Combinatorics of Body-bar-hinge Frameworks

Shin-ichi Tanigawa  
based on a handbook chapter with Csaba Király  

Tokyo  

June 6, 2018
Body-bar-hinge Frameworks

body-bar framework in $\mathbb{R}^3$

body-hinge framework in $\mathbb{R}^3$

body-bar framework in $\mathbb{R}^2$

body-hinge framework in $\mathbb{R}^2$
Why interesting?

- appear in lots of real problems \(\rightarrow\) Ileana’s talk
- rigidity characterization problem can be solved in any dimension.

<table>
<thead>
<tr>
<th></th>
<th>rigidity</th>
<th>global rigidity</th>
</tr>
</thead>
<tbody>
<tr>
<td>bar-joint</td>
<td>unsolved ((d \leq 2: \text{Laman}))</td>
<td>unsolved ((d \leq 2: \text{Jackson-Jordán05}))</td>
</tr>
<tr>
<td>body-bar</td>
<td>Tay84</td>
<td>Connelly-Jordán-Whiteley13</td>
</tr>
<tr>
<td>body-hinge</td>
<td>Tay89, Tay91, Whiteley88</td>
<td>Jordán-Király-T16</td>
</tr>
</tbody>
</table>
Body-bar Frameworks

- A $d$-dimensional body-bar framework is a pair $(G, b)$:
  - $G = (V, E)$: underlying graph;
  - $b$: a bar-configuration; $E \ni e \mapsto$ a line segment in $\mathbb{R}^d$. 

![Diagram of body-bar frameworks]
Rigidity, Infinitesimal Rigidity, Global Rigidity

- An equivalent bar-joint framework to \((G, b)\):

- Local rigidity (LR), infinitesimal rigidity (IR), global rigidity (GL) are defined through an equivalent bar-joint framework.

- All the basic results for bar-joint can be transferred e.g., infinitesimal rigidity \(\Rightarrow\) rigidity
Maxwell and Tay

Maxwell’s condition

If a $d$-dimensional body-bar framework $(G, b)$ is IR, then

$$|E(G)| \geq D|V(G)| - D$$

with $D = \binom{d+1}{2}$.

for $d = 3$, $|E(G)| \geq 6|V(G)| - 6$
Maxwell and Tay

Maxwell’s condition

If a \(d\)-dimensional body-bar framework \((G, b)\) is IR, then

\[|E(G)| \geq D|V(G)| - D\]

with \(D = \binom{d+1}{2}\).

Maxwell’s condition (stronger version)

If a \(d\)-dimensional body-bar framework \((G, b)\) is IR, then \(G\) contains a spanning subgraph \(H\) satisfying

- \(|E(H)| = D|V(H)| - D\)
- \(\forall H' \subseteq H, |E(H')| \leq D|V(H')| - D\)
Maxwell and Tay

Maxwell’s condition

If a $d$-dimensional body-bar framework $(G, b)$ is IR, then

$$|E(G)| \geq D|V(G)| - D$$

with $D = \binom{d+1}{2}$.

Maxwell’s condition (stronger version)

If a $d$-dimensional body-bar framework $(G, b)$ is IR, then $G$ contains a spanning $(D, D)$-tight subgraph.

- $H$ is $(k, k)$-sparse $\iff \forall H' \subseteq H, |E(H')| \leq k|V(H')| - k$
- $H$ is $(k, k)$-tight $\iff (k, k)$-sparse $\& |E(H)| = k|V(H)| - k$
Maxwell and Tay

Maxwell’s condition

If a $d$-dimensional body-bar framework $(G, b)$ is IR, then

$$|E(G)| \geq D|V(G)| - D$$

with $D = \binom{d+1}{2}$.

Maxwell’s condition (stronger version)

If a $d$-dimensional body-bar framework $(G, b)$ is IR, then $G$ contains a spanning $(D, D)$-tight subgraph.

- $H$ is $(k, k)$-sparse $\iff \forall H' \subseteq H, |E(H')| \leq k|V(H')| - k$
- $H$ is $(k, k)$-tight $\iff (k, k)$-sparse & $|E(H)| = k|V(H)| - k$

Theorem (Tay84)

A generic $d$-dimensional body-bar framework $(G, b)$ is IR (or LR) $\iff G$ has a spanning $(D, D)$-tight subgraph.
(Better) Characterizations

**Theorem (Tutte61, Nash-Williams61, 64)**

TFAE for a graph $H$:

1. $H$ contains a spanning $(k, k)$-tight subgraph;
2. $H$ contains $k$ edge-disjoint spanning trees;
3. $e_G(\mathcal{P}) \geq k|\mathcal{P}| - k$ for any partition $\mathcal{P}$ of $V$, where $e_G(\mathcal{P})$ denotes the number of edges connecting distinct components of $\mathcal{P}$. 

![Graph 1](image1.png)

![Graph 2](image2.png)
Proof 1

Based on tree packing (Whiteley88):

```
\text{pined}
```
Proof 2

Inductive construction (Tay84):

**Theorem (Tay84)**

G is \((k, k)\)-tight if and only if G can be built up from a single vertex graph by a sequence of the following operation:

- **pinch** \(i\) \((0 \leq i \leq k - 1)\) existing edges with a new vertex \(v\), and add \(k - i\) new edges connecting \(v\) with existing vertices.

Each operation preserves rigidity.
Proof 3

Quick proof (T):

- Prove: a \((D, D)\)-sparse graph \(G\) with \(|E(G)| = D|V(G)| - D - k\) has \(k\) dof.
- Take any edge \(e = uv\);
- By induction, \((G - e, b)\) has \(k + 1\) dof.
- Try all possible bar realizations of \(e\)
- If dof does not decrease, body \(u\) and body \(v\) behave like one body
- \(\Rightarrow (G/e, b)\) has \(k + 1\) dof.
- However, \(G/e\) contains a spanning \((D, D)\)-sparse subgraph \(H\) with \(|E(H)| = D|V(H)| - D - k\), whose generic body-bar realization has \(k\) dof by induction, a contradiction.
Body-hinge Frameworks

- A $d$-dimensional body-hinge framework is a pair $(G, h)$:
  - $G = (V, E)$: underlying graph;
  - $h$: hinge-configuration; $E \ni e \mapsto$ a $(d - 2)$-dimensional segment in $\mathbb{R}^d$

- LR, IR, GR are defined by an equivalent bar-joint framework.

Body-hinge framework in $\mathbb{R}^2$
Reduction to Body-bar (Whiteley'88)

- A hinge ≈ five bars passing through a line

- Body-hinge framework \((G, h)\) ≈ body-bar framework \(((D - 1)G, b)\)
  - \(kG\): the graph obtained by replacing each edge with \(k\) parallel edges

Maxwell's condition

If a \(d\)-dimensional body-hinge framework \((G, h)\) is IR, then \(((D - 1)G, b)\) contains \(D\) edge-disjoint spanning trees.
Reduction to Body-bar (Whiteley88)

- A hinge ≈ five bars passing through a line

- Body-hinge framework \( (G, h) \) ≈ body-bar framework \( ((D - 1)G, b) \)
  - \( kG \): the graph obtained by replacing each edge with \( k \) parallel edges

Maxwell’s condition

If a \( d \)-dimensional body-hinge framework \( (G, h) \) is IR, then \( (D - 1)G \) contains \( D \) edge-disjoint spanning trees.
Theorem (Tay 89,91, Whiteley 88)

A generic $d$-dimensional body-hinge framework $(G, b)$ is LR (IR) $\iff (D - 1)G$ contains $D$ edge-disjoint spanning trees.

- Proof 1 can be applied
  - an equivalent body-bar framework is non-generic
- Body-bar-hinge frameworks (Jackson-Jordán09)
- Q. Any quick proof (without tree packing)?
Molecular Frameworks

- **square** of $G$: $G^2 = (V(G), E(G)^2)$
  - $E(G)^2 = \{uv : d_G(u, v) \leq 2\}$

![Diagram of $G$ and $G^2$]
Molecular Frameworks

- **square of** $G$: $G^2 = (V(G), E(G)^2)$
  - $E(G)^2 = \{ uv : d_G(u, v) \leq 2 \}$

- **molecular framework**: a three-dimensional body-hinge framework in which hinges incident to each body are **concurrent**.
  - $G^2 \iff$ a molecular framework $(G, h)$
Molecular Frameworks

- **square** of $G$: $G^2 = (V(G), E(G)^2)$
  - $E(G)^2 = \{ uv : d_G(u, v) \leq 2 \}$

- **molecular framework**: a three-dimensional body-hinge framework in which hinges incident to each body are concurrent.
  - $G^2 \iff$ a molecular framework $(G, h)$

molecular framework $(G, h)$ is LR $\Rightarrow$

5$G$ contains six edge-disjoint spanning trees.
Theorem (Katoh-T11)

generic molecular framework \((G, h)\) is LR \(\iff\)

\[ 5G \text{ contains six edge-disjoint spanning trees.} \]

- a refined version: a characterization of rigid component decom.
  - fast algorithms for computing static properties of molecules
    - Ileana’s talk
  - graphical analysis of molecular mechanics

- a rank formula of \(G^2\) in the 3-d rigidity matroid (Jackon-Jordán08)
  - Open: a rank formula of a subgraph of \(G^2\)
Plate-bar Frameworks

- a $d$-dim. $k$-plate-bar framework
  - vertex = $k$-plate ($k$-dim. body)
  - edge = a bar linking $k$-plates
- $k = d$: body-bar framework
- $k = 0$: bar-joint framework

**Theorem (Tay 89, 91)**

A generic ($d^2$)-plate-bar framework in $\mathbb{R}^d$ contains a ($D_1; D_2$)-tight spanning subgraph.

**Corollary**: a characterization of identified body-hinge framework.

**Open**: characterization of the rigidity of generic ($d^3$)-plate-bar framework for large $d$. 
Plate-bar Frameworks

- a $d$-dim. $k$-plate-bar framework
  - vertex = $k$-plate ($k$-dim. body)
  - edge = a bar linking $k$-plates
- $k = d$: body-bar framework
- $k = 0$: bar-joint framework

**Theorem (Tay 89, 91)**

A generic $(d - 2)$-plate-bar framework in $\mathbb{R}^d$ is LR $\iff$ $G$ contains a $(D - 1, D)$-tight spanning subgraph.

- **Corollary**: a characterization of identified body-hinge framework.
- **Open**: characterization of the rigidity of generic $(d - 3)$-plate-bar framework for large $d$. 
Body-pin Frameworks

- A $d$-dimensional **body-pin framework** is a pair $(G, p)$:
  - $G$: underlying graph;
  - $p : E(G) \rightarrow \mathbb{R}^d$: a pin-configuration.

- a pin $\approx d$ bars

**Maxwell’s condition**

If a 3-dimensional body-pin framework $(G, p)$ is rigid, then $3G$ contains six edge-disjoint spanning trees.
Beyond Maxwell

Conjecture

A generic three-dimensional body-pin framework is rigid iff

\[ \sum_{\{X, X'\} \in \binom{\mathcal{P}}{2}} h_G(X, X') \geq 6(|\mathcal{P}| - 1) \]

for every partition \( \mathcal{P} \) of \( V \), where \( \binom{\mathcal{P}}{2} \) denotes the set of pairs of subsets in \( \mathcal{P} \) and

\[ h_G(X, X') = \begin{cases} 
6 & \text{if } d_G(X, X') \geq 3 \\
5 & \text{if } d_G(X, X') = 2 \\
3 & \text{if } d_G(X, X') = 1 \\
0 & \text{if } d_G(X, X') = 0.
\end{cases} \]

- If \( h_G \) were defined to be \( h_G(X, X') = 6 \) for \( d_G(X, X') = 2 \), it is Maxwell.
Symmetric Body-bar-hinge Frameworks

- $C_s$: a reflection group
- A $C_s$-symmetric body-bar(-hinge) framework $(G, b)$
Symmetric Body-bar-hinge Frameworks

- $C_s$: a reflection group
- A $C_s$-symmetric body-bar(-hinge) framework $(G, b)$

- the underlying quatiant signed graph $G^\sigma$
- $L_0$: the set of loops "fixed by the action"
Theorem (Schulze-T14)

A "generic" body-bar \((G, b)\) with reflection symmetry is IR in \(\mathbb{R}^3\) \(\iff\) 
\(G^\sigma - L_0\) contains edge-disjoint

- three spanning trees, and
- three non-bipartite pseudo-forests.

- **pseudo-tree**: each connected component has exactly one cycle
- **bipartite**: if every cycle has even number of minus edges
Theorem (Schulze-T14)

A ”generic” body-bar \( (G, b) \) with reflection symmetry is IR in \( \mathbb{R}^3 \) if \( G^\sigma - L_0 \) contains edge-disjoint

- three spanning trees, and
- three non-bipartite pseudo-forests.

- periodic (crystallographic) infinite body-bar frameworks (Borcea-Streinu-T15, Ross14, Schulze-T14, T15)
  - Proof 1 works only if the underlying symmetry is \( \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \).
  - Proof 3 works for any case

- body-hinge frameworks with symmetry
  - Proof 1 works if \( \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \).
  - open for other cases
Bar-joint Frameworks with Boundaries

- body-bar framework with boundaries: some of bodies are linked by bars to the external (fixed) environment
- = a body-bar framework with a designated body (corresponding to the external environment)
Characterization with non-generic boundaries

Theorem (Katoh and T13)

\( G \): a graph with a designated vertex \( v_0 \);
\( E_0 \): the set of edges in \( G \) incident to \( v_0 \);
\( b^0(e) \): a line segment for \( e \in E_0 \).

Then one can extend \( b^0 \) to \( b \) s.t. \((G, b)\) is IR \iff

\[
e_G(\mathcal{P}) \geq D|\mathcal{P}| - \sum_{X \in \mathcal{P}} \dim \text{span}\{\tilde{b}(e) : e \in E_0(X)\}
\]

for every partition \( \mathcal{P} \) of \( V(G) \setminus \{v_0\} \), where
\( E_0(X) \) is the set of edges in \( E_0 \) incident to \( X \) and
\( \tilde{b}(e) \) is the Plücker coordinate of the line segment \( b(e) \).

- subspace-constrained system
Basic Tree Packing

- $G = (V, E)$: a graph with a designated vertex $v_0$;
- $E_0$: the set of edges in $G$ incident to $v_0$;
- $x_e$: a vector in $\mathbb{R}^k$ for each $e \in E_0$.

A packing of edge-disjoint trees $T_1, \ldots, T_s$ is basic if each $v \in V \setminus \{v_0\}$ receives a base of $\mathbb{R}^k$ from $v_0$ through $T_1, \ldots, T_s$.

**Theorem (Katoh-T13)**

$\exists$ a basic packing $\iff e_G(\mathcal{P}) \geq k|\mathcal{P}| - \sum_{X \in \mathcal{P}} \dim \text{sp}\{x_e : e \in E_0(X)\}$ $(\forall \mathcal{P})$
Other Variants

- generic infinite frameworks (Kiston-Power13)
- different normed space (Kiston-Power13)
- body-bar frameworks with direction-length constraints (Jackson-Nguyen15)
  - a characterization is still open
- angle constrained (Haller et al.12)
Global Rigidity

Theorem (Hendrickson92)

If a generic bar-joint framework is globally rigid in $\mathbb{R}^d$, then the underlying graph is a complete graph, or $(d + 1)$-connected and redundantly rigid.

- sufficient in $d \leq 2$ (Jackson-Jordán05)
- may not in $d \geq 3$ (Connelly)
Theorem (Connelly, Jordán, and Whiteley13)

A generic \( d \)-dimensional body-bar framework \((G, b)\) is GR \(\iff\)
\[ \forall e \in E(G), \ G - e \text{ contains } D \text{ edge-disjoint spanning trees}. \]

- **Proof 1**: Inductive construction (Frank and Szegö03)
- **Proof 2**: The underlying graph of an equivalent bar-joint framework is vertex-redundantly rigid.
  - A generic bar-joint framework is GR if the underlying graph is vertex-redundantly rigid. (T15)
- **Proof 3**: the same approach as Proof 3 for IR
Orientation Theorem

A characterization of $\ell$-edge-redundantly rigid body-bar frameworks.

**Theorem (Frank80)**

TFAE for a graph.

- After deleting any $\ell$ edges it contains $k$ edge-disjoint spanning trees
- it admits an $r$-rooted $(k, \ell)$-edge-connected orientation for $r \in V(G)$.

A digraph $D$ is $r$-rooted $(k, \ell)$-edge-connected if for any $v \in V(G)$,

- there are $k$ arc-disjoint paths from $r$ to $v$;
- there are $\ell$ arc-disjoint paths from $v$ to $r$. 
**Body-hinge**

### Theorem (Jordán, Király, T16)

A generic $d$-dimensional body-hinge framework $(G, b)$ is GR $\Leftrightarrow$

$\forall e \in E(DG)$, $DG - e$ contains $D$ edge-disjoint spanning trees.
Body-hinge

Theorem (Jordán, Király, T16)

A generic $d$-dimensional body-hinge framework $(G, b)$ is GR $\iff$ 
\[ \forall e \in E(DG), \ DG - e \text{ contains } D \text{ edge-disjoint spanning trees.} \]

Corollary

A family of graphs which satisfy Hendrickson’s condition but are not GR

- Take a graph $H$ that contains six edge-disjoint spanning trees but $H - e$ does not for some $e \in E(H)$.
- Construct an equivalent bar-joint framework by replacing each body with a dense subgraph.
Open: Global Rigidity of $G^2$