Prime Ideals in Burnside Rings

Burnside and Mackey Functors Revisited

Zachary Hall

28th September 2021

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we set $B(G)$, the Burnside ring of G, to be the ring of all finite G-sets with operations

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It is useful to note that

$$B(G) = \text{span}_{\mathbb{Z}}\{G/H | H \leq G\}.$$

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$$G/H \times G/K = \sum_{g \in [H \backslash G/K]} G/(H \cap {}^{g}K) \notin \mathcal{P}.$$

At least one of these summands must not be in $\mathcal{P}$, and since $G/(H \cap {}^{g}K) \preceq G/H, G/K$, we have a contradiction unless $G/H = G/K$ and so we have proved 2. □

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$$\begin{aligned}
\varphi_U(G/H) &= |\{gH | u.gH = gH,\ \forall u \in U\}| \\
&= |\{gH | g^{-1}ugH = H,\ \forall u \in U\}| \\
&= |\{gH | U^g H = H\}| \\
&= |\{gH | U^g \leq H\}|
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$$\Rightarrow \varphi_U(G/H) \neq 0 \Leftrightarrow U \lesssim H.$$

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$$\Rightarrow \varphi_U(X) \equiv \varphi_W(X) \text{ mod } p \quad \forall X \in B(G)$$

and

$$\varphi_U(G/W) \equiv \varphi_W(G/W) = |N_G(W) : W| \not\equiv 0 \text{ mod } p$$

since $G/W \notin \mathcal{P}$.

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Take U^p to be the smallest subgroup of U such that U/U^p is a p-group,

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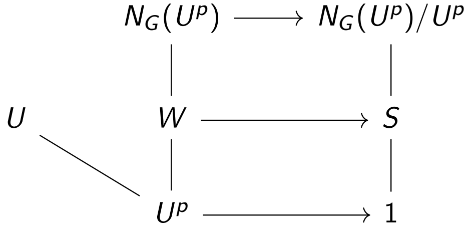

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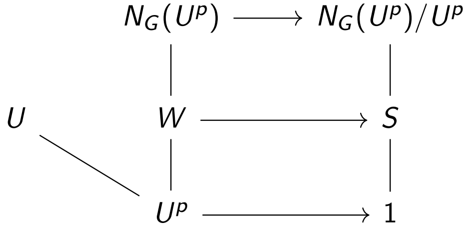

where S is a Sylow-p-subgroup of $N_G(U^p)/U^p$, W the preimage of S and the arrows are the quotient map.

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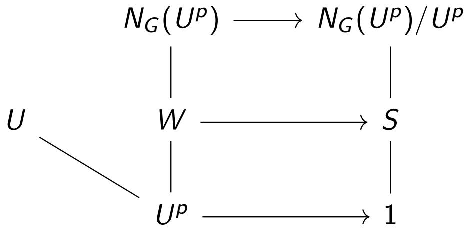

where S is a Sylow-p-subgroup of $N_G(U^p)/U^p$, W the preimage of S and the arrows are the quotient map.
From this we conclude that $\mathcal{P}_{U,p} = \mathcal{P}_{W,p}$.

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Definition

For G a profinite group, we take $\widehat{B}(G)$, the completed Burnside ring of G to be the ring of almost finite G-spaces.

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1. The requirement for there to be a unique minimal transitive G-space not in the prime ideal,
2. That U^p may not be open.

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However, we do have the same result below (the proof is similar to the finite case), with the same ordering as before.

Lemma

Let $\mathcal{P}$ be a prime ideal in $\widehat{B}(G)$. Then the set

$$\{G/H | H \leq_O G, G/H \notin \mathcal{P}\}$$

has at most one minimal element, if any exist.

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has at most one minimal element, if any exist.
In the case one does exist then for this minimal $T = G/U$ we have

$$\mathcal{P} = \{X \in \widehat{B}(G) | \varphi_U(X) \equiv 0 \bmod p\}$$

where p is the characteristic of the quotient ring.

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$$\begin{aligned} T \times X &= \varphi_U(X)T + \sum_{K \lesssim U} m_H G/K \\ &\equiv \varphi_U(X)T \bmod \mathcal{P}. \end{aligned}$$

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As discussed the minimal element argument is nearly identical to the finite case. The remainder of the argument is as follows. Let $T = G/U$ be the minimal element as above, $X \in \widehat{B}(G)$

$$T \times X = \varphi_U(X)T + \sum_{K \lesssim U} m_H G/K$$
$$\equiv \varphi_U(X)T \bmod \mathcal{P}.$$

$$X \in \mathcal{P} \Leftrightarrow \varphi_U(X) \equiv 0 \bmod \mathrm{char}(\widehat{B}(G)/\mathcal{P})$$
$$\Leftrightarrow X \in \{Y | \varphi_U(X) \equiv 0 \bmod p\}.$$

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If $\mathcal{P}$ is a prime ideal such that there exists an infinite chain of the form $\{G/H | H \leq_O G, G/H \notin \mathcal{P}\}$ then this set has no minimum.

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Definition

If $\mathcal{P}$ is a prime ideal of $\widehat{B}(G)$ for some profinite group G, then we call $\mathcal{P}$ large if there is a minimal $G/H \notin \mathcal{P}$ and small otherwise.

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$$\begin{aligned} G/H \times G/K &= \sum_{g \in [H \backslash G/K]} G/(H \cap {}^g K) \\ &= \sum_{g \in [H \backslash G/K]} G/(H \cap K) \\ &= |[H \backslash G/K]| G/(H \cap K) \end{aligned}$$

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This proves the second statement and the first follows from a simple contradiction.

Example

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$$\varphi_H(G/K) = \begin{cases} |G : K| & \textit{if } H \leq K \\ 0 & \textit{otherwise.} \end{cases}$$

$$\Rightarrow \mathcal{P}_{H,0} = \operatorname{span}_{\mathbb{Z}}\{G/K | H \nleq K\} + \{X | \varphi_H(X) = 0,\ X = \sum_{H \leq K} G/K\ X_K\}$$

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$$\mathcal{P}_{U,p} = \mathrm{span}_{\mathbb{Z}}\{G/K | K <_O G\} + p\mathbb{Z}G/G.$$

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In fact we can increase the strength of this to any pro-p group that has no self normalizing subgroups since

$$|N_G(K) : K| \text{ divides } \varphi_U(G/K)$$

$\Rightarrow p \nmid \varphi_U(G/K) \Rightarrow N_G(K) = K$. This observation allows us to extend to any pro-p group.

Theorem (H.)

The above theorem holds for any pro-p group.

As for the small prime ideals, whether they exist or not is unknown, but the following results are advancing towards researching this.

Lemma

For $\mathcal{P}$ a small prime ideal, then there does not exist $N \trianglelefteq_O G$ such that ker(Fix_N^G)$\subseteq \mathcal{P}$.

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An additional result in the finite case is that a group G is solvable if and only if the spectrum is connected which is if and only if 0 and 1 are the only idempotents in $\widehat{B}(G)$. A similar result may be possible in the infinite case.