Rigidity of Sticky Disks

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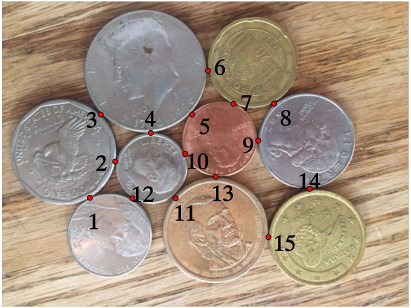
An Experiment

Take a small bunch of coins, all as different as possible, put them on a table, and push them together, without overlap, to maximize the number of contacts as in the figure below:



Record the number n of coins, and the number m of contacts. Here $n = 9$, and $m = 15$. Can you guess the relation between n and m?

Notation and Rigidity Theory that we know

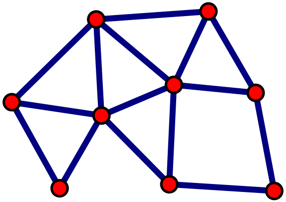
A framework $(G, \mathbf{p})$ is a graph G together with a configuration $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_n})$ in the plane where the edges of G connect the points of $\mathbf{p}$ with fixed length bars. Think of the bars as imposing constraints on the configuration, and rigidity is determined by the result of those constraints locally.

A framework $(G, \mathbf{p})$ is *flexible* if the given configuration can be deformed continuously changing its shape, but while its bar lengths remain constant. Otherwise it is called *rigid*.

The underlying graph G is simple (no multiple edges or loops). Each edge is called a bar.

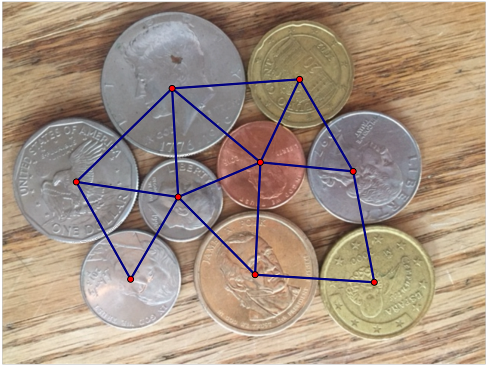
An example

The following is an example of a rigid framework, where $n = 9$ vertices, and $m = 2 \cdot 9 - 3 = 15$ bars.



Circle Packings

A circle packing is a (usually finite) collection of circular disks (coins) with disjoint interiors, possibly touching on their boundaries. We form a framework $(G, \mathbf{p})$ from the packing by placing a vertex $\mathbf{p}_i$ at the center of the $i-th$ disk and placing a bar between two vertices if the corresponding disks touch on their common boundary as in the Figure below.



The generic hypothesis

Definition

A set of real numbers $x_1, \ldots, x_n$ is called *generic* if they do not satisfy any non-zero polynomial with integral (or equivalently rational) coefficients.

Definition

A configuration $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_n})$ is called *generic in the plane* if its set of $2n$ coordinates are generic.

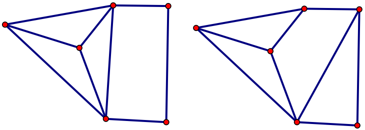
Theorem (Basic Generic Result)

If a framework $(G, \mathbf{p})$ is rigid at a generic configuration $\mathbf{p}$, it is rigid at all generic configurations $\mathbf{q}$.

The Basic Rigidity Theorem

Theorem (Pollaczek-Geiringer 1927, Laman 1970)

If a graph G has $n \geq 2$ vertices and $m = 2n - 3$ bars (edges), such that every subgraph with n' vertices has at most $2n' - 3$ edges, then for every generic configuration $\mathbf{p}$, the bar framework $(G, \mathbf{p})$ is rigid.



For example, both graphs above have $n = 6$ vertices and $m = 2n - 3 = 9$ edges, but the one on the left is not generically rigid since the left 4 vertices span a graph with $6 > 2 \cdot 4 - 3 = 5$ edges. While the rigid graph on the right has no such subgraph.

A Brief Aside about Hilda Geiringer

Hilda Pollaczek-Geiringer's rigidity papers

Brigitte Servatius

Worcester Polytechnic Institute


1893–1973

Jan Peter Schäfermeyer recently made the rigidity community aware of the work of Hilda Pollaczek-Geiringer who published some papers on 2- and 3-dimensional trusses in the 1920's and 30's. [9, 10]; [4].
While Laman's paper has no bibliography, Hilda cites

Making other things generic

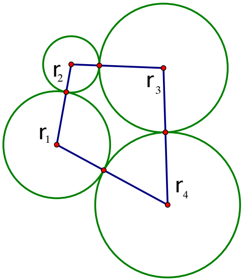
A corollary of the techniques used in the Basic Generic Theorem, is the following:

Corollary

If a set of m bar lengths of a bar framework $(G, \mathbf{p})$ in the plane with n vertices are generic, then $m \leq 2n - 3$, and it is rigid if and only if $m = 2n - 3$.

The Sad Story: Packing frameworks are almost never generic

The alternating $\pm$ sum of edge lengths in an even cycle of a packing graph is 0.



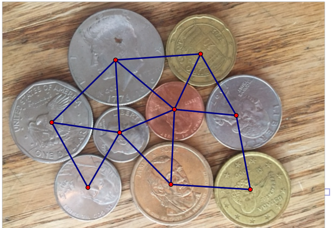
For example in the Figure, the bar lengths
$l_{12} = r_1 + r_2, l_{23} = r_2 + r_3, l_{34} = r_3 + r_4, l_{41} = r_4 + r_1$, so
$l_{12} - l_{23} + l_{34} - l_{41} = 0.$

Sticky disk packings

Suppose we have a packing of n disks in the plane with m contacts. Fix the radii and maintain the contacts. The rigidity of such configurations of packings is equivalent to the rigidity of the underlying contact graph. The following appeared in Proc. Royal Soc. A, Feb. 2019.

Theorem (Coin Theorem: Connelly, Gortler, Theran 2019)

If the n radii of a planar disk packing are generic, and have m contacts, then then $m \leq 2n-3$*, and it is rigid if and only if* $m = 2n-3$.



What is unexpected and what is not.

For general frameworks, configurations are of dimension $2n$ with trivial rotations and translations contributing 3 dimensions to the space of configurations. So if the number of edges is $2n-3$, that is just the right number of constraints to prevent all the other non-trivial motions.

The Theorem about packings is such that the dimension of the radii n is roughly about half the dimension of the space of configurations and it is unexpected.

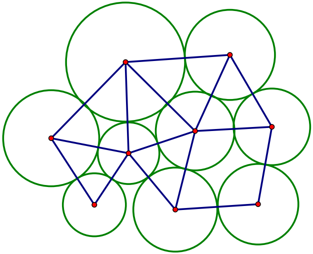
Indeed, most embedded frameworks in the plane are NOT the frameworks of a packing.

Key idea

Theorem (Degrees of Freedom DOF)

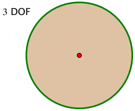
The space $\mathfrak{M}_G \subset \mathbb{R}^{3n}$ of configurations of packings with contact planar graph G, (with n disks and m edges) is a smooth manifold of dimension $3n - m$.

In the Figure, $n = 9$ and $m = 15$, so the dimension of $\mathfrak{M}_G$ is $3 \cdot 9 - 15 = 12$. As each circle is added to the figure in order as shown next, the degrees of freedom are shown.



Counting the degrees of freedom

We add in the disks one at a time counting the degrees of freedom as we go:



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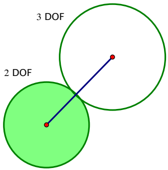

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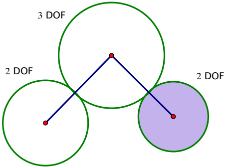

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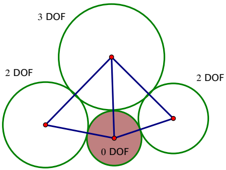

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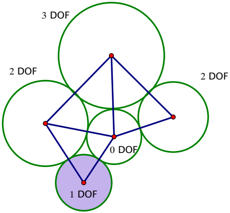

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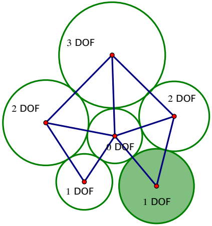

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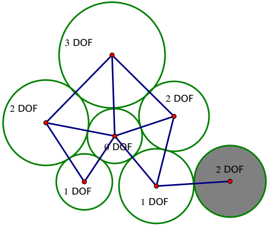

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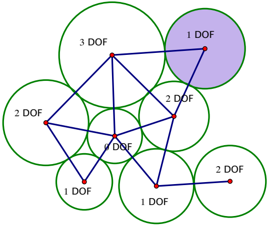

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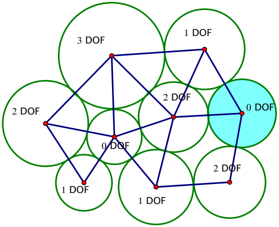


So in this example, the total degrees of freedom is
$3 + 2 + 2 + 0 + 1 + 1 + 2 + 1 + 0 = 12 = 9 + 3$

Proving rigidity

Notice that the degrees of freedom and smoothness of $\mathfrak{M}_G$ is independent of how generic $\mathbf{r} = (r_1, \ldots, r_n)$ is. But when $\mathbf{r}$ is generic, Sard's theorem can be used to show that the corresponding configurations are of the"right" dimension and therefore rigid when the number of contacts is $m = 2n - 3$.

The projection map from $\mathfrak{M}_G$, $\pi : \mathbb{R}^{2n} \times \mathbb{R}^n \supset \mathfrak{M}_G \to \mathbb{R}^n$ at a regular value (i.e. when $\mathbf{r}$ is generic) has inverse image a manifold of dimension $3n - (2n - 3) = n + 3$. This means that π is surjective, and thus the $n + 3$ corresponds to the degrees of freedom of the n radii plus only three trivial rigid motions in the plane. This implies that G, has at most $2n - 3$ contacts, and when it has $2n - 3$ contacts, $(G, \mathbf{p})$ is rigid.

Why it goes to the end

What happens if you choose a degree 4 vertex/disk?

Answer: Don't choose it. Only choose degree 3 or less vertices. If there are only degree 4 vertices left, then there are $2n$ or more edges and the radii would not have been chosen generically.

Dimension 3

The DOF Theorem is false in $\mathbb{R}^3$, for example for the usual FCC lattice.

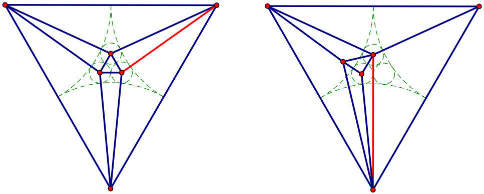
A theorem (Koebe, Andreev, Thurston) says that for, say, any planar graph, there is a realization such that it is the contact graph for a packing. Rigidity is another a matter.

We propose a visual proof of this in the description below.

The analogue of coin theorem is not known in dimension 3.

Moving Triangulations Around with Flips

A *flip* in a triangulation is where an edge is removed and reinserted the opposite way in the quadrilateral, as in the following figure.



The Figure on the left is a triangulation of a triangle and the triangulation of a packing. The Figure on the right is another triangulation of a triangle but not coming from a triangulation of a packing.

Continuous Flips

We say that an edge in a triangulation can be used for an *allowed flip* if the vertices at both ends are incident to at least four edges (i.e. of degree at least four). Thus after the flip, all vertices are of degree at least three.

Theorem (Connelly, Gortler (last week))

Given a finite circle packing in the plane, whose graph is a triangulation of a triangle, any allowed flip of the graph can be achieved by continuously deforming the packing through packings, where all the contacts are preserved except those corresponding to the flipped edge.

Proving KAT Theory

It is known that for any two triangulations of a triangle, with the same number of triangles, one can be obtained from the other by a sequence of flips. (Wagner (1936)) Our proof is different from and independent of KAT theory. So we get the KAT result as a corollary.

Definition

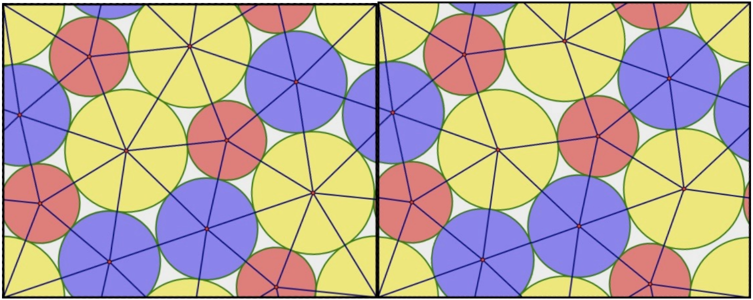
We say that graph of packing of disks in a (flat) torus is a triangulation if its universal cover is a triangulation.

Conjecture

For any disk packing of a torus whose graph is a triangulation, any allowable flip can be achieved continuously as in the plane.

Flipping in a torus

Can you see where the flip, that starts here, ends up? (Two flips are going on simultaneously.)(Picture by Maurice Pierre)



Packing 53 (left) was found by Thomas Fernique. The altered version (right) is shown. Both are with their contact graphs. The rectangular borders of each packing are also their fundamental regions.