The Rigidity of Infinite Frameworks in Euclidean and Polyhedral Normed Spaces

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A *real normed space* is a vector space $X$ over $\mathbb{R}$ together with a map $\| \cdot \| : X \to [0, \infty)$ such that for all $x, y \in X$ and $\lambda \in \mathbb{R}$:

- $\|x\| = 0 \iff x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$. 
The Euclidean norm $\| \cdot \|_2$ on $\mathbb{R}^d$ is given by

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For a centrally symmetric polytope $\mathcal{P} \subseteq \mathbb{R}^d$ with facets $\pm F_1, \ldots, \pm F_n$ we can define the norm $\| \cdot \|_{\mathcal{P}}$ on $\mathbb{R}^d$ by

$$\|x\|_{\mathcal{P}} = \max_{1 \leq k \leq n} |\langle \hat{F}_k, x \rangle|$$

where $\hat{F} \in \mathbb{R}^d$ is the unique vector that defines the hyperspace that the face $F$ lies on.
Asimow-Roth for normed spaces

The following is a famous result from *The Rigidity of Graphs* by L. Asimow and B. Roth and an equivalent result for polyhedral normed spaces from *Finite and Infinitesimal Rigidity with Polyhedral Norms* by Derek Kitson.

**Theorem**

Let \((G, p)\) be a finite, affinely spanning and regular framework in \((\mathbb{R}^d, \| \cdot \|_2)\) or \((\mathbb{R}^d, \| \cdot \|_P)\). Then TFAE:

- \((G, p)\) is infinitesimally rigid
- \((G, p)\) is continuously rigid (all deformations are rigid motions)
- \((G, p)\) is locally rigid (all equivalent frameworks within a neighbourhood of \(p\) are congruent).

What would be an equivalent result for infinite frameworks in either space?
**** Figure: Infinitesimally rigid but continuously flexible in \((\mathbb{R}^2, \| \cdot \|_2)\). This framework is infinitesimally flexible for all generic positions.****
Frameworks

We shall always assume that $(X, \| \cdot \|)$ is a finite dimensional real normed space with an open set of smooth points.

Definition

A framework in $(X, \| \cdot \|)$ is a pair $(G, p)$ where $G$ is a simple graph (i.e. no loops, repeated edges and undirected) and $p \in X^{V(G)}$. 
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For a framework we will define the *rigidity map* to be

\[
f_G : X^{V(G)} \to \mathbb{R}^{E(G)}, \ (x_v)_{v \in V(G)} \mapsto (\|x_v - x_w\|)_{vw \in E(G)}.\]
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$$f_G : X^{V(G)} \to \mathbb{R}^{E(G)}, \quad (x_v)_{v \in V(G)} \mapsto (\|x_v - x_w\|)_{vw \in E(G)}.$$

Definition

We say an edge $vw \in E(G)$ of $(G, p)$ is *well-positioned* if $p_v - p_w$ is a smooth point and we say $(G, p)$ is *well-positioned* if all edges $(G, p)$ are well-positioned.
For a well-positioned edge $vw \in E(G)$ we define the linear functional $\varphi_{v,w} : X \rightarrow \mathbb{R}$ to be the support functional of $p_v - p_w$. 
Support functionals

For a well-positioned edge \( vw \in E(G) \) we define the linear functional \( \varphi_{v,w} : X \to \mathbb{R} \) to be the support functional of \( p_v - p_w \).

For \((\mathbb{R}^d, \| \cdot \|_2)\):

\[
\varphi_{v,w}(\cdot) = \left\langle \frac{p_v - p_w}{\|p_v - p_w\|}, \cdot \right\rangle.
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For polyhedral normed space $(\mathbb{R}^d, \| \cdot \|_P)$:

$$\varphi_{v,w}(\cdot) = \left\langle \hat{F}, \cdot \right\rangle$$

where $\| p_v - p_w \|_P = \left\langle \hat{F}, p_v - p_w \right\rangle$. 
Notation

The space of infinitesimal flexes:

\[ \mathcal{F}(G, p) = \left\{ u \in X^V(G) : \varphi_{v, w}(u_v - u_w) = 0 \text{ for all } vw \in E(G) \right\} \]
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The space of trivial flexes:

$$\mathcal{T}(p) = \left\{ (\gamma'_{p_v}(0))_{v \in V(G)} \in X^{V(G)} : \gamma \text{ is a smooth rigid body motion} \right\}$$
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Orbit of \( p \):

\[ \mathcal{O}_p := \left\{ (h(p_v))_{v \in V(G)} : h \text{ is an isometry of } (X, \| \cdot \|) \right\} \]
Equicontinuity

Let $F$ be a family of continuous curves $f : I \rightarrow X$ for some interval $I$ and some normed space $X$. We say that $F$ is equicontinuous at $t_0 \in I$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$t \in (-\delta + t_0, \delta + t_0) \Rightarrow \|f(t_0) - f(t)\| < \epsilon$$

for all $f \in F$. If $F$ is equicontinuous at all $t \in I$ then $F$ is equicontinuous.
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**Definition**

We say that a family $\alpha = (\alpha_v)_{v \in V(G)}$ of continuous paths $\alpha_v : (-1, 1) \rightarrow X$ is an *equicontinuous finite flex* of $(G, p)$ in $(X, \| \cdot \|)$ if:

- $\alpha_v(0) = p_v$ for all $v \in V(G)$
- $\|\alpha_v(t) - \alpha_w(t)\| = \|p_v - p_w\|$ for all $vw \in E(G)$ and $t \in (-1, 1)$
- $\alpha$ is equicontinuous.
For $X^V(G)$ we define the \textit{generalised metric} (i.e. a metric that allows infinite distances between points) $d_{V(G)}$ where

$$d_{V(G)}(x, y) := \sup_{v \in V(G)} \|x_v - y_v\|.$$
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We now define for all $p \in X^{V(G)}$ and $r > 0$ the open balls of $X^{V(G)}$

$$B_r(p) := \left\{ q \in X^{V(G)} : d_{V(G)}(p, q) < r \right\}.$$
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For more information on generalised metric spaces see \textit{A Course in Metric Geometry} by Dmitri Burago, Yuri Burago and Sergei Ivanov.
Rigidity for infinite frameworks

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A framework $(G, p)$ is *locally rigid* (with respect to the $d_{V(G)}$-topology on $X^V(G)$) if there exists $r > 0$ such that $f_G^{-1}[f_G(p)] \cap B_r(p) = \mathcal{O}_p \cap B_r(p)$. 
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A framework $(G, p)$ is **infinitesimally rigid** if $\mathcal{F}(G, p) = \mathcal{T}(p)$.

Definition
A framework $(G, p)$ is **locally rigid** (with respect to the $d_{\mathcal{V}(G)}$-topology on $X^\mathcal{V}(G)$) if there exists $r > 0$ such that $f_G^{-1}[f_G(p)] \cap B_r(p) = \mathcal{O}_p \cap B_r(p)$.

Definition
A framework $(G, p)$ is **equicontinuously rigid** if all equicontinuous finite flexes are rigid body motions.
Local rigidity implies equicontinuous rigidity

**Proposition**

Let \((G, p)\) be locally rigid in \((X, \| \cdot \|)\), then \((G, p)\) is equicontinuously rigid.
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**Local rigidity implies equicontinuous rigidity**

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So how does this link to infinitesimal rigidity?

![Figure](image.png)

**Figure**: Locally and equicontinuously rigid but infinitesimally and continuously flexible in \((\mathbb{R}^2, \| \cdot \|_2)\).
Bounded infinitesimal rigidity

We say that $u \in \mathcal{F}(G, p)$ is a bounded flex if

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We denote $b\mathcal{F}(G, p)$ to be the space of bounded flexes of $(G, p)$. 
We say that $u \in \mathcal{F}(G, p)$ is a \textit{bounded flex} if

$$\sup_{v \in V(G)} \|u_v\| < \infty.$$ 

We denote $b\mathcal{F}(G, p)$ to be the space of bounded flexes of $(G, p)$.

\textbf{Definition}

We say that a well-positioned framework $(G, p)$ is \textit{bounded infinitesimally rigid} if $b\mathcal{F}(G, p) \subseteq \mathcal{T}(p)$. 
**Equivalence of rigidity for Euclidean spaces**

**Theorem**

Let \((G, p)\) be an affinely spanning framework in a \(d\)-dimensional Euclidean space such that

- The points of the placement \(p\) are uniformly discrete in \(X\)
- for some \(r > 0\) we have that \(b\mathcal{F}(G, q)\) is linearly isomorphic to \(b\mathcal{F}(G, p)\) for all \(q \in B_r(p)\);

then the following are equivalent:

- \((G, p)\) is bounded infinitesimally rigid
- \((G, q)\) is locally rigid for all \(q \in B_r(p)\).
Equivalence of rigidity for Euclidean spaces

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then the following are equivalent:

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- $(G, q)$ is locally rigid for all $q \in B_r(p)$.

It is an open question whether there is any way of choosing placements such that the condition on linear isomorphisms of bounded flex spaces on an open neighbourhood is automatic.
Figure: A generic framework in \((\mathbb{R}^2, \| \cdot \|_2)\) that is infinitesimally and continuously rigid but locally flexible.
Equivalence of rigidity for polyhedral normed spaces

Definition

We say a framework \((G, p)\) is uniformly well-positioned if there exists \(r > 0\) such that \((G, q)\) is well-positioned for all \(q \in B_r(p)\).
Equivalence of rigidity for polyhedral normed spaces

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We say a framework \((G, p)\) is uniformly well-positioned if there exists \(r > 0\) such that \((G, q)\) is well-positioned for all \(q \in B_r(p)\).

Theorem

Let \((G, p)\) be a uniformly well-positioned framework in a polyhedral normed space \(\left(\mathbb{R}^d, \| \cdot \|_P \right)\) then the following are equivalent:

- \((G, p)\) is bounded infinitesimally rigid
- \((G, p)\) is locally rigid
- \((G, p)\) is equicontinuously rigid.
Equivalence of rigidity for polyhedral normed spaces

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- \((G, p)\) is bounded infinitesimally rigid
- \((G, p)\) is locally rigid
- \((G, p)\) is equicontinuously rigid.

The result is important as checking if a framework is uniformly well-positioned is much easier than checking if all frameworks in a neighbourhood of a placement are bounded infinitesimally rigid.
Special case: \((\mathbb{R}^d, \| \cdot \|_\infty)\)

The max norm \(\| \cdot \|_\infty\) on \(\mathbb{R}^d\):

\[
\|(a_1, \ldots, a_d)\|_\infty := \max_{1 \leq k \leq d} |a_k| = \max_{1 \leq k \leq d} |\langle e_k, (a_1, \ldots, a_d) \rangle|,
\]

where \(e_1, \ldots, e_d\) is the standard basis of \(\mathbb{R}^d\).
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**Theorem**

Let \((G, p)\) be a uniformly well-positioned framework in \((\mathbb{R}^d, \| \cdot \|_\infty)\) then the following are equivalent:

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- \((G, p)\) is bounded infinitesimally rigid
- \((G, p)\) is locally rigid
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Figure: (Left) Unit ball of \((\mathbb{R}^2, \| \cdot \|_\infty)\); (right) a framework in \((\mathbb{R}^2, \| \cdot \|_\infty)\) that is infinitesimally, equicontinuously and locally rigid.
Thank you for listening!
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Questions?