On Mackey functors for TDLC-groups

WORK IN PROGRESS WITH N. MAZZA AND B. NUCINKIS

BURNSIDE AND MACKEY FUNCTORS REVISITED, 28 SEPTEMBER 2021

Totally Disconnected Locally compact groups

Definition: A locally compact (Hausdorff) group G is **totally disconnected** if the largest connected set containing the identity e_G is the singleton $\{e_G\}$.

Characterisation of TDLC-topology (van Dantzig, 1934)

Let G be a topological group and $\mathscr{C}\mathscr{O}(G) = \{O \leq G \mid \text{compact and open}\}$.
The group G is TDLC if, and only if, $\mathscr{C}\mathscr{O}(G)$ forms a neighbourhood basis at the identity e_G.

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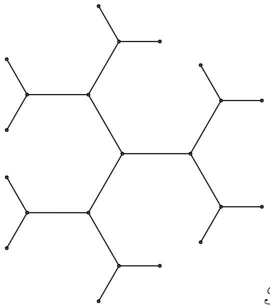
Motivation: structure theory of locally compact groups

- the theory of connected LC-groups
- the theory of TDLC-groups
- the theory of extension of groups

Examples

- Discrete groups and Profinite groups;
- Non-Archimedean local fields. E.g., $\mathbb{Q}_p$.
- Linear groups over non-Archimedean local fields. E.g., $SL_n(\mathbb{Q}_p)$.
- Automorphism groups of locally finite graphs.



$\mathscr{T}_3$

Cohomology theory (Castellano–Weigel, 2016)

Let G be a TDLC-group and M a (left) $\mathbb{Q}[G]$-module. We say that M is **discrete** if the map

$$G \times M \longrightarrow M$$

is continuous whenever M carries the discrete topology.

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Projective discrete $\mathbb{Q}[G]$-modules

- Let Ω be a (left) G-set such that $G_\omega = \{g \in G \mid g \cdot \omega = \omega\} \in \mathscr{C}\mathscr{O}$ for every $\omega \in \Omega$. Then

$$\mathbb{Q}[\Omega] \cong \coprod_{\omega \in G \backslash \Omega} \mathbb{Q}[G/G_\omega]$$

is projective (and we call it **proper discrete permutation** $\mathbb{Q}[G]$**-module**).

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Finiteness conditions in rational discrete cohomology

Type FP_n

A TDLC-group G is said to have **type** FP_n if there is a projective resolution

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Q} \longrightarrow 0$$

where each discrete $\mathbb{Q}[G]$-module P_i is finitely generated.

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Rational discrete cohomological dimension

A TDLC-group G is said to have $\mathrm{cd}_{\mathbb{Q}}(G) = n$ if there is a projective resolution

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Examples:

- G has type $\mathrm{FP}_1 \iff G$ is compactly generated.
- $\mathrm{cd}_{\mathbb{Q}}(G) = 0 \iff G$ is profinite.

Bredon cohomology

Let G be a TDLC-group and $\mathscr{C}\mathscr{O}$ the family of its compact open subgroups.
The orbit category $\mathfrak{O}_{\mathscr{C}\mathscr{O}}(G)$ consists of the following data:

objects: G/K, where $K \in \mathscr{C}\mathscr{O}$

morphisms: $G/L \overset{G\text{-map}}{\longrightarrow} G/K$ for $L, K \in \mathscr{C}\mathscr{O}$

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A **Bredon module** is a contravariant functor $T\colon \mathfrak{O}_{\mathscr{C}\mathscr{O}}(G) \to {}_R\mathrm{mod}$.
For instance, the Bredon module $\underline{R}$ is defined to be the constant functor $G/K \mapsto R$.

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where $R[G/L, G/K]$ is the free R-module on the basis $[G/L, G/K]_{\mathfrak{O}_{\mathscr{C}\mathscr{O}}}$.

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- A Bredon module M is free $\Leftrightarrow$ M is the direct sum of modules $R[-, G/K]$, for $K \in \mathscr{C}\mathscr{O}$.
- A Bredon module M is projective $\Leftrightarrow$ M is the direct summand of a free Bredon module.

Finiteness conditions in Bredon cohomology

Type FP_n in Bredon cohomology

A TDLC-group G is said to have **type** $\mathfrak{O}_{\mathscr{C}\mathscr{O}} - \mathrm{FP}_n$ **over** R if there is a projective resolution

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Bredon cohomological dimension (over R)

A TDLC-group G is said to have $\mathfrak{D}_{\mathscr{C}\mathscr{O}} - \mathrm{cd}_R(G) = n$ if there is a Bredon projective resolution

$$0 \to P_n \to \cdots \to P_0 \to \underline{R} \to 0$$

of finite length $n \in \mathbb{N}$ and any other Bredon projective resolution of $\underline{R}$ has length at least n.

Comparison of cohomology theories

Let $R = \mathbb{Q}$.

Question

Is the rational discrete cohomology of G equivalent to the Bredon cohomology of G over $\mathbb{Q}$?

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Finiteness conditions can help:

- Type FP_0 is an empty condition in the rational discrete cohomology.
- Type $\mathfrak{O}_{\mathscr{C}\mathscr{O}} - \mathrm{FP}_0 \Leftrightarrow \exists U_1, \ldots, U_n \in \mathscr{C}\mathscr{O}(G)$ s.t. any $K \in \mathscr{C}\mathscr{O}$ is subconjugated to some U_i.

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New question

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Question: Let G be a TDLC-group of type $\mathfrak{O}_{\mathscr{C}\mathscr{O}} - \mathrm{FP}_n$. Does G have type FP_n?

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COHOMOLOGICAL DIMENSION OF MACKEY FUNCTORS FOR INFINITE GROUPS

CONCHITA MARTINEZ-PÉREZ and BRITA E. A. NUCINKIS

Abstract

We consider the cohomology of Mackey functors for infinite groups and define the Mackey-cohomological dimension $\mathrm{cd}_{\mathfrak{M}}G$ of a group G. We will relate this dimension to other cohomological dimensions such as the Bredon-cohomological dimension $\mathrm{cd}_{\mathfrak{F}}G$ and the relative cohomological dimension $\mathfrak{F}$-cd$G$. In particular, we show that for virtually torsion free groups the Mackey-cohomological dimension is equal to both $\mathfrak{F}$-cd$G$ and the virtual cohomological dimension.

1. *Introduction*

Mackey-functors for finite groups have been around for a long time since they give an abstraction of the properties enjoyed by natural functors for finite groups such as group cohomology, the Burnside ring, the representation ring, algebraic K-theory or topological K-theory for classifying spaces to name a few. The motivation for this work on Mackey functors for finite groups was representation theory, see [**19–21**] as well as equivariant cohomology theory [**5**, **6**]. The study of Mackey functors for infinite groups is a fairly recent phenomenon, see, for example, [**12**] for a less general definition. In connection with the Baum–Connes conjecture, Bredon homology with coefficients in Mackey functors, especially with coefficients in the representation ring, seem to be of importance [**15**].

Let G be a group and denote by $\mathfrak{F}$ the family of finite subgroups of G. We denote by $\mathfrak{O}_{\mathfrak{F}}G$ the orbit category, which has as objects cosets G/K, where $K \in \mathfrak{F}$ and where morphisms are G-maps $G/L \to G/K$ for $G/L, G/K \in \mathfrak{O}_{\mathfrak{F}}G$. The most common definition of a Mackey functor is a pair of functors

$$(M^*, M_*) : \mathfrak{O}_{\mathfrak{F}}G \to \mathcal{A}b,$$

where M^* is contravariant, M_* is covariant and which coincide on objects. Furthermore they satisfy a certain pull-back condition, which we will describe later. A different but equivalent definition turns out to be better suited for our purposes. We shall introduce this in Section 3. The category of Mackey functors, like any category of functors to abelian groups, is an abelian category. Moreover, it can be shown that a sequence of Mackey functors is exact if and only if its evaluation on each object is exact. Also, the category of Mackey functors has enough projectives and hence there is the notion of cohomology of Mackey functors and of cohomological dimension $\mathrm{cd}_{\mathfrak{M}_{\mathfrak{F}}}G$.

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Our motivation comes from classifying spaces for proper actions and their algebraic mirror, Bredon cohomology. Bredon functors are slightly less complicated gadgets. A Bredon functor, or Bredon module, is a contravariant functor $T : \mathfrak{O}_{\mathfrak{F}}G \to \mathcal{A}b$ and there is a natural way to define cohomology and the cohomological dimension $\mathrm{cd}_{\mathfrak{F}}G$ of a group G. This is the projective dimension in the Bredon category of the constant functor $\underline{\mathbb{Z}}$. A classifying space for proper actions, denoted $\underline{E}G$ is a G-CW-complex X satisfying the following: the fixed point complex X^K is contractible if K is a finite subgroup of G and empty otherwise. Constructions by Milnor [**14**] and Segal [**18**] imply that these always exist, but these constructions give us very large models. We denote by $\mathrm{gd}_{\mathfrak{F}}G$ the minimal dimension of a model for an $\underline{E}G$. By taking fixed points, the augmented cellular chain complex of an $\underline{E}G$ gives us a projective resolution of Bredon functors

$$C_*(X^{(-)}) \twoheadrightarrow \underline{\mathbb{Z}}$$

and hence $\mathrm{cd}_{\mathfrak{F}}G \leqslant \mathrm{gd}_{\mathfrak{F}}G$. The work of Dunwoody [**3**] for dimension one and Lück [**9**] for higher dimensions implies that unless $\mathrm{cd}_{\mathfrak{F}}G = 2$, $\mathrm{cd}_{\mathfrak{F}}G = \mathrm{gd}_{\mathfrak{F}}G$. Furthermore, there are examples where $\mathrm{cd}_{\mathfrak{F}}G = 2$ but $\mathrm{gd}_{\mathfrak{F}}G = 3$ (see [**1**]). In Section 3, we shall compare the Bredon cohomology with the cohomology of Mackey functors and will show (Corollary 3.9) that for every group G,

$$\mathrm{cd}_{\mathfrak{M}_{\mathfrak{F}}}G \leqslant \mathrm{cd}_{\mathfrak{F}}G.$$

It is, however, not clear which connection there is between Mackey cohomological dimension and the topology of G-spaces.

Another quantity of interest is the relative cohomological dimension $\mathfrak{F}$-cd$G$. This is defined as the length of the shortest relative projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. A relative projective resolution $P_* \twoheadrightarrow \mathbb{Z}$ is an exact sequence of $\mathbb{Z}G$-modules, which splits when restricted to each finite subgroup of G and where the P_i are direct summands of direct sums of modules induced up from finite subgroups. In particular, permutation modules $\mathbb{Z}[G/K]$ with K finite are relative projective. It can be shown that $\mathrm{cd}_{\mathbb{Q}}G \leqslant \mathfrak{F}\text{-cd}G$. For details on relative cohomology, see [**16**]. We shall show (Theorem 4.3) that always $\mathfrak{F}\text{-cd}G \leqslant \mathrm{cd}_{\mathfrak{M}_{\mathfrak{F}}}G$ and we therefore have the following chain of inequalities:

$$\mathrm{cd}_{\mathbb{Q}}G \leqslant \mathfrak{F}\text{-cd}G \leqslant \mathrm{cd}_{\mathfrak{M}_{\mathfrak{F}}}G \leqslant \mathrm{cd}_{\mathfrak{F}}G.$$

The main motivation for studying Mackey functors came from looking at the behaviour of $\mathfrak{F}$-cd$G$ and $\mathrm{cd}_{\mathfrak{F}}G$ for virtually torsion-free groups. A group G is said to be virtually torsion-free if it has a torsion-free subgroup H of finite index. The virtual cohomological dimension vcd$G$ is defined to be equal to the cohomological dimension of H over $\mathbb{Z}$. By Serre's theorem (see [**2**]) this is well defined. Serre's theorem also implies that whenever $\mathrm{vcd}G = n$ is finite, there is a model for $\underline{E}G$ of dimension $|G : H|n$. Furthermore, $\mathrm{vcd}G \leqslant \mathfrak{F}\text{-cd}G$ (see [**1**]). A question of interest, which has become known as Brown's conjecture, is whether we can always find a model for $\underline{E}G$ of dimension equal to vcd$G$. In [**7**], examples were exhibited, where this is not the case. In particular for these examples and positive integers m $3m = \mathrm{vcd}G = \mathfrak{F}\text{-cd}G < \mathrm{cd}_{\mathfrak{F}}G = \mathrm{gd}_{\mathfrak{F}}G = 4m$. As Mackey functors seem to have a more 'symmetric' structure and seem to behave more naturally under induction from finite index subgroups (see Theorem 3.3), one would expect that things are slightly more straightforward, which is indeed the case. Theorem 5.1 implies:

if G is a virtually torsion-free group, then
$$\mathrm{vcd}G = \mathfrak{F}\text{-cd}G = \mathrm{cd}_{\mathfrak{M}_{\mathfrak{F}}}G.$$

The Mackey category

The **Mackey category** $\mathfrak{M}_{\mathscr{C}\mathscr{O}}(G)$ consists of the following data:

- objects: transitive G-set G/K for some $K \in \mathscr{C}\mathscr{O}$
- morphisms: the set of morphisms $[G/L, G/K]_{\mathfrak{M}_{\mathscr{C}\mathscr{O}}(G)}$ is the free abelian group generated by equivalence classes of the so-called basic morphisms from G/L to G/K

Definition

A **basic morphism** $(\alpha_L, \alpha_K)\colon G/L \to G/K$ is given by a diagram of G-maps

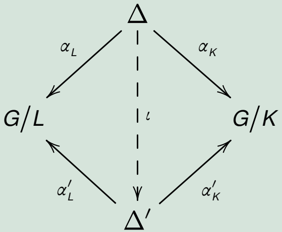
$$\begin{array}{ccc} \Delta & \xrightarrow{\alpha_K} & G/K \\ {\scriptstyle \alpha_L}\downarrow & \nearrow_{(\alpha_L,\alpha_K)} & \\ G/L & & \end{array}$$

where Δ is a finitely generated G-set.

Mackey category: equivalence classes of basic morphisms

Definition

Two basic morphisms (α_L, α_K) and (α'_L, α'_K) from G/L to G/K are **equivalent** if there exists an isomorphism $\iota\colon \Delta \to \Delta'$ of G-sets such that the following diagram

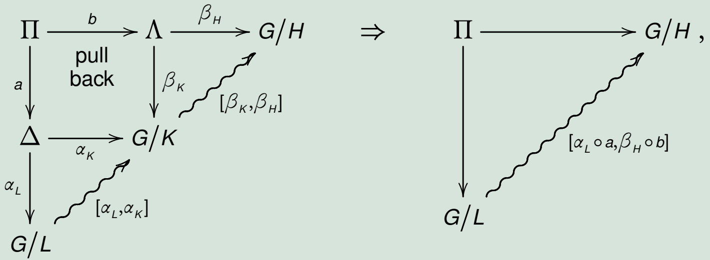


commutes; we denote the equivalence class of (α_x, α_y) by $[\alpha_x, \alpha_y]$.

Mackey category: composition of basic morphisms

Definition

The composition of two elements $[\alpha_L, \alpha_K]$ and $[\beta_K, \beta_H]$ is represented by a pull-back diagram:



Mackey functors for TDLC-groups

Let G be a TDLC-group. A **Mackey functor** for G is an additive contravariant functor

$$M\colon \mathfrak{M}_{\mathscr{C}\mathscr{O}}(G) \to {}_R\mathrm{mod}$$

We denote by $\mathrm{Mack}_{\mathscr{C}\mathscr{O}}(G)$ the category of Mackey functors for G.

Projective Mackey functors

- Let K be a compact open subgroup of G. The prototype of the projective object is then the Mackey functor

$$R[-, G/K]\colon \mathfrak{M}_{\mathscr{C}\mathscr{O}}(G) \to {}_R\mathrm{mod} \quad \text{defined by} \quad G/L \mapsto R[G/L, G/K]$$

where $R[G/L, G/K]$ is the free R-module on the basis $[G/L, G/K]_{\mathfrak{M}_{\mathscr{C}\mathscr{O}}(G)}$.

What do we need to understand

1 The **Burnside Mackey functor** B^G...

The Mackey cohomological dimension

The TDLC-group G has $\mathfrak{M}_{\mathscr{C}\mathscr{O}}\mathrm{cd}_R(G) = n$ if there is a projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow B^G \longrightarrow 0$$

by $\mathfrak{M}_{\mathscr{C}\mathscr{O}}(G)$-modules and any other projective resolution of B^G has length at least n.

Remark: If G is profinite, then B^G should be represented by $R[-, G/G]_{\mathfrak{M}_{\mathscr{C}\mathscr{O}}(G)}$. Therefore, B^G is projective and $\mathfrak{M}_{\mathscr{C}\mathscr{O}}\mathrm{cd}_R(G) = 0$.

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2. Does the chain of inequalities below hold?

$$\mathrm{cd}_{\mathbb{Q}}(G) \leq \mathfrak{M}_{\mathscr{C}\mathscr{O}}\mathrm{cd}_{\mathbb{Q}}(G) \leq \mathfrak{O}_{\mathscr{C}\mathscr{O}}\mathrm{cd}_{\mathbb{Q}}(G)$$

Mackey systems & TDLC-groups

For an arbitrary group G, a **Mackey system** is a pair $(\mathscr{C}, \mathscr{O})$ where

- $\mathscr{C}$ is a set of subgroups of G closed under conjugation and finite intersection, and,
- for every $H \in \mathscr{C}$, there is a set

$$\mathscr{O}(H) \subseteq \{U \in \mathscr{C} \mid U \leq H\} \text{ of subgroups of } H$$

satisfying the following conditions for every $H \in \mathscr{C}$ and $U \in \mathscr{O}(H)$:

1. $|H : U| < \infty$;
2. $\mathscr{O}(U) \subseteq \mathscr{O}(H)$;
3. $\mathscr{O}(gHg^{-1}) = g\mathscr{O}(H)g^{-1}$, for all $g \in G$;
4. $U \cap V \in \mathscr{O}(V)$ for all $V \in \mathscr{C}$ such that $V \leq H$.

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4. $U \cap V \in \mathscr{O}(V)$ for all $V \in \mathscr{C}$ such that $V \leq H$.

Definition:

If G is a TDLC-group, we may define $\mathscr{C}$ as the set of compact open subgroups of G and, for $H \in \mathscr{C}$, $\mathscr{O}(H)$ as the set of open subgroups of H. We refer to it as the **natural Mackey system** for G.

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Almost finite G-spaces

A G**-space** is a Hausdorff topological space X with a continuous G-action.
We say that X is **almost finite** if

- X is discrete;
- the stabiliser G_x is compact open for every $x \in X$;
- X^U, regarded as $N_G(U)/U$-space, has finitely many orbits for every $U \in \mathscr{C}\mathscr{O}$.

The category $\mathscr{A}\mathscr{F}_G$ consists of the following data:

objects: almost finite G-spaces

morphisms: $f\colon X \xrightarrow{G\text{-map}} Y$ such that each fiber $f^{-1}(y)$, $y \in Y$, is an almost finite G_y-space

Thanks for the attention