Co-boundary operators for infinite frameworks

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Outline

A bar-joint framework in a normed space X consists of a graph $G = (V, E)$ and map $q : V \to X$, $v \mapsto q_v$.

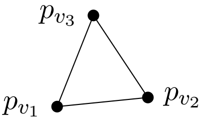
Suppose that, for each edge $vw \in E$, $q_v - q_w$ is a smooth point in X.

The (vw, v)-entry of the rigidity matrix is the unique support functional for $q_v - q_w$.

Varying the norm on X gives rise to different rigidity matrices.

Example

Let $G = K_3$ and $p : V \to \mathbb{R}^2$, $v \mapsto p_v = (p_v^x, p_v^y)$.



The rigidity matrix is a $|E| \times 2|V|$-matrix:

$$\begin{array}{cc} & \begin{array}{cccccc} (v_1;x) & (v_1;y) & (v_2;x) & (v_2;y) & (v_3;x) & (v_3;y) \end{array} \\ \begin{array}{c} v_1v_2 \\ v_1v_3 \\ v_2v_3 \end{array} & \begin{pmatrix} p_{v_1}^x - p_{v_2}^x & p_{v_1}^y - p_{v_2}^y & p_{v_2}^x - p_{v_1}^x & p_{v_2}^y - p_{v_1}^y & 0 & 0 \\ p_{v_1}^x - p_{v_3}^x & p_{v_1}^y - p_{v_3}^y & 0 & 0 & p_{v_3}^x - p_{v_1}^x & p_{v_3}^y - p_{v_1}^y \\ 0 & 0 & p_{v_2}^x - p_{v_3}^x & p_{v_2}^y - p_{v_3}^y & p_{v_3}^x - p_{v_2}^x & p_{v_3}^y - p_{v_2}^y \end{pmatrix} \end{array}$$

Let $G = (V, E)$ be a simple graph and let X and Y be normed linear spaces over $\mathbb{K}$.

To each ordered pair $(v, w) \in V \times V$ assign a linear map $\varphi(v, w) : X \to Y$ such that

- $\varphi(v, w) = -\varphi(w, v)$,
- $\varphi(v, w) = 0$ whenever $vw \notin E$.

We will call the pair (G, φ) a framework.

Notation:

- $\Delta(G) = \sup_{v \in V} \deg(v)$
- $\|\varphi\|_\infty = \sup_{vw \in E} \|\varphi(v, w)\|_{op}$

A co-boundary matrix for a framework (G, φ) has rows indexed by E, columns indexed by V, and entries

$$c_{e,v} = \begin{cases} \varphi(v,w) & \text{if } e = vw, \\ 0 & \text{otherwise.} \end{cases}$$

Example

- Incidence matrices for directed graphs.
- Rigidity matrices for bar-joint frameworks.

The co-boundary matrix for (G, φ) takes the form,

$$\begin{array}{cc} & \begin{array}{ccccccccc} & & & v & & & & w & \end{array} \\ vw & \begin{pmatrix} & & & \vdots & & & & \vdots & & \\ 0 & \cdots & 0 & \varphi(v,w) & 0 & \cdots & 0 & -\varphi(v,w) & 0 & \cdots \\ & & & \vdots & & & & \vdots & & \end{pmatrix} \end{array}$$

Goal: To understand co-boundary matrices for infinite frameworks.

For an index set I $(= V$ or $E)$ and normed space X, we will consider the following spaces:

- $\ell^\infty(I;X) = \{(x_i)_{i\in I} : \sup_{i\in I} \|x_i\| < \infty\}$.
- $c_0(I;X) = \{(x_i)_{i\in I} : \forall\, \epsilon > 0,\ \exists\, I_0$ fin s.t. $\sup_{i\in I\backslash I_0} \|x_i\| < \epsilon\}$.
- $\ell^p(I;X) = \{(x_i)_{i\in I} : \sum_{i\in I} \|x_i\|^p < \infty\}$, $p \in [1,\infty)$.

Under suitable conditions, $C(G,\varphi)$ gives rise to the following linear maps:

- $C(G,\varphi) : \ell^\infty(V;X) \to \ell^\infty(E;Y)$.
- $C(G,\varphi) : c_0(V;X) \to c_0(E;Y)$.
- $C(G,\varphi) : \ell^p(V;X) \to \ell^p(E;Y)$, $p \in [1,\infty)$.

Questions:

- When is $C(G,\varphi)$ a bounded operator?
- When is it a compact operator?
- When is it bounded below?
- Can we compute its operator norm?

Related work:

- Maddox, Infinite matrices of operators. Lecture Notes in Mathematics, 786. Springer, Berlin, 1980.
- Mohar and Woess, A survey of spectra of infinite graphs. Bull. London Math. Soc. 1989.
- Agrawal, Berge, Colbert-Pollack, Martinez-Avenano, Sliheet, Norms, kernels and eigenvalues of some infinite graphs. 2018. arXiv:1812.08276v1

Theorem

Let Z be a subspace of $\ell^\infty(V;X)$ which contains $c_{00}(V;X)$. TFAE:

(i) $\varphi : V \times V \to L(X,Y)$ *is a bounded function.*

(ii) $C(G,\varphi) : Z \to \ell^\infty(E;Y)$ *is a bounded operator.*

Moreover,

$$\|C(G,\varphi)\|_{op} = 2\|\varphi\|_\infty.$$

Theorem

The following statements are equivalent.

(i) $\varphi : V \times V \to L(X, Y)$ *is a bounded function.*

(ii) $C(G, \varphi) \in B(\ell^1(V; X), \ell^\infty(E; Y))$.

(iii) $C(G, \varphi)$ *maps* $\ell^1(V; X)$ *into* $\ell^\infty(E; Y)$.

Moreover,

$$\|C(G, \varphi)\|_{op} = \|\varphi\|_\infty.$$

Theorem

Let $C(G,\varphi)$ be a co-boundary matrix. If G is locally finite then TFAE:

(i) $\varphi : V \times V \to L(X,Y)$ *is a bounded function.*

(ii) $C(G,\varphi)$ *maps* $c_0(V;X)$ *into* $c_0(E;Y)$.

(iii) $C(G,\varphi) \in B(c_0(V;X), c_0(E;Y))$.

Moreover,

$$\|C(G,\varphi)\|_{op} = 2\|\varphi\|_\infty.$$

Theorem

Let $p \in [1, \infty)$. If $\Delta(G) < \infty$ then TFAE:

(i) $\varphi : V \times V \to L(X, Y)$ *is a bounded function.*

(ii) $C(G, \varphi) \in B(\ell^p(V; X), \ell^p(E; Y))$.

(iii) $C(G, \varphi)$ *maps* $\ell^p(V; X)$ *into* $\ell^p(E; Y)$.

Moreover,

$$2^{1-\frac{1}{p}} \|\varphi\|_\infty \leq \|C(G, \varphi)\|_{op} \leq 2^{1-\frac{1}{p}} \|\varphi\|_\infty \Delta(G)^{\frac{1}{p}}.$$

Theorem

Suppose $\Delta(G) < \infty$ and $\varphi : V \times V \to L(X, Y)$ is bounded.

(i) *If $G_k \to G$ as $k \to \infty$ then,*

$$\|C(G, \varphi)\|_{op} = \lim_{k\to\infty} \|C(G_k, \varphi)\|_{op}.$$

(ii) *If $\mathcal{S}$ and $\mathcal{S}'$ denote respectively the set of all subgraphs and the set of all finite subgraphs of G then,*

$$\|C(G, \varphi)\|_{op} = \sup_{G_0 \in \mathcal{S}} \|C(G_0, \varphi)\|_{op} = \sup_{G_0 \in \mathcal{S}'} \|C(G_0, \varphi)\|_{op}.$$

The operator norms refer to the cases:

(a) $C(G, \varphi) \in B(c_0(V; X), c_0(E; Y))$, *and,*

(b) $C(G, \varphi) \in B(\ell^p(V; X), \ell^p(E; Y))$, *where $p \in [1, \infty)$.*

Theorem

Let Z *be a subspace of* $\ell^\infty(V;X)$ *which contains* $c_{00}(V;X)$*.*
If $\varphi : V \times V \to L(X,Y)$ *vanishes at infinity then the operator* $C(G,\varphi) : Z \to \ell^\infty(E;Y)$ *is compact.*

Theorem

Suppose one of the following conditions holds.

(i) $C(G,\varphi) \in K(c_0(V;X), c_0(E;Y))$.

(ii) $C(G,\varphi) \in K(\ell^p(V;X), \ell^p(E;Y))$*, where* $p \in [1,\infty)$*.*

Then $\varphi : V \times V \to L(X,Y)$ *vanishes at infinity.*

Corollary

If G has bounded degree then TFAE:

(i) $\varphi : V \times V \to L(X,Y)$ *vanishes at infinity.*

(ii) $C(G,\varphi) \in K(c_0(V;X), c_0(E;Y))$.

(iii) $C(G,\varphi) \in K(\ell^p(V;X), \ell^p(E;Y))$, *where* $p \in [1,\infty)$.

Given framework (G, φ) , define

$$l : V \to \mathbb{R}, \quad l(v) = \sup_{vw \in E} \|\varphi(v, w)\|_{op}.$$

Theorem

Suppose one of the following conditions holds.

(a) $C(G, \varphi) : Z \to c_0(E; Y)$ *is bounded below, where* Z *is a subspace of* $\ell^\infty(V; X)$ *which contains* $c_{00}(V; X)$*.*

(b) G *has bounded degree and* $C(G, \varphi) : \ell^p(V; X) \to \ell^p(E; Y)$ *is bounded below, where* $p \in [1, \infty)$*.*

Then the function $l : V \to \mathbb{R}$ *is bounded away from zero.*

If $V_0 \subset V$ then denote by ∂V_0 the set of edges of G with exactly one vertex in V_0.

The isoperimetric constant for G is the value,

$$i(G) = \inf_{V_0 \text{ finite}} \frac{|\partial V_0|}{|V_0|},$$

where the infimum is taken over all finite subsets V_0 of V.

Denote by $\chi(V;X)$ the set of finitely supported vectors with constant non-zero entries.

Theorem

Let G be a locally finite graph and let $p \in [1,\infty)$.

If $\varphi : V \times V \to L(X,Y)$ is bounded then,

$$\inf\{\|C(G,\varphi)z\|_p : z \in \chi(V;X),\ \|z\|_p = 1\} \leq i(G)^{\frac{1}{p}}\,\|\varphi\|_\infty.$$

Corollary

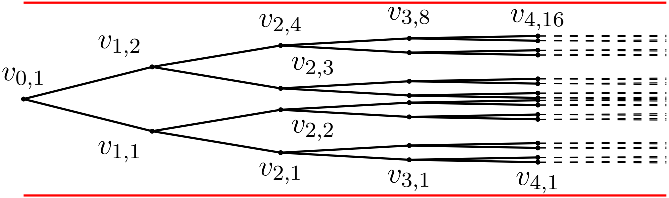
Let $p \in [1, \infty)$.

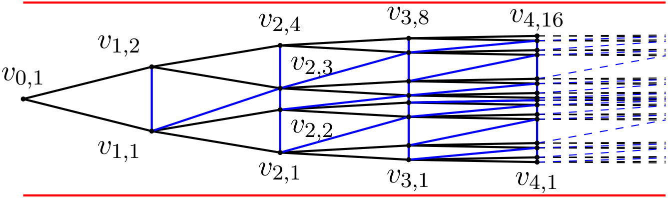
If $\Delta(G) < \infty$ *and* $i(G) = 0$ *then the operator,*

$$C(G, \varphi) : \ell^p(V; X) \to \ell^p(E; Y),$$

is not bounded below.

Bounded below





Given a framework (G, φ) and a unit vector $x \in X$ we define,

$$i(G, \varphi; x) = \inf_{V_0 \text{ finite}} \left(\sup_{e=vw \in \partial V_0} \|\varphi(v, w)(x)\| \right),$$

where the infimum is taken over all finite subsets V_0 of $V(G)$.

(Note that $i(G, \varphi; x) \leq \inf_{v \in V} l(v)$).

Theorem

Let $p \in [1, \infty)$ and let $x \in X$ be a unit vector.

If $\Delta(G) < \infty$ then,

$$\inf\{\|C(G,\varphi)z\|_p : z \in \chi(V; \mathbb{K}^d), \|z\|_p = 1\} \leq i(G,\varphi;x)\Delta(G)^{\frac{1}{p}}.$$

Corollary

Let $p \in [1, \infty)$.

If $\Delta(G) < \infty$ and $i(G, \varphi; x) = 0$ for some unit vector $x \in X$ then the operator,

$$C(G, \varphi) : \ell^p(V; X) \to \ell^p(E; Y),$$

is not bounded below.

Thank you