The $x + y$ Factor, 28th May

Problem by Dr Jonny Evans

The rational number $\frac{22}{7}$ is quite a good approximation to π: it comes out at about 3.1429.

In fact, it's the best approximation you can get amongst fractions with denominator 7. In fact, better still: it's the best approximation you can get with fractions whose denominator is 7 or less.

As we allow bigger and bigger denominators, we don't do better for quite a long time: $\frac{35}{11} = 3.1818...$, $\frac{47}{15} = 3.1333...$, neither of which is a better approximation than $\frac{22}{7}$.

It's not until we get to $\frac{355}{113}$ that we can do any better (at which point we get a staggeringly good 3.1415929...). As you keep allowing bigger and bigger denominators, you keep finding these fractions that do better than any fraction before them: they're called **best rational approximations**. The best rational approximations of π are hard to compute, so my question is about an easier number: the golden ratio. This is

$$\frac{1+\sqrt{5}}{2} = 1.6180339887...$$

My challenge is this. By trial and error, find the first few "best rational approximations" of the golden ratio. Do you observe a pattern? Can you explain what you observe?