The $x + y$ Factor, June 3rd, 2020

Problem by Dr Bernd Schulze

Consider a collection of rigid bars which are connected at their ends by flexible joints which allow bending in any direction in $\mathbb{R}^d$. We call such a structure a d-dimensional *(bar-joint) framework*.

Mathematically, we may model a framework as a pair (G, p), where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a map that assigns a point in $\mathbb{R}^d$ to each vertex of G.

Loosely speaking, such a framework is *rigid* if it cannot be deformed continuously into another non-congruent framework while keeping the lengths of all edges (bars) fixed. Otherwise, the framework is said to be *flexible*.

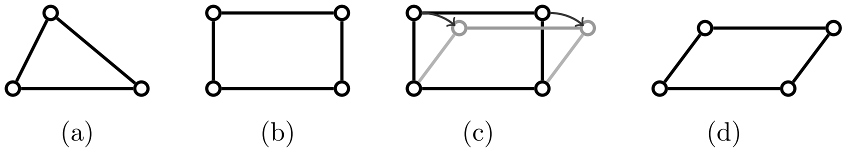

Figure 1: A rigid (a) and a flexible (b) framework in $\mathbb{R}^2$. The flex shown in (c) takes the framework in (b) to the framework in (d).

Problems

- Can you sketch a planar graph (so edges not allowed to cross), with 6 vertices and 9 straight edges, which is flexible as a bar-joint framework in $\mathbb{R}^2$?
- Can you sketch a planar graph, with 6 vertices and 8 straight edges, which is rigid as a bar-joint framework in $\mathbb{R}^2$?
- (Bonus question): Consider the graph of the triangular prism (i.e., the graph with 6 vertices and 9 edges consisting of two triangles a, b, c and a', b', c' with the three additional edges aa', bb' and cc'). Can you sketch this graph (with edges allowed to cross) so that it is flexible as a bar-joint framework in $\mathbb{R}^2$?

Note: You can make your own physical models to explore this if you like! You can use strips of cardboard (or even paper) to make bars, and split pins to make joints.