The icosahedra of edge length 1

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Simplicial surfaces

j. w. K.-H. Brakhage, A. Niemeyer, W. Plesken, A. Strzelczyk et al.

build surfaces from triangles belonging to very few congruence classes

- simplicial surfaces as combinatorial objects
- simplicial surfaces as Euclidean two-dim. (compact) manifolds with singularities
- embeddings of abstract simplicial surfaces into Euclidean 3-space

K.-H. Brakhage, A. Niemeyer, W. Plesken, A. Strzelczyk,
Simplicial surfaces controlled by one triangle,
17th Int. Conference on Geometry and Graphics, 4–8 Aug. 2016, Beijing

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Classify embeddings of icosahedron in $\mathbb{R}^3$ with 12 distinct vertices admitting non-trivial symmetry

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$\rightsquigarrow$ 35 inequivalent rigid icosahedra, 1 curve of flexible icosahedra

K.-H. B., A. C. N., W. P., D. R., A. S., *The icosahedra of edge length 1*, J. Algebra, in press

web page:

`http://algebra.data.rwth-aachen.de/Icosahedra/visualplusdata.html`

Software

We used:

- Magma
- Maple: Involutive
- C++/Python: GINV
- Bertini

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Also under development:

- simplicial-surfaces in GAP (M. Baumeister, A. Niemeyer)

Icosahedron

Combinatorial automorphism group $A \cong C_2 \times A_5$ generated by

$$\begin{aligned} a &:= (1,2)(3,4)(5,7)(6,8)(9,11)(10,12), \\ b &:= (1,10)(3,9)(2,12)(4,11)(5,6)(7,8), \\ c &:= (1,7)(2,3)(4,11)(5,12)(6,8)(9,10), \\ d &:= (1,12)(3,9)(2,10)(4,11)(5,7)(6,8) \end{aligned}$$

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d generates centre of A, interchanges combinatorially opposite vertices

20 faces:	orbit of $\{1,2,3\}$,
30 edges:	orbit of $\{1,2\}$,
30 diagonals of combinatorial distance 2:	orbit of $\{3,4\}$,
6 diagonals of combinatorial distance 3:	orbit of $\{1,12\}$

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$\pi : A \to \mathrm{GL}(12,\mathbb{R})$ natural representation of A by permutation matrices

Theorem

The subgroups U of A with more than one element that arise as symmetry group of an icosahedron fall into 11 conjugacy classes:

Automorphism group $U \leq A = C_2 \times A_5$	*Number of icosahedra*
$C_2 \times A_5$	2
$C_2 \times D_{10}$	4
$C_2 \times D_6$	2
$D_{10} \quad (\not\leq A_5)$	3
$D_6 \quad (\not\leq A_5)$	2
${C_2}^2 \quad (\ni d)$	1
${C_2}^2 \quad (\not\ni d, \not\leq A_5)$	5
${C_2}^2 \quad (\leq A_5)$	1
$C_2 \quad (\leq A_5)$	5
$C_2 \quad (\not\ni d, \not\leq A_5)$	10
$C_2 \quad (= \langle d \rangle)$	∞

$S := \mathrm{Stab}_A$	$\mathrm{Syl}_2(S)$	d_G	$r_{1,G}$	r_G	$r_{f,G}$	Trace relation
${C_2}^2$	$\langle a, d\rangle \begin{smallmatrix}\bullet & - \\ & + \end{smallmatrix}$	8	4	2	1	$\lambda^4 - \frac{76}{3}\lambda^3 + 238\lambda^2 - \frac{4964}{5}\lambda + \frac{23767}{15}$
$C_2 \times A_5$	$\langle a, b, d\rangle$	2	2	2	2	$\lambda^2 - 15\lambda + 45$
$C_2 \times D_{10}$	$\langle a, d\rangle \begin{smallmatrix} - & \bullet \\ & + \end{smallmatrix}$	2	2	2	2	$\lambda^2 - 15\lambda + \frac{269}{5}$
${C_2}^2$	$\langle a, bd\rangle \begin{smallmatrix} - & + \\ & + \end{smallmatrix}$	2	2	2	2	$\lambda^2 - \frac{71}{5}\lambda + \frac{10561}{225}$
$C_2 \times D_{10}$	$\langle a, d\rangle \begin{smallmatrix} - & + \\ & + \end{smallmatrix}$	4	2	2	2	$\lambda^4 - 18\lambda^3 + \frac{583}{5}\lambda^2 - \frac{1658}{5}\lambda + \frac{9101}{25}$
$C_2 \times D_6$	$\langle a, d\rangle \begin{smallmatrix} - & + \\ & + \end{smallmatrix}$	4	4	2	2	$\lambda^4 - 26\lambda^3 + 243\lambda^2 - 970\lambda + 1397$
${C_2}^2$	$\langle a, bd\rangle \begin{smallmatrix} - & + \\ & + \end{smallmatrix}$	24	10	6	3	$\lambda^{12} - \frac{5179\cdot 2^2}{3^2\cdot 5^2}\lambda^{11} \pm \cdots$
${C_2}^2$	$\langle a, b\rangle \begin{smallmatrix} - & - \\ & - \end{smallmatrix}$	30	18	6	1	$\lambda^5 - \frac{117}{2}\lambda^4 \pm \cdots$
C_2	$\langle a\rangle \quad -$	172	48	20	5	$\lambda^{43} - \frac{73\cdot 7\cdot 11\cdot 461687}{2^2\cdot 3^3\cdot 5^2\cdot 29\cdot 79}\lambda^{42} \pm \cdots$
D_{10}	$\langle ad\rangle \quad +$	2	2	2	2	$\lambda^2 - \frac{44}{3}\lambda + \frac{2131}{45}$
D_6	$\langle ad\rangle \quad +$	2	2	2	2	$\lambda^2 - \frac{68}{5}\lambda + \frac{1111}{25}$
C_2	$\langle ad\rangle \quad +$	36	12	8	4	$\lambda^{18} - \frac{1106}{9}\lambda^{17} \pm \cdots$
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Classification

Choose origin as centre of mass of the 12 (equilibrated) vertices

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Gram matrices: equivalence = conjugacy by permutation matrices

$$(g, G) \longmapsto \pi(g)^{tr}\, G\, \pi(g) \;=\; (G_{ig,jg})_{1\le i,j\le 12}$$

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Lemma. Gram matrix G with automorphism group $U \le A$. There exists a faithful orthogonal repres. $\delta : U \to O_3(\mathbb{R})$ and $M \in \mathbb{R}^{3\times 12}$ such that

$$\delta(g)\, M \;=\; M\, \pi(g) \qquad \text{for all } g \in U, \qquad G := M^{tr}\, M\,.$$

Classification

Minimal subgroups U of $A \cong C_2 \times A_5$ up to conjugacy:

$\langle abc \rangle \cong C_3, \qquad \langle ac \rangle \cong C_5, \qquad \langle a \rangle \cong C_2, \qquad \langle d \rangle \cong C_2, \qquad \langle ad \rangle \cong C_2$

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Faithful orthogonal representations of degree 3 of C_2: generator maps to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

9 cases to consider

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Determine the U-invariant Gram matrices!

Classification

Let e_1, e_2, ..., e_{12} be the standard basis of $\mathbb{R}^{12\times 1}$, $\quad R := \mathbb{Q}[y_1, \ldots, y_n]$.

Def. ideal I gen. by $\quad (e_i - e_j)^{tr}\, G\, (e_i - e_j) - 1, \quad \{i, j\} \in \{1, 2\}^A$, and 4×4 minors of G, where y_i are the entries of G corresp. to U-orbits

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Def. A maximal ideal $m \trianglelefteq R$ associated to I is *relevant* if

(a) rank $(G_{i,j} + m) \in (R/m)^{12\times 12}$ at most 3,

(b) $U = \{\, g \in A \mid \pi(g)^{tr}\, (G_{i,j} + m)\, \pi(g) = (G_{i,j} + m) \,\}$,

(c) $\exists\, \iota : R/m \to \mathbb{R}$ such that $(\iota(G_{i,j} + m))$ is positive semidefinite.

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$d_{G(m)} = [R/m : \mathbb{Q}]$
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Example: case $\delta(a) = \mathrm{diag}(1, -1, -1)$

coordinate matrix M

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 & z_9 & z_{10} & z_{11} & z_{12} \end{pmatrix}$$

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recall $\delta(a)\, M = M\, \pi(a)$

$$\begin{pmatrix} x_1 & x_1 & x_2 & x_2 & x_3 & x_4 & x_3 & x_4 & x_5 & x_6 & x_5 & x_6 \\ y_1 & -y_1 & y_2 & -y_2 & y_3 & y_4 & -y_3 & -y_4 & y_5 & y_6 & -y_5 & -y_6 \\ z_1 & -z_1 & z_2 & -z_2 & z_3 & z_4 & -z_3 & -z_4 & z_5 & z_6 & -z_5 & -z_6 \end{pmatrix}$$

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1st and 2nd column not equal, w.l.o.g. $z_1 = 0$, then $4\, y_1^2 - 1 = 0$

$$\begin{pmatrix} x_1 & x_1 & x_2 & x_2 & x_3 & x_4 & x_3 & x_4 & x_5 & x_6 & x_5 & x_6 \\ \frac{1}{2} & -\frac{1}{2} & y_2 & -y_2 & y_3 & y_4 & -y_3 & -y_4 & y_5 & y_6 & -y_5 & -y_6 \\ 0 & 0 & z_2 & -z_2 & z_3 & z_4 & -z_3 & -z_4 & z_5 & z_6 & -z_5 & -z_6 \end{pmatrix}$$

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$\{3, 4\}$ is "orthogonal diagonal" of $\{1, 2\}$

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edge $\{10, 12\}$, $\quad 4y_6^2 + 4z_6^2 - 1 = 0$

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centre of mass condition

$$\begin{pmatrix} x_1 & x_1 & x_2 & x_2 & x_3 & x_4 & x_3 & x_4 & x_5 & -\sum_{i=1}^{5} x_i & x_5 & -\sum_{i=1}^{5} x_i \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & y_3 & y_4 & -y_3 & -y_4 & -\zeta\, s & \frac{1}{2}\, c & \zeta\, s & -\frac{1}{2}\, c \\ 0 & 0 & z_2 & -z_2 & z_3 & z_4 & -z_3 & -z_4 & \zeta\, c & \frac{1}{2}\, s & -\zeta\, c & -\frac{1}{2}\, s \end{pmatrix}$$

13 unknowns, $\quad c^2 + s^2 - 1 = 0$

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Primary decomposition (Magma):

$$I = \underbrace{\bigcap_{i=1}^{48} m_i}_{\deg 2} \cap \underbrace{\bigcap_{i=49}^{58} m_i}_{\deg 4} \cap \underbrace{\bigcap_{i=59}^{62} m_i}_{\deg 6} \cap \underbrace{\bigcap_{i=63}^{66} m_i}_{\deg 8} \cap \underbrace{\bigcap_{i=67}^{68} m_i}_{\deg 16} \cap \underbrace{\bigcap_{i=69}^{70} m_i}_{\deg 48} \cap \underbrace{m_{71}}_{\deg 120} \cap \underbrace{m_{72}}_{\deg 688}$$

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$$I = \underbrace{\bigcap_{i=1}^{48} m_i}_{\deg 2} \cap \underbrace{\bigcap_{i=49}^{58} m_i}_{\deg 4} \cap \underbrace{\bigcap_{i=59}^{62} m_i}_{\deg 6} \cap \underbrace{\bigcap_{i=63}^{66} m_i}_{\deg 8} \cap \underbrace{\bigcap_{i=67}^{68} m_i}_{\deg 16} \cap \underbrace{\bigcap_{i=69}^{70} m_i}_{\deg 48} \cap \underbrace{m_{71}}_{\deg 120} \cap \underbrace{m_{72}}_{\deg 688}$$

deg 2: 20 maximal ideals with 12 distinct vertices
5 non-equivalent Gram matrices with automorphism group $C_2 \times A_5$, $C_2 \times D_{10}$, $C_2 \times D_{10}$, C_2^2, C_2^2

deg 4: 8 maximal ideals with 12 distinct vertices
no positive semidefinite Gram matrix

deg 6: R/m has no real embeddings

Example: case $\delta(a) = \mathrm{diag}(1, -1, -1)$

deg 8: 4 maximal ideals with 12 distinct vertices
2 non-equivalent Gram matrices with automorphism group $C_2 \times D_{10}$

deg 16: 2 maximal ideals with 12 distinct vertices
2 of the 4 real embeddings yield positive semidefinite Gram matrices
2 non-equivalent Gram matrices with automorphism group $C_2 \times D_6$

deg 48: 2 maximal ideals with 12 distinct vertices
6 of the 10 real embeddings yield positive semidefinite Gram matrices
3 non-equivalent Gram matrices with automorphism group C_2^2

deg. 120: 6 of the 18 real embeddings yield pos. semidef. Gram matrices
1 Gram matrix with automorphism group C_2^2

deg. 688: 20 of the 48 real embeddings yield pos. semidef. Gram matrices
5 non-equivalent Gram matrices with automorphism group C_2

Classification

Proposition. There exist $\varepsilon > 0$ and a non-constant real analytic map $\Phi : [0, \varepsilon) \to \mathbb{R}^{3\times 12}$ such that $\Phi(t)$ is the coordinate matrix of a $\langle d \rangle$-invariant icosahedron for all but finitely many $t \in [0, \varepsilon)$.

There exist infinitely many isometry types of $\langle d \rangle$-invariant icosahedra.

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Proof.
$p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_{15} = 0$ (lin. indep.) quadratic equations for entries $y_1, y_2, \ldots, y_{16}$ of coord. matrix M corresp. to $\langle d \rangle$-symmetry

$$\tau(y_1, \ldots, y_{16}) := \big(\det \mathrm{D}p|_1, \ldots, (-1)^{i-1} \det \mathrm{D}p|_i, \ldots, -\det \mathrm{D}p|_{16}\big)$$

Solution to initial value problem

$$\begin{cases} \phi'(t) &= \tau(\phi_1(t), \ldots, \phi_{16}(t)), \\ \phi(0) &= y^0. \end{cases}$$

for $\tau(y^0) \neq 0$ gives curve of (possibly degenerate) icosahedra.
Rule out degeneracies... $\square$

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Linear algebra

Lemma. Let $G \in \mathbb{R}^{n \times n}$ be symmetric, positive semidefinite, of rank k.

(a) There exists $M \in \mathbb{R}^{k \times n}$ such that $M^{tr}\, M = G$.

(b) G and M have the same row space.

(c) If $L \in \mathbb{R}^{k \times n}$ also satisfies $L^{tr}\, L = G$, then $\exists!\, g \in \mathbb{R}^{k \times k}$ such that $L = g\, M$. Moreover, g is orthogonal.

Proof.

(a) $(E_1, \ldots, E_n)$ orthogonal basis of eigenrows of G with eigenvalues $\lambda_1 > 0, \ldots, \lambda_k > 0, \lambda_{k+1} = \ldots = \lambda_n = 0$. Rows $\frac{1}{\sqrt{\lambda_i}}\, E_i$ define M (because $E_i^{tr}\, E_i$ repres. orthogonal projection of $\mathbb{R}^{1 \times n}$ onto $\langle E_i \rangle$).

(b) Obvious.

(c) Clear from (b).

$\square$