}(1 + \varpi_t) \\
l_{q,t} &= l_{q,t-1}(1 + \delta \varpi_t)
\end{align*}
\] (10)

\[(1 + \epsilon_t) \sim \log N(0, \sigma^2_o) \]
\[o_t \sim \text{Bernoulli}\left(\frac{1}{1+q_t}\right)\]
\[(1 + \varpi_t) \sim \log N(0, \sigma^2_q).\]

Now it becomes a bit more complicated...
Croston’s iETS(M,N,N)

Conditional expectation:

\[ E(y_{t+h}|t) = l_t E \left( \frac{1}{1 + q_{t+h}} \bigg| t \right). \]
Croston’s iETS(M,N,N)

Conditional expectation:

\[ E(y_{t+h}|t) = l_t E \left( \frac{1}{1 + q_{t+h}} \middle| t \right). \]

Not yet simplified:

\[ E(y_{t+h}|t) = l_t E \left( \frac{1}{1 + l_{q,t} \prod_{j=1}^{h-1} (1 + \delta \varepsilon_{t+j})(1 + \varepsilon_{t+h})} \middle| t \right). \]

We feel that this should be close to SBA.
Croston’s iETS(M,N,N)

Variance is currently mind blowing...
Croston’s iETS(M,N,N)

Variance is currently mind blowing...

But it should be based on the variance of $o_t$:

$$\sigma_o^2 = p_t(1 - p_t)$$
Variance is currently mind blowing...

But it should be based on the variance of $o_t$:

$$\sigma_o^2 = p_t(1 - p_t)$$

Meaning that the conditional variance of $y_{t+h}$ is:

$$V(y_{t+h}|t) = E\left(\frac{1}{1+q_{t+h}}|t\right) \left(1 - E\left(\frac{1}{1+q_{t+h}}|t\right)\right) l_t^2 + E\left(\frac{1}{1+q_{t+h}}|t\right) l_t^2 \sigma^2 \left(1 + \alpha^2(1 + \sigma^2) \sum_{j=1}^{h-1} (1 + \alpha^2 \sigma^2)\right).$$
Croston’s iETS(M,N,N)

Likelihood however can be done in two stages (assuming demand sizes and intervals are independent):

1. Likelihood for intervals;
Croston’s iETS(M,N,N)

Likelihood however can be done in two stages (assuming demand sizes and intervals are independent):

1. Likelihood for intervals;
2. Likelihood for demands.

Both of them are based on lognormal distributions.
Croston’s iETS(M,N,N)

Concentrated log-likelihoods.
For intervals (first stage):

\[ \ell(\theta_q, \hat{\sigma}_q^2|q_t) = - \frac{T_q}{2} \left( \log (2\pi e) + \log (\hat{\sigma}_q^2) \right) - \sum_{t=1}^{T_q} \log(q_t), \quad (11) \]
Croston’s iETS(M,N,N)

Concentrated log-likelihoods. For intervals (first stage):

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\ell(\theta_q, \hat{\sigma}_q^2 | q_t) = -\frac{T_q}{2} \left( \log (2\pi e) + \log (\hat{\sigma}_q^2) \right) - \sum_{t=1}^{T_q} \log(q_t), \quad (11)
\]

For demands (second stage):

\[
\ell(\theta, \hat{\sigma}^2 | y_t) = -\frac{T_1}{2} \left( \log (2\pi e) + \log (\hat{\sigma}^2) \right) - \sum_{o_t=1} \log(y_t)
+ \sum_{o_t=0} \log(1 - p_t) + \sum_{o_t=1} \log p_t, \quad (12)
\]
Croston’s iETS(M,N,N). Example

ETS(MNN)

- Series
- Fitted values
- Point forecast
- 95% prediction interval
- Forecast origin

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Intermittent state-space model for demand forecasting
Croston’s iETS(M,N,N). Example. Probabilities
Croston’s iETS. Sub-conclusion

- There is a statistical model underlying Croston’s method;
Croston’s iETS. Sub-conclusion

- There is a statistical model underlying Croston’s method;
- Conditional expectation should be closer to SBA;
- Conditional variance can be found analytically;
Croston’s iETS. Sub-conclusion

- There is a statistical model underlying Croston’s method;
- Conditional expectation should be closer to SBA;
- Conditional variance can be found analytically;
- There are still some problems with derivations.
iETS(M,N,N),

time varying probability,

TSB
ETS(M,N,N) + compound Bernoulli distribution:

\[ o_t \sim \text{Bernoulli}(p_t), \text{ where:} \]

\[
\begin{align*}
    p_t &= \frac{l_{p,t-1}(1 + \xi_t)}{l_{p,t-1}(1 + \delta \xi_t)} \\
    l_{p,t} &= l_{p,t-1}(1 + \delta \xi_t).
\end{align*}
\]
TSB iETS(M,N,N)

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\] (13)

\( p_t \) can be estimated as naïve probability: \( p_t = o_t \).
ETS(M,N,N) + compound Bernoulli distribution:

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We want to have conditional Beta\((a, b)\) distribution.

But this means that \( p_t \in (0, 1). \)
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\( p_t \) can be estimated as naïve probability: \( p_t = o_t \).

We want to have conditional Beta\((a, b)\) distribution.

But this means that \( p_t \in (0, 1) \).

We need boundary values!
TSB iETS(M,N,N)

Temporary fix – simple transfer function:

\[ p'_t = (1 - 2\kappa) p_t + \kappa, \]

where \( \kappa \) is some small number. For example, \( \kappa = 10^{-20}. \) This means that \( p'_t \in (\kappa, 1 - \kappa). \) So \( p'_t \sim \text{Beta}(a, b). \)
TSB iETS(M,N,N)

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Temporary fix – simple transfer function:

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where \( \kappa \) is some small number. e.g. \( \kappa = 10^{-20} \).

This means that \( p'_t \in (\kappa, 1 - \kappa) \).

So \( p'_t \sim \text{Beta}(a,b) \).
The fixed TSB iETS(M,N,N) is then:

\[

y_t = o_t l_{t-1} (1 + \epsilon_t) \\
l_t = l_{t-1} (1 + \alpha \epsilon_t) \\
p_t = \frac{p'-\kappa}{1-2\kappa} \\
p'_t = l_{p,t-1} (1 + \xi_t) \\
l_{p,t} = l_{p,t-1} (1 + \delta \xi_t) \\
(1 + \epsilon_t) \sim \log N(0, \sigma^2) \\
o_t \sim \text{Bernoulli}(p_t) \\
p'_t \sim \text{Beta}(a,b)
\]
Conditional expectation is simpler than in Croston:

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TSB iETS(M,N,N)

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Conditional variance is based on Bernoulli $p_{t+h|t}(1 - p_{t+h|t})$:

$$V(y_{t+h}|t) = \frac{l_{p,t-1} - \kappa}{1 - 2\kappa} \left( 1 - \frac{l_{p,t-1} - \kappa}{1 - 2\kappa} \right) l_t^2$$

$$+ \frac{l_{p,t-1} - \kappa}{1 - 2\kappa} l_t^2 \sigma^2 \left( 1 + \alpha^2 (1 + \sigma^2) \sum_{j=1}^{h-1} (1 + \alpha^2 \sigma^2) \right).$$
TSB iETS(M,N,N)

Concentrated log-likelihood in two stages.
For the probability (stage 1):

\[
\ell(\theta_p, a, b|p_t) = (a - 1) \sum_{t=1}^{T} \log(l_{p,t-1}(1 + \xi_t)) + (b - 1) \sum_{t=1}^{T} \log(1 - l_{p,t-1}(1 + \xi_t)) - T \log B(a, b),
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(15)
TSB iETS(M,N,N)

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- T \log B(a, b),
\]

(15)

For the demand sizes (stage 2):

\[
\ell(\theta, \hat{\sigma}^2|y_t) = -\frac{T}{2} \left( \log (2\pi e) + \log (\hat{\sigma}^2) \right) - \sum_{o_t=1} \log(y_t) \\
+ \sum_{o_t=0} \log(1 - p_t) + \sum_{o_t=1} \log p_t,
\]

(16)
TSB iETS(M,N,N). Example

ETS(MNN)

Series
Fitted values
Point forecast
95% prediction interval
Forecast origin

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Intermittent state-space model for demand forecasting
TSB iETS(M,N,N). Example. Probabilities

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Intermittent state-space model for demand forecasting
TSB iETS. Sub-conclusion

- There is a statistical model underlying TSB;
TSB iETS. Sub-conclusion

- There is a statistical model underlying TSB;
- Estimation problem solved;
- Works fine even with the proposed approximation;
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- There is a statistical model underlying TSB;
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- Works fine even with the proposed approximation;
- $p_t$ is unknown, problem with estimation;
- Problem with distribution of $p_t$;
- Multiplicative damped trend could be more appropriate.
Real time series example
Example on the real data

1. 58 intermittent time series,
2. One product, different branches, daily data,

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Intermittent state-space model for demand forecasting
Example on the real data

1. 58 intermittent time series,
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4. Holdout sample of 20 obs,
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   - Stable probability,
   - Croston’s probability,
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   - Croston’s probability,
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6. Croston’s method and TSB, ”tsintermittent” package in R.
Example on the real data

<table>
<thead>
<tr>
<th>Method</th>
<th>sPIS</th>
<th>sAPIs</th>
<th>ARMSE</th>
<th>Complex bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>iETS, stable</td>
<td>-609.2</td>
<td>2219.6</td>
<td>1.00</td>
<td>-46.3%</td>
</tr>
<tr>
<td>iETS, Croston</td>
<td>-442.0</td>
<td>2299.4</td>
<td>0.99</td>
<td>-48.4%</td>
</tr>
<tr>
<td>iETS, TSB</td>
<td>-538.2</td>
<td>2082.3</td>
<td>0.92</td>
<td>-46.1%</td>
</tr>
<tr>
<td>Croston’s method</td>
<td>-256.0</td>
<td>2158.9</td>
<td>1.03</td>
<td>-53.2%</td>
</tr>
<tr>
<td>TSB method</td>
<td>-279.6</td>
<td>2116.2</td>
<td>1.03</td>
<td>-52.8%</td>
</tr>
<tr>
<td>Zero forecast</td>
<td>-2363.6</td>
<td>2363.6</td>
<td>0.82</td>
<td>99.5%</td>
</tr>
</tbody>
</table>

Table: Intermittent demand data performance.
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- We proposed a very simple modification, that can be applied to any model;

- Connection between intermittent and conventional models;

- iETS is one of possible models;

- Conditional expectation can be correctly estimated;

- The same holds for the conditional variance;

- Prediction intervals for intermittent data;
Conclusions

- Croston and TSB have underlying iETS model;

- Estimation problem is now solved for them.
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• Croston and TSB have underlying iETS model;

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• Multiplicative models are available now;

• Model selection is also available;
Conclusions

- Croston and TSB have underlying iETS model;

- Estimation problem is now solved for them.

- Multiplicative models are available now;

- Model selection is also available;

- It can even be done between Stable / Croston / TSB;
Thank you for your attention!

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Snyder, R. D., 1985. Recursive Estimation of Dynamic Linear

Intermittent state-space model for demand forecasting


