Graph reconstruction from unlabeled edge lengths

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Length equivalent frameworks

We say that two frameworks (G, p) and (H, q) are *length-equivalent* (*under the bijection* ψ) if there is a bijection ψ between the edge sets of G and H such that for every edge e of G, the length of e in (G, p) is equal to the length of $\psi(e)$ in (H, q).

If G and H have the same number of vertices then we say that they have the same *order*.

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Generic unlabeled global rigidity

Theorem (S. Gortler, L. Theran, D. Thurston, 2018)

Let $G = (V, E)$ be globally rigid in $\mathbb{R}^d$ on at least $d + 2$ vertices, where $d \geq 2$, and let (G, p) be a d-dimensional generic realization of G. Suppose that (H, q) is another d-dimensional framework such that G and H have the same order and (G, p) is length-equivalent to (H, q). Then there is a graph isomorphism $\varphi : V(G) \to V(H)$ which induces ψ, that is, for which $\psi(uv) = \varphi(u)\varphi(v)$ for all $uv \in E$. In particular, G and H are isomorphic and the frameworks (G, p) and (H, q) are congruent after relabeling, i.e. (G, p) is congruent to $(G, q \circ \varphi)$.

Complete graphs

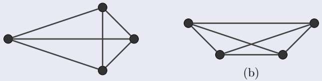

(a) (b)

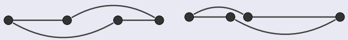
Figure 1: Two non-equivalent frameworks with the same edge measurement sets.

Weak and strong reconstructibility

Let (G, p) be a d-dimensional generic (complex) realization of the graph G. We say that (G, p) is *weakly reconstructible* if whenever (H, q) is a d-dimensional generic complex framework such that G and H have the same order and (G, p) is length-equivalent to (H, q), then H is isomorphic to G.

Let (G, p) be a d-dimensional generic (complex) realization of the graph G. We say that (G, p) is *strongly reconstructible* if for every d-dimensional generic complex framework (H, q) which is length-equivalent to (G, p) under some bijection ψ and has the same order, there is an isomorphism $\varphi : G \to H$ for which $\psi(uv) = \varphi(u)\varphi(v)$ for all $uv \in E$.

Four-cycle



(a) (b)

Figure 1: Realizations of C_4 with the same edge measurements where the mapping between corresponding edges does not arise from a graph isomorphism. This shows that C_4 is not strongly reconstructible in one dimension.

Weakly and strongly reconstructible graphs

A graph G is said to be *(generically) weakly reconstructible* (respectively *strongly reconstructible*) in $\mathbb{C}^d$ if every d-dimensional generic realization (G, p) of G is weakly (respectively strongly) reconstructible.

Theorem (S. Gortler, L. Theran, D. Thurston, 2018)

Let G be a graph on $n \geq d + 2$ vertices, where $d \geq 2$. Suppose that G is globally rigid in $\mathbb{R}^d$. Then G is strongly reconstructible in $\mathbb{C}^d$.

Families of non-globally rigid (non-rigid) WR graphs

We call a graph G on n vertices and m edges *maximally non-globally rigid* (resp. *maximally non-rigid*) if it is not globally rigid (resp. not rigid) but every graph on n vertices and with more than m edges is globally rigid (resp. rigid).

Theorem

Let G be a graph on n vertices and $d \geq 2$ a fixed dimension. Suppose that G is the only maximally non-globally rigid graph on n vertices (up to isomorphism). Then G is weakly reconstructible in $\mathbb{C}^d$.

Theorem

Let G be a graph on n vertices and $d \geq 1$ a fixed dimension. Suppose that G is the only maximally non-rigid graph on n vertices (up to isomorphism). Then G is weakly reconstructible in $\mathbb{C}^d$.

Maximally non-globally rigid graphs

Theorem

Let H be the extension of K_{n-1} by a vertex of degree d, where $d \geq 1$ and $n \geq d+2$. Then H is the unique maximal non-globally rigid graph in d dimensions on n vertices.

Theorem

Let H be the extension of K_{n-1} by a single vertex of degree $d-1$ for some $d \geq 1$ and $n \geq d+3$. Then H is the unique maximally non-rigid graph in d dimensions on n vertices.

Examples of WR graphs

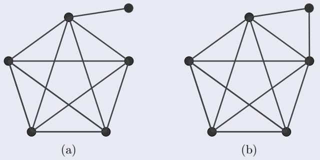


Figure 1: The maximum size non-rigid and non-globally rigid graphs in the plane on 6 vertices.

The measurement variety

Let G be a graph with n vertices and m edges. Fix an ordering of the edges and consider the (rigidity) map $m_{d,G}$ from $\mathbb{C}^{nd}$ to $\mathbb{C}^{m}$, where the i-th coordinate of $m_{d,G}(p)$ (which corresponds to some edge uv of G) is $m_{uv}(p)$, that is, the squared complex edge length of uv in the d-dimensional complex realization (G, p).

Definition

The *d-dimensional measurement variety* of a graph G (on n vertices), denoted by $M_{d,G}$, is the Zariski closure of $m_{d,G}(\mathbb{C}^{nd})$.

Weak reconstructibility and the measurement variety

We say that $M_{d,G}$ *uniquely determines* the graph G if whenever $M_{d,G} = M_{d,H}$ for some graph H with the same order as G, we have that H is isomorphic to G.

Theorem

Let G be a graph and $d \geq 1$ be fixed. The following are equivalent.
1. G is (generically) weakly reconstructible in $\mathbb{C}^d$.
2. There exists some generic d-dimensional framework (G, p) which is weakly reconstructible.
3. $M_{d,G}$ uniquely determines G.

Strong reconstructibility and the measurement variety

Theorem

Let G be a graph and $d \geq 1$ be fixed. The following are equivalent.

1. G is (generically) strongly reconstructible in $\mathbb{C}^d$.
2. There exists some generic d-dimensional framework (G, p) which is strongly reconstructible.
3. $M_{d,G}$ uniquely determines G and whenever $M_{d,G}$ is invariant under a permutation ψ of the edges of G, ψ is induced by a graph automorphism.
4. Whenever $M_{d,G} = M_{d,H}$ under an edge bijection ψ for some graph H with the same order as G, ψ is induced by a graph isomorphism.

The measurement variety and the rigidity matroid

Lemma

Let G be a graph on n vertices. Then

$$\dim(M_{d,G}) = r_d(G). \tag{1}$$

In particular, when $n \geq d+1$ we have $\dim(M_{d,G}) \leq nd - \binom{d+1}{2}$ and equality holds if and only if G is rigid in d dimensions.

Theorem

Let G be a graph with m edges. Then G is independent in d dimensions if and only if $M_{d,G} = \mathbb{C}^m$.

The measurement variety and the rigidity matroid

Theorem

Let $G = (V, E)$ be a graph, and suppose that $M_{d,G} = M_{d,G_1} \oplus M_{d,G_2}$ for some subgraphs $G_i = (V, E_i)$ for $i = 1, 2$. Then $\mathcal{R}_d(G) = \mathcal{R}_d(G_1) \oplus \mathcal{R}_d(G_2)$.

Theorem

Let G be a graph on n vertices with edges $e_1, \ldots, e_m$ and let $G' = G - e_m$. Then e_m is a bridge of $\mathcal{R}_d(G)$ if and only if $M_{d,G} = M_{d,G'} \oplus \mathbb{C}$.

The measurement variety and the rigidity matroid

Theorem

Let G and H be graphs with the same number of edges and suppose that $M_{d,G} = M_{d,H}$ under some edge bijection ψ. Then this edge bijection defines an isomorphism between the d-dimensional rigidity matroids of G and H.

Theorem

Let G be a graph that is uniquely (up to isomorphism) determined by its d-dimensional rigidity matroid among graphs on the same number of vertices. Then G is weakly reconstructible in d dimensions.

WR-1

Theorem

Let G be a graph that is not 2-connected and has at least two edges. Then G is weakly reconstructible in one dimension if and only if one of the following holds:
1. G is isomorphic to a 2-connected, weakly reconstructible graph H plus some isolated vertices.
2. G is isomorphic to the 1-sum of two connected vertex-transitive graphs.

Theorem

The Graph Isomorphism problem is polynomially reducible to the problem of testing weak reconstructibility in $\mathbb{C}^1$.

WR-2

Theorem

Let G be a non-redundant rigid graph in $\mathbb{R}^2$. Then G is weakly reconstructible in $\mathbb{C}^2$ if and only if G can be obtained by taking the 1-sum of two complete graphs K_r and K_s and then adding an edge, where $r, s \geq 2$, and if $s = 3$ (resp. $r = 3$) then $r = 2$ (resp. $s = 2$) holds.

Theorem

The Graph Isomorphism problem is polynomially reducible to the problem of testing weak reconstructibility in $\mathbb{C}^2$.

Bridge invariant graphs

Definition

Suppose that G has at least $d+2$ vertices and let e be a bridge in $\mathcal{R}_d(G)$. Then there is another edge e' that we can add to the flexible graph $G-e$ to obtain a graph $H = G - e + e'$ which is again rigid. In this case we say that H is obtained from G by a *bridge replacement* operation.

Definition

A graph G is called *bridge invariant* if every sequence of bridge replacement operations starting from G leads to a graph isomorphic to G.

Bridge invariant graphs in the pane

Theorem

Let G be a non-redundant rigid graph in $\mathbb{R}^2$ on $n \geq 3$ vertices. Then G is bridge invariant if and only if it satisfies one of the following properties:
1. G is isomorphic to a degree-2-extension of K_{n-1},
2. G is the cone graph of a connected graph obtained from two disjoint vertex-transitive graphs on at least three vertices by adding an edge e.

SR-1, SR-2

Theorem

Let G be a graph on at least four vertices and without isolated vertices. Then G is strongly reconstructible in $\mathbb{C}^1$ if and only if it is 3-connected.

Theorem

Let G be a graph on at least four vertices and without isolated vertices. Then G is strongly reconstructible in $\mathbb{C}^2$ if and only if it is globally rigid in $\mathbb{R}^2$.

Real reconstructibility

Theorem

Let (G, p) be a generic framework in $\mathbb{R}^1$ that is weakly (respectively strongly) reconstructible in $\mathbb{R}^1$. Then (G, p) is weakly (resp. strongly) reconstructible in $\mathbb{C}^1$.

Thank you