Generalized biset functors from a category of fusion systems

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In all the presentation, p denotes a fixed prime.

Definition

Let P be a finite p-group. A *fusion system over* P is a category $\mathcal{F}$ whose objects are the subgroups of P and whose morphism sets $\mathcal{F}(P_1, P_2)$, for $P_1, P_2 \leq P$, satisfy the following axioms:

(i) $\mathrm{Hom}_P(P_1, P_2) \subseteq \mathcal{F}(P_1, P_2) \subseteq \mathrm{Inj}(P_1, P_2)$

(ii) If $\varphi : P_1 \to P_2$ is a morphism in $\mathcal{F}$, then $\varphi : P_1 \to \varphi(P_1)$ and $\varphi^{-1} : \varphi(P_1) \to P_1$ are morphisms in $\mathcal{F}$.

Notation: We will display the underlying p-group in subscript: "Let $\mathcal{F}_P$ be a fusion system" means that we take a fusion system $\mathcal{F}_P$ over a finite p-group P.

Examples:

(1) The *inner fusion system* of P, $\mathcal{F}_P(P)$, whose morphisms are restrictions of elements in $\mathrm{Inn}(P)$.

(2) More generally, whenever P is a subgroup of a finite group G, we can define the *realizable* fusion system $\mathcal{F}_P(G)$, whose morphisms come from restrictions of elements in $\mathrm{Inn}(G)$.

(3) *Saturated fusion systems*: the most important class of fusion systems, which satisfy some extra axioms so that saturated fusion systems are fusion systems which "behave" like $\mathcal{F}_P(G)$ when $P \in \mathrm{Syl}_p(G)$.

Definition

Let G, H be finite groups. The *Burnside module of* G *and* H, denoted by $B(G, H)$, is defined to be the Grothendieck group of the set of isomorphism classes of finite (G, H)-bisets, with the additive law $+$ induced by disjoint union of bisets.

Composition law:

Given a (G, H)-biset X and an (H, K)-biset Y, one can consider the (G, K)-biset

$$X \circ Y := (X \times Y)/\sim$$

where $\sim$ is the equivalence relation on $X \times Y$ defined by $(x \cdot h, y) \sim (x, h \cdot y)$ for all $h \in H$.

This operation is distributive over $+$ and extends to a map:

$$\circ : B(G, H) \times B(H, K) \to B(G, K)$$

Let $\mathcal{F}_P$, $\mathcal{F}_Q$ be fusion systems.

Definition

An element $X \in B(P,Q)$ is said to be $(\mathcal{F}_P, \mathcal{F}_Q)$-*stable* if it is both:

- right $\mathcal{F}_Q$-stable, i.e.

$$X \circ [Q', \psi]_{Q'}^{Q} = X \circ [Q', \mathrm{incl}]_{Q'}^{Q} \qquad \forall Q' \leq Q,\ \forall \psi \in \mathcal{F}_Q(Q', Q)$$

  where $[Q', \psi]_{Q'}^{Q}$ is denotes the transitive biset associated to

$$\Delta(Q', \psi) := \{(\psi(x), x) | x \in Q'\} \ \leq Q \times Q' \ ;$$

- left $\mathcal{F}_P$-stable, i.e.

$$[\varphi(P'), \varphi^{-1}]_P^{P'} \circ X = [P', \mathrm{Id}]_P^{P'} \circ X \qquad \forall P' \leq P,\ \forall \varphi \in \mathcal{F}_P(P', P).$$

The $(\mathcal{F}_P, \mathcal{F}_Q)$-stable elements of $B(P,Q)$ form a submodule, which is called the *Burnside module of $\mathcal{F}_P$ and $\mathcal{F}_Q$* and denoted by $B(\mathcal{F}_P, \mathcal{F}_Q)$.

Finite groups, together with Burnside modules $B(G,H)$ as sets of homomorphisms, form a preadditive category, which is called the *biset category for finite groups*.

Question

Do fusion systems over finite p-groups, together with Burnside modules $B(\mathcal{F}_P, \mathcal{F}_Q)$, form a category? i.e. is there an identity morphism in each $B(\mathcal{F}_P, \mathcal{F}_P)$?

➙ When $\mathcal{F}_P := \mathcal{F}_P(P)$ is the inner fusion system, the multiplicative identity in $B(P,P)$ provides such a morphism.

➙ When $\mathcal{F}_P$ is saturated, the characteristic idempotent $\omega_{\mathcal{F}_P}$ in $B(\mathcal{F}_P, \mathcal{F}_P)_{(p)}$ gives an identity morphism, at the cost of an extension of scalars from $\mathbb{Z}$ to $\mathbb{Z}_{(p)}$.

From now on we will allow ourselves more scalars, by considering $RB(\mathcal{F}_P, \mathcal{F}_Q) := R \otimes_{\mathbb{Z}} B(\mathcal{F}_P, \mathcal{F}_Q)$ for any commutative ring R of characteristic 0.

Definition

Let $\mathcal{F}_P$ be a fusion system. Let $\omega \in RB(P, P)$ be an idempotent. We say that ω is an *R-characteristic idempotent* for $\mathcal{F}_P$ if it satisfies the following conditions:

(i) ω is *$\mathcal{F}_P$-generated*, i.e. it is an R-linear combination of transitive bisets of the form $[P', \varphi]_P^P$ for $P' \leq P$ and $\varphi \in \mathcal{F}_P(P', P)$;

(ii) ω is $(\mathcal{F}_P, \mathcal{F}_P)$-stable, i.e. $\omega \in RB(\mathcal{F}_P, \mathcal{F}_P)$;

(iii) $\varepsilon(\omega) \in R^{\times}$ where $\varepsilon : RB(P, P) \to R$ is the R-algebra homomorphism defined on bisets by $X \mapsto \frac{|X|}{|P|}$.

Theorem (Ragnarsson 2006, or Reeh 2016)

Every saturated fusion system on a p-group has a unique $\mathbb{Z}_{(p)}$-characteristic idempotent.

Theorem (Ragnarsson & Stancu 2013)

Any fusion system on a p-group having a $\mathbb{Z}_{(p)}$-characteristic idempotent is saturated.

Theorem ($\sim$ Boltje & Danz 2011)

Every fusion system $\mathcal{F}_P$ has a $\mathbb{Q}$-characteristic idempotent $\omega_{\mathcal{F}_P}$ which moreover satisfies

$$X \circ \omega_{\mathcal{F}_P} = X \qquad \text{and} \qquad \omega_{\mathcal{F}_P} \circ Y = Y$$

for each $X \in \mathbb{Q}B(\mathcal{F}_Q, \mathcal{F}_P)$ and $Y \in \mathbb{Q}B(\mathcal{F}_P, \mathcal{F}_Q)$.

From now on, k will be a fixed commutative ring containing $\mathbb{Q}$, so that the following definition makes sense.

Definition

We define the (k-linear) *biset category for fusion systems* $\mathfrak{Fus}$ by taking all fusion systems on finite p-groups as objects, and Burnside modules for fusion systems $kB(\mathcal{F}_P, \mathcal{F}_Q)$ as sets of homomorphisms.

➙ The category $\mathfrak{Fus}$ contains an isomorphic copy of the biset category for finite p-groups, as the full subcategory whose objets are inner fusion systems, denoted by $\mathfrak{Inn}$.

Recall the following definition (from S. Bouc):

Definition

Let R be a commutative ring with unit. A *biset functor* with values in R-Mod is an R-linear functor from the R-linear biset category for finite groups to R-Mod.

This motivates us to introduce "fusion" biset functors:

Definition

A *fusion biset functor* is a k-linear functor from $\mathfrak{Fus}$ to k-Mod.

➙ Together with natural transformations of functors, fusion biset functors form a category which we denote by $[\mathfrak{Fus}, k\text{-Mod}]$.

Remark

When restricted to $\mathfrak{Inn}$, a fusion biset functor is exactly a p-biset functor (with values in k-Mod), i.e. a classical biset functor defined on p-groups.

Notation: The category of k-linear p-biset functors together with natural transformations will be denoted by $[\mathfrak{Inn}, k\text{-Mod}]$.

➙ In fact there is a deeper connection between fusion biset functors and p-biset functors, coming from the fact that for any fusion system $\mathcal{F}_P$ we have:

$$\omega_{\mathcal{F}_P} \circ kB(P,P) \circ \omega_{\mathcal{F}_P} = kB(\mathcal{F}_P, \mathcal{F}_P)$$

Lemma

Let F be a fusion biset functor and $\mathcal{F}_P$ a fusion system. Then $F(\mathcal{F}_P)$ is a $kB(\mathcal{F}_P, \mathcal{F}_P)$-module isomorphic to

$$\omega_{\mathcal{F}_P} \cdot F(P) := \operatorname{Im}\left(F(\omega_{\mathcal{F}_P} : P \to P)\right)$$

via

$$F(\mathcal{F}_P) \underset{F(\omega_{\mathcal{F}_P} : P \to \mathcal{F}_P)}{\overset{F(\omega_{\mathcal{F}_P} : \mathcal{F}_P \to P)}{\rightleftarrows}} \omega_{\mathcal{F}_P} \cdot F(P)$$

Proof: Just apply F to the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\omega_{\mathcal{F}_P}} & \mathcal{F}_P \\ {\scriptstyle \omega_{\mathcal{F}_P}}\downarrow & \overset{\omega_{\mathcal{F}_P}}{\swarrow} & \downarrow{\scriptstyle \mathrm{Id}_{\mathcal{F}_P}} \\ P & \xrightarrow[\omega_{\mathcal{F}_P}]{} & \mathcal{F}_P \end{array}$$

□

These isomorphisms give rise to an equivalence of categories between $[\mathfrak{Fus}, k\text{-Mod}]$ and $[\mathfrak{Inn}, k\text{-Mod}]$.

Proposition

The restriction functor

$$\begin{array}{ccc} [\mathfrak{Fus}, k\text{-Mod}] & \longrightarrow & [\mathfrak{Inn}, k\text{-Mod}] \\ F & \longmapsto & F_{|\mathfrak{Inn}} \end{array}$$

together with the functor $[\mathfrak{Inn}, k\text{-Mod}] \to [\mathfrak{Fus}, k\text{-Mod}]$ *sending* F *on*

$$\begin{array}{cccccc} \overline{F} & : & \mathcal{F}_P & \longmapsto & \omega_{\mathcal{F}_P} \cdot F(P) \\ & & (X : \mathcal{F}_P \to \mathcal{F}_Q) & \longmapsto & F(X : P \to Q)_{|\omega_{\mathcal{F}_P} \cdot F(P)} \end{array}$$

define an equivalence of categories.

Let $\mathcal{C}$ be a k-linear category. We denote by $[\mathcal{C}, k\text{-Mod}]$ the category of k-linear functors from $\mathcal{C}$ to k-Mod.

Definition

A *simple* functor in $[\mathcal{C}, k\text{-Mod}]$ is a non-zero functor whose only subfunctors are itself and the zero functor.

Let Z be an object in $\mathcal{C}$. We start with an $\mathrm{End}_{\mathcal{C}}(Z)$-module V, and we consider the following functor :

$$L_{Z,V} := \mathrm{Hom}_{\mathcal{C}}(Z, -) \otimes_{\mathrm{End}_{\mathcal{C}}(Z)} V$$

It is a k-linear functor which satisfies $L_{Z,V}(Z) \simeq V$.

Proposition (adjunction)

The functor $\mathrm{End}_{\mathcal{C}}(Z)\text{-Mod} \to [\mathcal{C}, k\text{-Mod}]$ *defined by* $V \mapsto L_{Z,V}$ *is left adjoint of the evaluation functor* $\mathrm{ev}_Z : [\mathcal{C}, k\text{-Mod}] \to \mathrm{End}_{\mathcal{C}}(Z)\text{-Mod}$.

Proposition (existence)

Let V be a simple $\mathrm{End}_{\mathcal{C}}(Z)$-module. Then $L_{Z,V}$ has a unique maximal subfunctor $J_{Z,V}$, and the quotient $S_{Z,V} := L_{Z,V}/J_{Z,V}$ is a simple k-linear functor satisfying $S_{Z,V}(Z) \simeq V$.

Proposition (uniqueness)

If $F : \mathcal{C} \to k\text{-Mod}$ is a simple k-linear functor and if Z is an object of $\mathcal{C}$ such that $V := F(Z)$ is non-zero, then $F \cong S_{Z,V}$.

Now we know how the simple functors in $[\mathfrak{Inn}, k\text{-Mod}]$ and $[\mathfrak{Fus}, k\text{-Mod}]$ look like. The latter are of the form $S_{\mathcal{F}_P,V}$ for a fusion system $\mathcal{F}_P$ and a simple $kB(\mathcal{F}_P,\mathcal{F}_P)$-module V.

In fact, as $S_{\mathcal{F}_P,V}(P) =: W$ is non-zero, the uniqueness ensures that $S_{\mathcal{F}_P,V}$ is isomorphic to $S_{\mathcal{F}_P(P),W}$.

Corollary

The equivalence of categories between $[\mathfrak{Fus}, k\text{-Mod}]$ and $[\mathfrak{Inn}, k\text{-Mod}]$ takes simple functors to simple functors.

Corollary

Let $\mathcal{F}_P$ be a fusion system and V be a $kB(\mathcal{F}_P,\mathcal{F}_P)$-module. Then V is simple if and only if there exists a simple $kB(P,P)$-module W such that $V \simeq \omega_{\mathcal{F}_P} \cdot W$.

We have seen what can be said about simple fusion biset functors using general theory and little effort.

Question

Where can we go from here?

Some ideas:

➙ Follow the study of (p-)biset functors that has been performed by researchers such as Serge Bouc and his collaborators. For example, the study of evaluations of simple fusion biset functors provides information on the simple $kB(\mathcal{F}_P, \mathcal{F}_P)$-modules.

➙ Have a closer look on idempotents in $kB(P, P)$. In fact, we can generalize most of the above replacing $\mathfrak{Fus}$ by the idempotent completion of the biset category for p-groups.

Thank you for your attention!