A Cartan-Eilenberg stable elements formula for cohomological Mackey 2-Functors

Burnside and Mackey functors revisited

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- For a Mackey 2-functor $\mathbb{M}$,

$$\mathbb{M}(G) \simeq \operatorname*{bilim}_{P \in \mathcal{O}_S(G)^{\mathsf{op}}} \mathbb{M}(P).$$

  $\mathbb{M}(G), \mathbb{M}(P)$ are *additive categories*.

Cohomological Mackey 2-functors

1. The classical Cartan-Eilenberg formula
2. The Cartan-Eilenberg formula for Mackey 2-functors
3. p-monadic Mackey 2-functors and 2-sheaves

Global Mackey functors

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4. for any inclusions of groups $i\colon H \to G$ and $j\colon K \to G$:

$$M^*(j)M_*(i) = \sum_{KgH \in K\backslash G/H} M_*(i_{K\cap gHg^{-1}})M_*(c_g)M^*(j_{g^{-1}Kg\cap H})$$

where

- $j_{g^{-1}Kg\cap H}\colon g^{-1}Kg \cap H \to H$ and $i_{K\cap gHg^{-1}}\colon K \cap gHg^{-1} \to K$ are the natural inclusions
- $c_g\colon g^{-1}Kg \cap H \to K \cap gHg^{-1}$ is the conjugation by g

Global Mackey functors

Notation

- By a slight abuse of notation, the contravariant part of a Mackey functor M will also be denoted by M.
- The image of morphisms by the contravariant and covariant parts of a Mackey functor M are respectively noted

$$i^* = M^*(i) \text{ and } i_* = M_*(i)$$

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Example

The following functors are global cohomological Mackey functors:

- $H^*(-, \mathbb{Z})$, the (usual) group cohomology
- $H_*(-, \mathbb{Z})$, the (usual) group homology
- $\hat{H}^*(-, \mathbb{Z})$, the Tate cohomology

The Cartan-Eilenberg stable elements formula

From now on, we fix a prime p.

Theorem (Cartan-Eilenberg, 1956)

Let $M\colon \mathrm{gp}^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-Mod}$ be a global cohomological Mackey functor taking values in $\mathbb{Z}_{(p)}$-modules. Then for any group G and p-Sylow subgroup S of G, there is a canonical isomorphism

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Proof.

There is an explicit description of the limit:

$$\lim_{P \in \mathcal{F}_S(G)^{\mathrm{op}}} M(P) = \{x \in M(S) \mid \forall H, K \subset S, g \in G \text{ s.t. } gHg^{-1} \subset K, (c_g)_*(x_{|K}) = x_{|H}\}$$

□

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The canonical morphism $M(G) \to \lim_{P \in \mathcal{F}_S(G)^{\mathrm{op}}} M(P)$ is induced by the restriction i^* along the inclusion $i\colon S \to G$.

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$$\begin{aligned} i^* i_*(x) &= \sum_{g \in S \backslash G / S} (i_{S \cap gSg^{-1}})_* (c_g)_* i^*_{S \cap g^{-1}Sg}(x) \\ &= \sum_{g \in S \backslash G / S} (i_{S \cap gSg^{-1}})_* i^*_{S \cap gSg^{-1}}(x) \\ &= \sum_{g \in S \backslash G / S} [S : S \cap gSg^{-1}] x = [G : S] x \end{aligned}$$

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and for any $y \in M(G)$,

$$i_* i^*(x) = [G : S]x$$

Since $[G : S]$ is prime to p, the morphism induced by i^* is an isomorphism. □

Mackey 2-functors

Definition (Balmer-Dell'Ambrogio, 2020)

A Mackey 2-functor $\mathbb{M}$ is a contravariant 2-functor

$$\mathbb{M}\colon \mathbf{gpd}^{\mathrm{op}} \to \mathbf{Add}$$

from the 2-category of finite groupoids to the 2-category of additive categories, satisfying the following four axioms:

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   is an equivalence.
2. **(Adjoints)** For any faithful morphism of groupoids $i\colon H \to G$, the image $i^* = \mathbb{M}(i)$ has a left adjoint $i_!$ and a right adjoint i_*

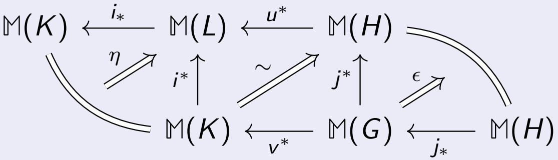
   $$i_! \dashv i^* \dashv i_*$$

Definition

3. **(Beck-Chevalley property)** For any bipullback square of groupoids

$$\begin{array}{ccc} L & \xrightarrow{u} & H \\ {\scriptstyle i}\downarrow & \overset{\sim}{\nearrow} & \downarrow{\scriptstyle j} \\ K & \xrightarrow[v]{} & G \end{array}$$

the following pasting diagram define an isomorphism $v^* j_* \simeq i_* u^*$



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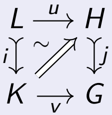

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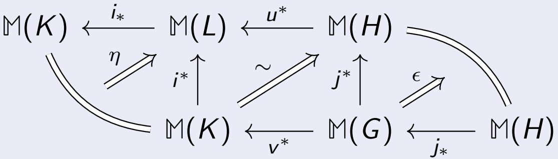

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4. **(Ambidexterity)** For any faithful morphism of groupoids $i \colon H \to G$,

$$i_! \simeq i^*$$

Cohomological Mackey 2-functors

Definition (Balmer-Dell'Ambrogio 2021)

A (rectified) Mackey 2-functor $\mathbb{M}$ is cohomological if for any inclusion of groups (= connected groupoids) $i \colon H \to G$,

$$\mathrm{Id}_{\mathbb{M}(G)} \overset{\eta}{\Longrightarrow} i_! i^* \overset{\mathrm{Id}}{\Longrightarrow} i_* i^* \overset{\epsilon}{\Rightarrow} \mathrm{Id}_{\mathbb{M}(G)} = [G \colon H] \, \mathrm{Id}_{\mathrm{Id}_{\mathbb{M}(G)}} .$$

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Example

The following mappings define cohomological Mackey 2-functors $\mathbb{M}$:

- $\mathbb{M}(G) = \mathrm{mod}(\Bbbk G)$, the category of modules on G.
- $\mathbb{M}(G) = \mathrm{D}(\Bbbk G)$, the derived category of modules on G.
- $\mathbb{M}(G) = \mathrm{coMack}(G)$, the category of G-local cohomological Mackey functors.

p-Monadic Mackey 2-functors

Definition

A Mackey 2-functor is p-monadic if for any inclusion of groups $i\colon H \to G$ of index prime to p, the adjunction $i_! \dashv i^*$ is monadic, that is, the canonical morphism

$$\mathbb{M}(G) \to \mathbb{M}(H)^{i^* i_!}$$

to the category of Eilenberg-Moore of the monad $i^* i_!$ is an equivalence.

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Proposition (Balmer-Dell'Ambrogio 2021)

Let $\mathbb{M}$ be a cohomological Mackey 2-functor taking values in $\mathbb{Z}_{(p)}$-linear and idempotent-complete categories. Then $\mathbb{M}$ is p-monadic.

Cartan-Eilenberg formula for Mackey 2-functors

Theorem (M., 2021)

Let $\mathbb{M}$ be a p-monadic Mackey 2-functor. Then for any group G with p-Sylow S, there is a canonical equivalence

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- By p-monadicity, there is an equivalence $\mathbb{M}(G) \simeq \mathbb{M}(S)^{i^* i_!}$, where $i \colon S \to G$ is the canonical inclusion.

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- By a 2-finality argument, the descent diagram can be replaced by the orbit category $\mathcal{O}_S(G)$ in the bilimit.

□

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Answer: p-monadic Mackey 2-functors are 2-sheaves, they satisfy a gluing condition with respect to the inclusion maps of order prime to p.

2-Sheaves on finite groupoids

Proposition

The 2-category **gpd** *of finite groupoids is endowed with a 2-topology of Grothendieck, the p-local topology:*

- *A covering morphism of a group G is a morphism $i\colon H \to G$ of index prime to p.*
- *These morphisms satisfy a stability property (slightly weaker than stability by bipullbacks)*

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Proposition

The restriction from the 2-category of 2-sheaves on finite groupoids with the p-local topology to the 2-category of 2-sheaves on finite p-groupoids (with the induced topology).

$$\mathbf{Sh}(\mathbf{gpd}) \to \mathbf{Sh}(p\text{-}\mathbf{gpd})$$

is a biequivalence.

Mackey 2-functors as 2-sheaves

Theorem

Let $\mathbb{M}\colon \mathbf{gpd}^{\mathrm{op}} \to \mathbf{Add}$ *be a 2-functor whose restriction to p-groupoids is a Mackey 2-functor. Then the following statements are equivalent:*

- $\mathbb{M}$ *is a p-monadic Mackey 2-functor.*
- $\mathbb{M}$ *is a 2-sheaf for the p-local topology on the 2-category of groupoids.*

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Corollary

The restriction

$$\mathrm{2Mack}_p(\mathbf{gpd}) \to \mathrm{2Mack}(p\text{-}\mathbf{gpd})$$

of p-monadic Mackey 2-functor on groupoids to p-groupoids is a biequivalence of 2-categories.

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  Yes, by the biequivalence.
- Are all p-monadic Mackey 2-functors cohomological ?
  No: any non-cohomological Mackey 2-functor on p-groupoids induce a p-monadic non-cohomological Mackey 2-functor.

Conclusion

- Notions and results from the theory of Mackey functors may be categorified to apply to Mackey 2-functors (in this presentation the notion of "cohomological" and the Cartan-Eilenberg formula).
- A p-monadic Mackey 2-functor is entirely determined by its restriction to p-groupoids.
- Looking at 2-sheaves over other classes of groups (for instance, profinite groups) may be interesting.