Mackey functors on saturated fusion systems

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Index

1. Mackey functors
2. Action of the Burnside ring

Mackey functors

Mackey functor (Dress definition).

Definition

A Mackey functor for a finite group G over a ring $\mathcal{R}$ is a pair $M = (M_*, M^*)$ satisfying the following 3 conditions:

1. **Functoriality:** M_* (M^*) is a (contravariant) functor from G-set to $\mathcal{R}$-mod and $M_*(X) = M^*(X) = M(X)$ for every finite G-set X.
2. **Pull-back commutativity:** For every pull-back diagram of finite G-sets

$$\begin{array}{ccc} \Omega & \xrightarrow{\alpha} & Y \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow[\delta]{} & Z \end{array}$$

   we have $M_*(\gamma) M^*(\alpha) = M^*(\delta) M_*(\beta)$.

3. **Additivity:** Given finite G-sets X and Y then applying M_* to the natural inclusions $X \to X \sqcup Y \leftarrow Y$ leads to an isomorphism $M(X) \oplus M(Y) \cong M(X \sqcup Y)$.

Examples

Examples

- The Burnside Mackey functor that associates to each subgroup its corresponding burnside ring.
- The group cohomology (homology) Mackey functor that associates to every subgroup its n-th cohomology (homology) group over a fixed $\mathbb{Z}G$-module.
- The Mackey algebra.

Difficulties of extending to fusion systems.

Problem

Since Fusion systems of the form $\mathcal{F}_S(G)$ do not "see" elements of G but just the morphisms they induce by conjugation a problem arises when trying to extend the Mackey functors on groups to Mackey functors on fusion systems.
For instance given a Mackey functor M on a group G, a subgroup $H \leq G$ and element $x, y \in G$ such that $x = yz$ for some $z \in C_G(H)$ then we have that the G-set morphisms $c_x, c_y : G/H \to G/{}^xH$ are different and, therefore, we could have $M_*(c_x), M_*(c_y) : M(G/H) \to M(G/{}^xH)$ with $M_*(c_x) \neq M_*(c_y)$.

Conjugation invariant Mackey algebra.

Fact

The $\mathbb{Z}$-module formed by isomorphism classes of (H, K)-bisets spanned (via disjoint union) by bisets of the form $[J, c_x]_H^K$ and with $H, K \leq S$ ranging over all subgroups of S forms a ring with product given by composition of bisets (product of non composable bisets is 0).

Definition

We call such ring the **conjugation invariant Mackey algebra** of G over $\mathbb{Z}$ and denote it by $\boldsymbol{\mu}_{\mathbb{Z}}^{c}(\mathrm{G})$. Given a commutative ring $\mathcal{R}$ we also define

$$\boldsymbol{\mu}_{\boldsymbol{\mathcal{R}}}^{c}(\mathrm{G}) := \mathcal{R} \otimes_{\mathbb{Z}} \mu_{\mathbb{Z}}^{c}(G).$$

Conjugation invariant Mackey functors.

Definition

A **conjugation invariant Mackey functor** for a group G over a ring $\mathcal{R}$ is a left $\mu^c_{\mathcal{R}}(G)$-module.

Remark

The category of conjugation invariant Mackey functors is embedded in the category of Mackey functors via the association $\Psi : \mu^c_{\mathcal{R}}(G)\text{-mod} \to \mathrm{Mack}_{\mathcal{R}}(G)$ satisfying

$$\Psi(M)(G/H) = [H, \mathrm{Id}]_H^H N,$$
$$\Psi(M)_*(\iota c_x) = [H, c_x]_H^K : M(S/H) \to M(S/K),$$
$$\Psi(M)^*(\iota c_x) = [{}^xH, c_{x^{-1}}]_K^H : M(S/K) \to M(S/H).$$

for every $H, K \leq S$ and $x \in S$ with $H \leq K^x$.

Mackey algebra of a fusion system.

Definition

Given a p-group S and a fusion system $\mathcal{F}$ on S (not necessarily saturated) we define $\mu_{\mathbb{Z}}(\mathcal{F})$ the **Mackey algebra of $\mathcal{F}$ on $\mathbb{Z}$** as the ring (with composition) having as elements isomorphism classes of (H, K)-bisets spanned (via disjoint union) by bisets of the form $[J, \varphi]_H^K$ with $H, K \leq S$ ranging over all subgroups of S, with $J \leq H$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(J, K)$. As before we define for any commutative ring $\mathcal{R}$ the Mackey algebra

$$\mu_{\mathcal{R}}(\mathcal{F}) := \mathcal{R} \otimes_{\mathbb{Z}} \mu_{\mathbb{Z}}(\mathcal{F})$$

Mackey functor on a fusion system.

Definition

A **Mackey functor for $\mathcal{F}$** on $\mathcal{R}$ is a left $\mu_{\mathcal{R}}(\mathcal{F})$-module.

Remark

Notice that for every $\mathcal{F} \subseteq \mathcal{G}$ we have that $\mu_{\mathcal{R}}(\mathcal{F}) \subseteq \mu_{\mathcal{R}}(\mathcal{G})$. Moreover we also have the identity $\mu_{\mathcal{R}}^{c}(S) = \mu_{\mathcal{R}}(\mathcal{F})$ whenever $\mathcal{F} = \mathcal{F}_S(S)$ (i.e. The conjugation invariant Mackey functors are exactly the Mackey functors over $\mathcal{F}_S(S)$).

Examples.

Examples

- Any global Mackey functor (for example $H^n(-, k)$) can be restricted to a Mackey functor over a fusion system $\mathcal{F}$.
- Any conjugation invariant Mackey functor on S can be seen as a Mackey functor on $\mathcal{F}_S(S)$.
- The Burnside Mackey functor that sends every $H \leq S$ to $[H, \mathrm{Id}]_H^H \mu_{\mathcal{R}}(\mathcal{F}) [S, \mathrm{Id}]_S^S$.

Some properties of Mackey functors on fusion systems.

Proposition

There is a 1-1 correspondence between pairs (H, V) with $H \leq S$ (taken up to $\mathcal{F}$-isomorphism) and V a simple $\mathcal{R}Out_{\mathcal{F}}(H)$-module (taken up to isomorphism) and isomorphism classes of simple Mackey functors.

Proposition

Given a fusion system $\mathcal{F}$ over a p-group S and a field k such that p is invertible in k and $|Out_{\mathcal{F}}(H)|$ is invertible in k for every $H \leq S$ then $\mu_k(\mathcal{F})$ is semisimple.

Proposition

Given fusion systems $\mathcal{F} \subseteq \mathcal{G}$ an induction and restriction between Mackey functors on $\mathcal{F}$ and on $\mathcal{G}$ can be defined via usual change of rings for modules. induction is always left adjoint to restriction. Moreover if we restrict to $\mathcal{G}$-centrics and impose $\mathcal{F} = \mathcal{F}_S(S)$ then induction is also right adjoint.

Action of the Burnside ring

Burnside ring.

Definition

Given a finite group S we define $\mathrm{B}(\mathrm{G})$ the **Burnside ring of** G as the Grothendieck group of the monoid having as elements isomorphism classes of finite G-sets. Addition here is given by disjoint union and multiplication by cartesian product.
In addition, given a commutative ring $\mathcal{R}$ we define

$$\mathrm{B}(\mathrm{G}, \boldsymbol{\mathcal{R}}) := \mathcal{R} \otimes_{\mathbb{Z}} B(G).$$

Burnside ring.

Fact

The Burnside ring $B(G,\mathcal{R})$ can be embedded in the center of $\mu^c_{\mathcal{R}}(G)$ through the linear mapping

$$[G/H] \to \sum_{K\leq G} \sum_{x\in[K\backslash G/H]} [K \cap^x H, Id]^K_K$$

Remark

Another way of describing that map is saying that a G-set Ω acts on the conjugation invariant Mackey functor M by applying to every $M(H)$ the biset

$$\Omega^H \circlearrowright = [H, \mathrm{Id}]^H_S \circ (\Omega \times \{e\}) \circ [\{e\}, \mathrm{Id}]^{\{e\}}_H .$$

Where $e \in G$ denotes the identity element.

$\mathcal{F}$-stable sets.

Definition

An S-set Ω is called **$\mathcal{F}$-stable** if and only if exists an $\mathcal{F}$-stable (S, S)-biset Ψ such that $\Omega = \Psi / \sim$ where $\sim$ is the equivalence class that relates right S-orbits.

Burnside ring for fusion systems.

Definition

When $\mathcal{F}$ is saturated Sune Reeh defines $\mathrm{B}(\mathcal{F})$ the **Burnside ring of $\mathcal{F}$** as the subring of $B(S, \mathbb{Z}_p)$ formed by the isomorphism classes of $\mathcal{F}$-stable S-sets. Given $\mathcal{R}$ a commutative ring in which p is invertible we define

$$\mathrm{B}(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{R}}) := \mathcal{R} \otimes_{\mathbb{Z}} B(\mathcal{F}).$$

Remark

There is in fact no need for p to be invertible in $\mathcal{R}$ but if it wasn't then $B(\mathcal{F}, \mathcal{R})$ wouldn't have a unit.

Action of Burnside ring.

Proposition

$B(\mathcal{F}, \mathcal{R})$ can be embedded in the center of $\mu_{\mathcal{R}}(\mathcal{F})$. In particular there is an action of $B(\mathcal{F}, \mathcal{R})$ on every Mackey functor

Action of Burnside ring.

Proof.

For every $\Omega \in B(\mathcal{F}, \mathcal{R})$ and every Mackey functor M on $\mathcal{F}$ we define the action of Ω on M as the action of Ω seen as an element of $B(S, \mathcal{R})$ on M seen as a conjugation invariant Mackey functor.
Since we already know that the action of Ω commutes with the action of bisets of the form $[H, \mathrm{Id}]_H^K$ and $[H, \mathrm{Id}]_K^H$ and that this action leads to an injective ring morphism onto then we only need to prove that the action of Ω commutes with the action of bisets of the form $[H, \varphi]_H^{\varphi(H)}$.
In other words we need to prove that the identity

$$\Omega^{\varphi(H)} \circlearrowright \circ [H, \varphi]_H^{\varphi(H)} = [H, \varphi]_H^{\varphi(H)} \circ \Omega^H \circlearrowright .$$

□

Action of Burnside ring.

Proof.

Through simple computation we have that

$$[H,\varphi]_H^{\varphi(H)} \circ [H,\mathrm{Id}]_S^{\varphi(H)} = [H,\varphi]_S^{\varphi(H)}\,,$$
$$[\{e\}\,,\mathrm{Id}]_{\varphi(H)}^{\{e\}} \circ [H,\varphi]_H^{\varphi(H)} = [\{e\}\,,\mathrm{Id}]_H^{\{e\}}\,,$$

and since Ω is $\mathcal{F}$-stable then

$$[\varphi(H)\,,\mathrm{Id}]_S^{\varphi(H)} \circ (\Omega \times \{e\}) = [H,\varphi]_S^{\varphi(H)} \circ (\Omega \times \{e\})\,,$$

□

Action of Burnside ring.

Proof.

Joining these results we have that

$$\begin{aligned}
\Omega^{\varphi(H)} \circlearrowleft \circ [H,\varphi]_H^{\varphi(H)} &= \left([\varphi(H), \mathrm{Id}]_S^{\varphi(H)} \circ (\Omega \times \{e\}) \circ [\{e\}, \mathrm{Id}]_{\varphi(H)}^{\{e\}}\right) \circ \\
&\quad \circ [H,\varphi]_H^{\varphi(H)}, \\
&= [\varphi(H), \mathrm{Id}]_S^{\varphi(H)} \circ (\Omega \times \{e\}) \circ [\{e\}, \mathrm{Id}]_H^{\{e\}}, \\
&= [H,\varphi]_S^{\varphi(H)} \circ (\Omega \times \{e\}) \circ [\{e\}, \mathrm{Id}]_H^{\{e\}}, \\
&= [H,\varphi]_H^{\varphi(H)} \circ \left([H, \mathrm{Id}]_S^{\varphi(H)} \circ (\Omega \times \{e\}) \circ [\{e\}, \mathrm{Id}]_H^{\{e\}}\right), \\
&= [H,\varphi]_H^{\varphi(H)} \circ \Omega^H \circlearrowleft .
\end{aligned}$$

□

Action of Burnside ring.

Fact

A similar result can be obtained replacing Reeh's Burnside ring with the Burnside ring Proposed by Diaz and Libman. This analogous result however limits us to $\mathcal{F}$-centric subgroups of S and to centric Mackey functors (i.e. Mackey functors M such that $[H, Id]_H^H M = 0$ whenever H is not $\mathcal{F}$-centric).

Thank you for listening.

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