Global rigidity of linearly constrained frameworks

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Linearly Constrained Frameworks

A d-**dimensional linearly constrained framework** is a triple (G, p, q) where $G = (V, E, L)$ is a looped simple graph, $p : V \to \mathbb{R}^d$ and $q : L \to \mathbb{R}^d$.

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Two d-dimensional linearly constrained frameworks (G, p, q) and $(G, \tilde{p}, q)$ are **equivalent** if

$$\begin{aligned} \|p_i - p_j\|^2 &= \|\tilde{p}_i - \tilde{p}_j\|^2 \text{ for all } v_i v_j \in E, \text{ and} \\ p_i \cdot q_j &= \tilde{p}_i \cdot q_j \text{ for all incident pairs } v_i \in V \text{ and } e_j \in L. \end{aligned}$$

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A framework (G, p, q) is **generic** if the coordinates of (p, q) are algebraically independent over $\mathbb{Q}$.

Relations to other frameworks

Lemma

(a) Let G be a simple graph. Then G is generically rigid in $\mathbb{R}^d$ if and only if the looped simple graph obtained by adding $\binom{d+1}{2}$ 'independent' loops to G is generically rigid as a linearly constrained framework in $\mathbb{R}^d$.

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(b) G is independent as a bar-joint framework on some surface S in R^d if and only if the linearly constrained framework $(G^{[d-2]}, p, q)$, is independent as a linearly constrained framework in $\mathbb{R}^d$, where $G^{[d-2]}$ is the looped simple graph obtained by adding $d-2$ loops at each vertex of G, p is generic on S and the directions of the loops lat v are chosen to constrain v to lie in the tangent plane to S at $p(v)$.

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(c) G is generically globally rigid as a bar-joint framework in $\mathbb{R}^2$ if and only if the generic linearly constrained framework (G^*, p, q), has exactly two equivalent realisations as a linearly constrained framework in $\mathbb{R}^2$, where G^* is the looped simple graph obtained by adding two loops at each end vertex of an edge of G.

Rigidity in $\mathbb{R}^2$

Theorem [Streinu and Theran, 2010]

Let (G, p, q) be a generic linearly constrained framework in $\mathbb{R}^2$. Then (H, p, q) is rigid if and only if G has a spanning subgraph $H = (V, E, L)$ such that
(a) $|E| + |L| = 2|V|$,
(b)$|F| \leq 2|V_F|$ for all $F \subseteq E \cup L$ and
(c) $|F| \leq 2|V_F| - 3$ for all $\emptyset \neq F \subseteq E$.

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Note Katoh and Tanigawa (2013) gave a characterisation of rigidity for a linearly constrained framework (G, p, q) in which only p is required to be generic.

Global Rigidity in $\mathbb{R}^2$

Theorem [Hendrickson 1992, Connelly 2005, Jackson and Jordán 2005]

Let (G, p) be a generic bar-joint framework in $\mathbb{R}^2$. Then (G, p) is globally rigid if and only if $G = K_1$, K_2 or K_3, or G is 3-connected and redundantly rigid.

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G is **redundantly rigid** if $G - f$ is rigid for all edges and loops f.

Necessary conditions for global rigidity in $\mathbb{R}^d$

A looped simple graph $G = (V, E, L)$ is d-**balanced** if,for all $S \subseteq V$ with $|S| = d$, each component of $G - S$ contains a loop.

Theorem

Suppose (G, p, q) is a generic globally rigid linearly constrained framework in $\mathbb{R}^d$. Then each connected component of G is either a single vertex with at least d loops or is d-balanced and redundantly rigid in $\mathbb{R}^d$.

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Equilibrium Stresses

An **equilibrium stress** for a linearly constrained framework (G, p, q) in $\mathbb{R}^d$ is a map $\omega : E \to \mathbb{R}$ with the property that, for all $v_i \in V$,

$$\sum_{v_j \in V} \omega_{ij}(p(v_i) - p(v_j)) \in \langle q(\ell_j) \,:\, \ell_j \text{ is a loop incident to } v_i \rangle$$

(where ω_{ij} is taken to be equal to $\omega(e)$ if $e = v_i v_j \in E$ and to be equal to 0 if $v_i v_j \notin E$, and the subspace generated by the empty set is taken to be $\{\mathbf{0}\}$).

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The **stress matrix** $\Omega(\omega)$ corresponding to ω is the $|V| \times |V|$-matrix in which the off diagonal entry in row v_i and column v_j is $-\omega_{ij}$, and the diagonal entry in row v_i is $\sum_{v_j \in V} \omega_{ij}$.

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Note Converse is false

Let $G = (V, E, L)$ be a looped simple graph. The *d*-**dimensional 1-extension operation** forms a new looped simple graph from G by deleting an edge or loop $f \in E \cup L$ and adding a new vertex v and $d + 1$ new edges or loops incident to v, with the provisos that each end vertex of f is incident to exactly one new edge, and, if $f \in L$, then there is at least one new loop incident to v.

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Lemma

The d-dimensional 1-extension operation preserves the property of having a full rank equilibrium stress for generic linearly constrained frameworks in $\mathbb{R}^d$.

Recursive Construction

Given a looped simple graph G, let $G^{[k]}$ be the graph obtained by adding k loops at each vertex of G.

Theorem

A connected looped simple graph is 2-balanced and redundantly rigid in $\mathbb{R}^2$ if and only if it can be obtained from $K_1^{[3]}$ by recursively applying the operations of 2-dimensional 1-extension and adding a new edge or loop.

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Theorem

Suppose G is a connected looped simple graph with at least two vertices and (G, p, q) is a generic realisation of G in $\mathbb{R}^2$. Then the following statements are equivalent:
(a) (G, p, q) is globally rigid;
(b) G is 2-balanced and redundantly rigid in $\mathbb{R}^2$;
(c) (G, p, q) has a full rank equilibrium stress.

Extension to $\mathbb{R}^d$?

The following result gives a characterisation of generic rigidity for linearly constrained frameworks in $\mathbb{R}^d$ when each vertex is constrained to lie in an affine subspace of sufficiently small dimension compared to d.

Theorem [Cruickshank, Guler, Jackson, Nixon 2018]

Suppose G is a looped simple graph and d, t are positive integers with $d \geq \max\{2t, t(t-1)\}$. Then $G^{[d-t]}$ is rigid in $\mathbb{R}^d$ if and only if G has a spanning subgraph $H = (V, E, L)$ such that
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Problem Does an analogous result hold for global rigidity?