TMUA Paper 2 Workshop

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Pairs $(n, 2n+1)$

Consider the following pairs of integers:

$$(3,7),\ \ (5,11),\ \ (7,15),\ \ (9,19),\ \ (11,23),\ \ (19,39),\ \ (25,51).$$

Which of the following statements is true about a pair $(n, 2n+1)$ of odd numbers?

I. **If** n is prime **then** $2n+1$ is prime.

II. For an odd number n: $2n+1$ is prime **only if** n is prime.

III. A **necessary and sufficient condition** for n to be prime is that $2n+1$ be prime.

IV. For any $n > 1$, at least one of $n, 2n+1$ must be prime.

V. **There exists** an odd prime number p such that $2p+1$ is prime.

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Counter-examples

To show that I-IV are false, we only need to find one **counter-example** in each case.

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Can you define the word "counter-example"?

A counter-example to a mathematical statement P is an object (e.g. a number) which **satisfies the hypotheses** of P, but does **not satisfy the conclusions** of P.

If ... then

Consider the following pairs of integers:

$$\text{A. } (3,7),\ \text{B. } (5,11),\ \text{C. } (7,15),\ \text{D. } (9,19),$$

$$\text{E. } (11,23),\ \text{F. } (19,39),\ \text{G. } (25,51).$$

Which of these is a counter-example to the statement:

"If n is prime then $2n+1$ is prime."

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Both C and F are counter-examples.

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Both C and F are counter-examples.

Note that G is **not** a counter-example, because 25 is not prime.

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Which of these is a counter-example to the statement:

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Both C and F are counter-examples.

Note that G is **not** a counter-example, because 25 is not prime. (The statement doesn't assert anything if n is not prime.)

Only if

Consider the following pairs of integers:

$$A.\ (3,7),\ B.\ (5,11),\ C.\ (7,15),\ D.\ (9,19),$$

$$E.\ (11,23),\ F.\ (19,39),\ G.\ (25,51).$$

Which of these is a counter-example to the statement:

"For an odd number n: $2n+1$ is prime only if n is prime."

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Which of these is a counter-example to the statement:

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Only D is a counter-example.

Note that (once again) G is **not** a counter-example, because 51 is not prime.

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Which of these is a counter-example to the statement:

"For an odd number n: $2n + 1$ is prime only if n is prime."

Only D is a counter-example.

Note that (once again) G is **not** a counter-example, because 51 is not prime. (The statement **doesn't say anything** about pairs where $2n + 1$ is not prime.)

Necessary and sufficient condition

Consider the following pairs of integers:

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Which of these is a counter-example to the statement:

"A necessary and sufficient condition for n to be prime

is that $2n+1$ be prime."

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Necessary and sufficient condition $=$ if and only if. So statement III is just I and II combined.

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Which of these is a counter-example to the statement:

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is that $2n+1$ be prime."

Necessary and sufficient condition = if and only if. So statement III is just I and II combined.

Hence the counter-examples are C, D and F.

For any... at least one

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Which of these is a counter-example to the statement:

"For any $n > 1$, at least one of $n, 2n+1$ must be prime."

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Which of these is a counter-example to the statement:

"For any $n > 1$, at least one of $n, 2n+1$ must be prime."

The counter-examples are the pairs where **neither** n and $2n+1$ is prime.

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So the only counter-example in the list is G.

Some interesting facts

A prime number p such that $2p + 1$ is prime is called a **Sophie Germain prime**. (The number $2p + 1$ is a **safe** prime.)

It is known that there are infinitely many prime numbers.

However, it is **not known** whether there are infinitely many Sophie Germain primes!

If $p > 3$ is a prime then it must be of the form $6k \pm 1$. It is easy to see that if $2p + 1$ is prime then p must be of the form $6k - 1$. So the possibilities are:

$$5, 11, 17, 23, 29, \ldots$$

Mathematical notation is your friend

The various phrases like "necessary condition", "if... then..." etc. can be very concisely written using the following symbols:

$$\Rightarrow, \quad \Leftarrow, \quad \Leftrightarrow$$

Some people find these symbols easier than the often confusing language. The phrases:

- "If A then B"
- "A only if B"
- "A implies B"
- "A is a sufficient condition for B"
- "B is a necessary condition for A"

can all be written $A \Rightarrow B$.

Finding errors in proofs

Most proofs are presented as a series of logical deductions: using the above notation, they are of the form

$$P_1 \;\Rightarrow\; P_2 \;\Rightarrow\; \ldots \;\Rightarrow P_n$$

where P_1, P_2 etc. are mathematical statements; P_1 is the starting-point, and P_n is the desired conclusion.

If all of the steps are correct then you have proved that $P_1 \Rightarrow P_n$.

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When checking a proof which consists of right-ward pointing arrows $\Rightarrow$, pay particular care:

- ◇ when dividing by terms which may be zero;
- ◇ that the final assertion is not of the form $P_n \Rightarrow P_1$.

If and only if statements

Some proofs are instead presented in the form:

$$P_1 \;\Leftrightarrow\; P_2 \;\Leftrightarrow\; \ldots \;\Leftrightarrow P_n.$$

If all steps are correct then you have proved $P_1 \Leftrightarrow P_n$.

If no statements like "if and only if" are included in the proof, then it should probably be read as a proof of the form $P_1 \Rightarrow \ldots \Rightarrow P_n$.

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When checking a proof which is of the form $P_1 \Leftrightarrow \ldots \Leftrightarrow P_n$, pay particular care when squaring.

Every negation is somewhat annoying

Many TMUA questions ask you to negate a given mathematical statement.
It's important to remember that when negating a statement of the form:

"**For every** *a* which satisfies [hypotheses], the following [conclusions] hold,"

that this statement would be disproved if you found just **one** *a* which satisfies the hypotheses, but not the conclusions.

So the negation is:

Similar rules apply when negating a statement of the form "There exists some".

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So the negation is:

"**There exists some** *a* which satisfies [hypotheses] but not [conclusions]."

Similar rules apply when negating a statement of the form "There exists some".