Group pairings with equivalent rigidity properties for bar-joint and point-hyperplane frameworks

Bernd Schulze
(with Katie Clinch, Anthony Nixon and Walter Whiteley)

Lancaster University

June 13, 2019

Outline

1. Rigidity of Euclidean, spherical and point-hyperplane frameworks
2. Symmetric frameworks
3. Pairing symmetry groups in $\mathbb{S}^d$ and $\mathbb{R}^d$

Rigidity of Euclidean frameworks

- Framework in a space M: (G, p), where $G = (V, E)$ is a finite simple graph and $p : V \to M$ is a map.

Rigidity of Euclidean frameworks

- Framework in a space M: (G, p), where $G = (V, E)$ is a finite simple graph and $p : V \to M$ is a map.
- We first consider Euclidean frameworks (G, p) in $\mathbb{R}^d$.

Rigidity of Euclidean frameworks

- Framework in a space M: (G, p), where $G = (V, E)$ is a finite simple graph and $p : V \to M$ is a map.
- We first consider Euclidean frameworks (G, p) in $\mathbb{R}^d$.
- Given (G, p), is every framework (G, q) in an open neighborhood of p satisfying the same length constraints for the edges:

$$\|p_i - p_j\| = \text{const} \qquad (ij \in E),$$

congruent to (G, p)?

Rigidity of Euclidean frameworks

- Framework in a space M: (G, p), where $G = (V, E)$ is a finite simple graph and $p : V \to M$ is a map.
- We first consider Euclidean frameworks (G, p) in $\mathbb{R}^d$.
- Given (G, p), is every framework (G, q) in an open neighborhood of p satisfying the same length constraints for the edges:

$$\|p_i - p_j\| = \text{const} \qquad (ij \in E),$$

  congruent to (G, p)?
- If so, then (G, p) is called (locally) rigid. Otherwise (G, p) is called (locally) flexible.

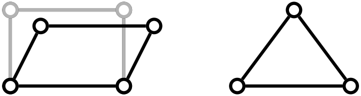


Figure: A flexible and a rigid framework in $\mathbb{R}^2$.

Infinitesimal rigidity of Euclidean frameworks

- It is common to analyse rigidity by taking the derivative of the square of each length constraint, which leads to the linear system:

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle = 0 \qquad (ij \in E).$$

We then check the dimension of the solution space with variables $\dot{p}$.

Infinitesimal rigidity of Euclidean frameworks

- It is common to analyse rigidity by taking the derivative of the square of each length constraint, which leads to the linear system:

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle = 0 \qquad (ij \in E).$$

  We then check the dimension of the solution space with variables $\dot{p}$.

- $\dot{p} : V \to \mathbb{R}^d$ is an infinitesimal motion of (G, p) if $\dot{p}$ satisfies the equations above.

Infinitesimal rigidity of Euclidean frameworks

- It is common to analyse rigidity by taking the derivative of the square of each length constraint, which leads to the linear system:

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle = 0 \qquad (ij \in E).$$

  We then check the dimension of the solution space with variables $\dot{p}$.

- $\dot{p} : V \to \mathbb{R}^d$ is an infinitesimal motion of (G, p) if $\dot{p}$ satisfies the equations above.

- (G, p) is called infinitesimally rigid if the dimension of the space of infinitesimal motions of (G, p) is equal to $\binom{d+1}{2}$ (assuming that the points $p(V)$ affinely span $\mathbb{R}^d$).

Infinitesimal rigidity of Euclidean frameworks

- It is common to analyse rigidity by taking the derivative of the square of each length constraint, which leads to the linear system:

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle = 0 \qquad (ij \in E).$$

  We then check the dimension of the solution space with variables $\dot{p}$.

- $\dot{p} : V \to \mathbb{R}^d$ is an infinitesimal motion of (G, p) if $\dot{p}$ satisfies the equations above.

- (G, p) is called infinitesimally rigid if the dimension of the space of infinitesimal motions of (G, p) is equal to $\binom{d+1}{2}$ (assuming that the points $p(V)$ affinely span $\mathbb{R}^d$).

- The matrix $R(G, p)$ corresponding to this linear system above is the rigidity matrix.

Infinitesimal rigidity of Euclidean frameworks

- It is common to analyse rigidity by taking the derivative of the square of each length constraint, which leads to the linear system:

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j \rangle = 0 \qquad (ij \in E).$$

  We then check the dimension of the solution space with variables $\dot{p}$.

- $\dot{p} : V \to \mathbb{R}^d$ is an infinitesimal motion of (G, p) if $\dot{p}$ satisfies the equations above.

- (G, p) is called infinitesimally rigid if the dimension of the space of infinitesimal motions of (G, p) is equal to $\binom{d+1}{2}$ (assuming that the points $p(V)$ affinely span $\mathbb{R}^d$).

- The matrix $R(G, p)$ corresponding to this linear system above is the rigidity matrix.

- For regular configurations p (i.e., $R(G, p)$ has maximum rank), rigidity is equivalent to infinitesimal rigidity.

Infinitesimal rigidity of spherical frameworks

- Next, we consider spherical frameworks (G, p) in $\mathbb{S}^d$.

Infinitesimal rigidity of spherical frameworks

- Next, we consider spherical frameworks (G, p) in $\mathbb{S}^d$.
- In $\mathbb{S}^d$ the 'distance' between two points is determined by their inner product. Hence we are interested in the solutions of:

$$\langle p_i, p_j \rangle = \text{const} \qquad (ij \in E).$$

Infinitesimal rigidity of spherical frameworks

- Next, we consider spherical frameworks (G, p) in $\mathbb{S}^d$.
- In $\mathbb{S}^d$ the 'distance' between two points is determined by their inner product. Hence we are interested in the solutions of:

$$\langle p_i, p_j \rangle = \text{const} \qquad (ij \in E).$$

- Since p_i is constrained to be on $\mathbb{S}^d$, we also have

$$\langle p_i, p_i \rangle = 1 \qquad (i \in V).$$

Infinitesimal rigidity of spherical frameworks

- Next, we consider spherical frameworks (G, p) in $\mathbb{S}^d$.
- In $\mathbb{S}^d$ the 'distance' between two points is determined by their inner product. Hence we are interested in the solutions of:

$$\langle p_i, p_j \rangle = \text{const} \qquad (ij \in E).$$

- Since p_i is constrained to be on $\mathbb{S}^d$, we also have

$$\langle p_i, p_i \rangle = 1 \qquad (i \in V).$$

- Again, taking derivatives, we obtain the following system of first-order inner product constraints:

$$\begin{aligned} \langle p_i, \dot{p}_j \rangle + \langle p_j, \dot{p}_i \rangle &= 0 \qquad (ij \in E) \\ \langle p_i, \dot{p}_i \rangle &= 0 \qquad (i \in V). \end{aligned}$$

Infinitesimal rigidity of spherical frameworks

- Next, we consider spherical frameworks (G, p) in $\mathbb{S}^d$.
- In $\mathbb{S}^d$ the 'distance' between two points is determined by their inner product. Hence we are interested in the solutions of:

$$\langle p_i, p_j \rangle = \text{const} \qquad (ij \in E).$$

- Since p_i is constrained to be on $\mathbb{S}^d$, we also have

$$\langle p_i, p_i \rangle = 1 \qquad (i \in V).$$

- Again, taking derivatives, we obtain the following system of first-order inner product constraints:

$$\begin{aligned} \langle p_i, \dot{p}_j \rangle + \langle p_j, \dot{p}_i \rangle &= 0 \qquad (ij \in E) \\ \langle p_i, \dot{p}_i \rangle &= 0 \qquad (i \in V). \end{aligned}$$

- $\dot{p} : V \to \mathbb{R}^{d+1}$ is called an infinitesimal motion of (G, p) if it satisfies this system of linear constraints, and (G, p) is infinitesimally rigid if the dimension of its space of infinitesimal motions is equal to $\binom{d+1}{2}$ (assuming the points $p(V)$ linearly span $\mathbb{R}^{d+1}$).

Transfer between $\mathbb{S}^d_{>0}$ and $\mathbb{A}^d$ (or $\mathbb{R}^d$)

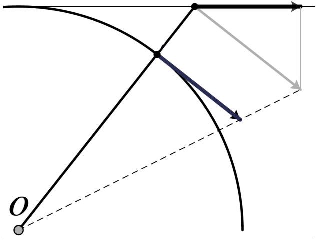


Figure: The transfer of infinitesimal motions between $\mathbb{S}^d_{>0}$ and $\mathbb{A}^d$

Theorem (S. and Whiteley, 2012): A bar-joint framework (G, p) is infinitesimally rigid in $\mathbb{A}^d$ if and only if $(G, \phi \circ p)$ is infinitesimally rigid in $\mathbb{S}^d_{>0}$, where ϕ is the central projection from the origin O.

Transfer between $\mathbb{S}^d_{>0}$ and $\mathbb{A}^d$ (or $\mathbb{R}^d$)

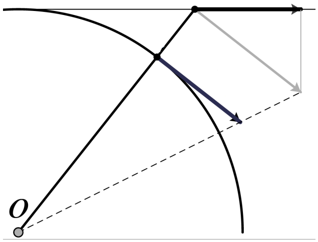


Figure: The transfer of infinitesimal motions between $\mathbb{S}^d_{>0}$ and $\mathbb{A}^d$

Theorem (S. and Whiteley, 2012): A bar-joint framework (G, p) is infinitesimally rigid in $\mathbb{A}^d$ if and only if $(G, \phi \circ p)$ is infinitesimally rigid in $\mathbb{S}^d_{>0}$, where ϕ is the central projection from the origin O.

Moreover, infinitesimal rigidity properties of (G, p) in $\mathbb{S}^d$ remain unchanged if any subset of the joints are inverted in O.

Point-line frameworks

- Jackson and Owen introduced the notion of a point-line framework in $\mathbb{R}^2$.

  Such a framework consists of points and lines in the plane which are linked by point-point distance constraints, point-line distance constraints, and line-line angle constraints.

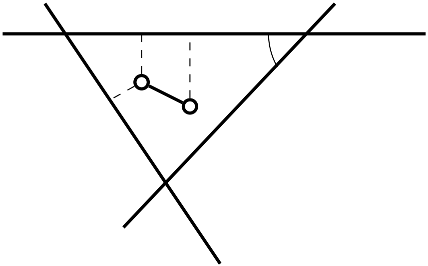


Figure: Point-line framework in $\mathbb{R}^2$.

Point-line frameworks

- Jackson and Owen introduced the notion of a point-line framework in $\mathbb{R}^2$. Such a framework consists of points and lines in the plane which are linked by point-point distance constraints, point-line distance constraints, and line-line angle constraints.

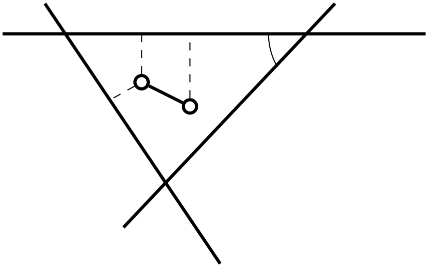


Figure: Point-line framework in $\mathbb{R}^2$.

- This was recently generalised to point-hyperplane frameworks in $\mathbb{R}^d$:

Point-hyperplane frameworks

- A point-hyperplane framework in $\mathbb{R}^d$ is a triple (G, p, ℓ):
  - $G = (V_P \cup V_H, E)$ is a graph. (V_P point vertices; V_H hyperplane vertices);
  - $p : V_P \to \mathbb{R}^d$;
  - $\ell = (a, r) : V_H \to \mathbb{S}^{d-1} \times \mathbb{R}$.
- Each j in V_H is mapped to the hyperplane $\{x \in \mathbb{R}^d : \langle a_j, x\rangle + r_j = 0\}$.

Point-hyperplane frameworks

- A point-hyperplane framework in $\mathbb{R}^d$ is a triple (G, p, ℓ):
  - $G = (V_P \cup V_H, E)$ is a graph. (V_P point vertices; V_H hyperplane vertices);
  - $p : V_P \to \mathbb{R}^d$;
  - $\ell = (a, r) : V_H \to \mathbb{S}^{d-1} \times \mathbb{R}$.
- Each j in V_H is mapped to the hyperplane $\{x \in \mathbb{R}^d : \langle a_j, x\rangle + r_j = 0\}$.
- An infinitesimal motion $(\dot{p}, \dot{\ell})$ of (G, p, ℓ) satisfies:

$$\langle p_i - p_j, \dot{p}_i - \dot{p}_j\rangle = 0 \qquad (ij \in E_{PP}) \qquad (1)$$

$$\langle p_i, \dot{a}_j\rangle + \langle \dot{p}_i, a_j\rangle + \dot{r}_j = 0 \qquad (ij \in E_{PH}) \qquad (2)$$

$$\langle a_i, \dot{a}_j\rangle + \langle \dot{a}_i, a_j\rangle = 0 \qquad (ij \in E_{HH}) \qquad (3)$$

$$\langle a_i, \dot{a}_i\rangle = 0 \qquad (i \in V_H). \qquad (4)$$

Point-hyperplane frameworks

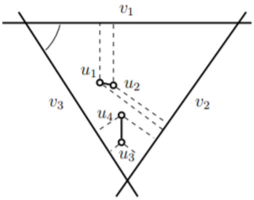
- A point-hyperplane framework in $\mathbb{R}^d$ is a triple (G, p, ℓ):
  - $G = (V_P \cup V_H, E)$ is a graph. (V_P point vertices; V_H hyperplane vertices);
  - $p : V_P \to \mathbb{R}^d$;
  - $\ell = (a, r) : V_H \to \mathbb{S}^{d-1} \times \mathbb{R}$.
- Each j in V_H is mapped to the hyperplane $\{x \in \mathbb{R}^d : \langle a_j, x\rangle + r_j = 0\}$.
- An infinitesimal motion $(\dot{p}, \dot{\ell})$ of (G, p, ℓ) satisfies:

$$\begin{aligned} \langle p_i - p_j, \dot{p}_i - \dot{p}_j\rangle &= 0 & (ij \in E_{PP}) \quad (1)\\ \langle p_i, \dot{a}_j\rangle + \langle \dot{p}_i, a_j\rangle + \dot{r}_j &= 0 & (ij \in E_{PH}) \quad (2)\\ \langle a_i, \dot{a}_j\rangle + \langle \dot{a}_i, a_j\rangle &= 0 & (ij \in E_{HH}) \quad (3)\\ \langle a_i, \dot{a}_i\rangle &= 0 & (i \in V_H). \quad (4) \end{aligned}$$

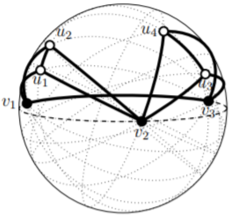
- (G, p, ℓ) is infinitesimally rigid if the dimension of the space of its infinitesimal motions is equal to $\binom{d+1}{2}$ (assuming points and hyperplanes affinely span $\mathbb{R}^d$).

Transfer between $\mathbb{S}^d$ and $\mathbb{R}^d$

- **Theorem (Eftekhari, Jackson, Nixon, S., Tanigawa, Whiteley, 2019):** Let $G = (V, E)$ be a graph and $X \subseteq V$. TFAE:
  - (a) G can be realised as an infinitesimally rigid bar-joint framework on $\mathbb{S}^d$ such that the points assigned to X lie on the equator.
  - (b) G can be realised as an infinitesimally rigid point-hyperplane framework in $\mathbb{R}^d$ such that each vertex in X is realised as a hyperplane and each vertex in $V \setminus X$ is realised as a point.



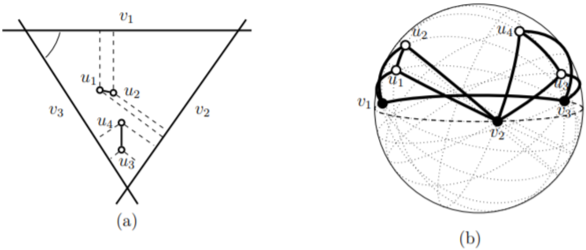
(a)



(b)

Transfer between $\mathbb{S}^d$ and $\mathbb{R}^d$

- **Theorem (Eftekhari, Jackson, Nixon, S., Tanigawa, Whiteley, 2019):** Let $G = (V, E)$ be a graph and $X \subseteq V$. TFAE:
  - (a) G can be realised as an infinitesimally rigid bar-joint framework on $\mathbb{S}^d$ such that the points assigned to X lie on the equator.
  - (b) G can be realised as an infinitesimally rigid point-hyperplane framework in $\mathbb{R}^d$ such that each vertex in X is realised as a hyperplane and each vertex in $V \setminus X$ is realised as a point.



- (a),(b) are also equivalent to:
  - (c) G can be realised as an infinitesimally rigid bar-joint framework in $\mathbb{R}^d$ such that the points assigned to X lie on a hyperplane.

Symmetric frameworks

Symmetric frameworks

- **Def.:** Let Γ be a finite group. A graph G is Γ-symmetric (with respect to θ) if there exists a homomorphism (an action) $\theta : \Gamma \to \mathrm{Aut}(G)$.

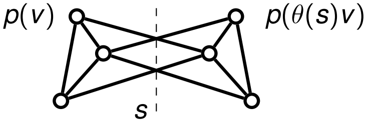
Symmetric frameworks

- **Def.:** Let Γ be a finite group. A graph G is Γ-symmetric (with respect to θ) if there exists a homomorphism (an action) $\theta : \Gamma \to \mathrm{Aut}(G)$.
- **Def.:** A framework (G, p) is Γ-symmetric (with respect to θ and τ) if

$$\tau(\gamma)p(v) = p(\theta(\gamma)v) \text{ for all } v \in V(G) \text{ and } \gamma \in \Gamma,$$

  where $\tau : \Gamma \to O(\mathbb{R}^d)$.

- **Example:**



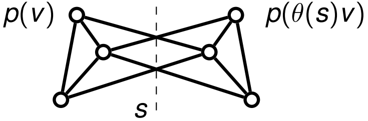
Symmetric frameworks

- **Def.:** Let Γ be a finite group. A graph G is Γ-symmetric (with respect to θ) if there exists a homomorphism (an action) $\theta : \Gamma \to \mathrm{Aut}(G)$.
- **Def.:** A framework (G, p) is Γ-symmetric (with respect to θ and τ) if

$$\tau(\gamma)p(v) = p(\theta(\gamma)v) \text{ for all } v \in V(G) \text{ and } \gamma \in \Gamma,$$

where $\tau : \Gamma \to O(\mathbb{R}^d)$.

- **Example:**



- The definitions of a Γ-symmetric spherical framework or point-hyperplane framework are analogous.
(Hyperplane vertices map to hyperplane vertices, and point vertices to point vertices!)

Forced vs incidental symmetry

Two basic approaches to the rigidity analysis of symmetric frameworks:

1. Forced symmetry:

   The framework must maintain symmetry with respect to a specific group throughout its motion.

2. Incidental symmetry:

   The framework starts in a symmetric position, but may move in unrestricted ways.

Forced vs incidental symmetry

Two basic approaches to the rigidity analysis of symmetric frameworks:

1. Forced symmetry:

   The framework must maintain symmetry with respect to a specific group throughout its motion.

2. Incidental symmetry:

   The framework starts in a symmetric position, but may move in unrestricted ways.

Combinatorial results for 'Γ-regular' frameworks (where θ acts freely on $V(G)$):

- Forced bar-joint in $\mathbb{R}^2$: $\mathcal{C}_s$, $\mathcal{C}_n$, $n \in \mathbb{N}$, and $\mathcal{C}_{(2n+1)v}$, $n \in \mathbb{N}$
- Forced bar-joint in $\mathbb{S}^2$: $\mathcal{C}_s$, $\mathcal{C}_n$, $n \in \mathbb{N}$, $\mathcal{C}_i$, $\mathcal{C}_{nv}$, n odd, $\mathcal{C}_{nh}$, n odd, and $\mathcal{S}_{2n}$, n even.
- Incidental bar-joint in $\mathbb{R}^2$: $\mathcal{C}_s$, $\mathcal{C}_n$, n odd.

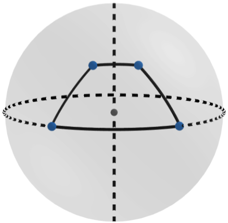
(Work by Ikeshita, Jordán, Kaszanitzky, Malestein, Nixon, S., Tanigawa, Theran, etc.)

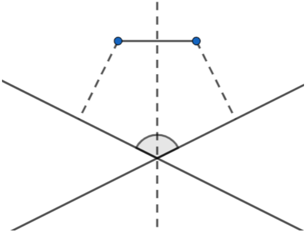
Symmetric transfer between $\mathbb{S}^d$ and $\mathbb{R}^d$

Theorem (Clinch, Nixon, S., Whiteley, 2019+): Let $G = (V, E)$ be a graph, $X \subseteq V$, and $\tau(\Gamma)$ be a symmetry group in $\mathbb{R}^d$. TFAE:

(a) G can be realised as an infinitesimally rigid Γ-symmetric bar-joint framework on $\mathbb{S}^d$ (with respect to θ and $\tilde{\tau}$) such that the points assigned to X lie on the equator.

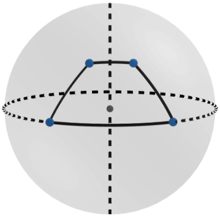
(b) G can be realised as an infinitesimally rigid Γ-symmetric point-hyperplane framework in $\mathbb{R}^d$ (with respect to θ and τ) such that each vertex in X is realised as a hyperplane and each vertex in $V \setminus X$ is realised as a point.

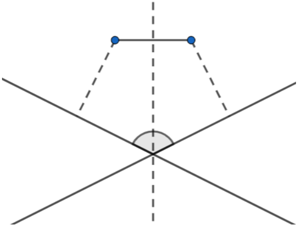




Remarks on symmetric transfer

- This transfer takes ‘Γ-X-regular’ spherical frameworks to Γ-regular point-hyperplane frameworks.
- For $X = \emptyset$, we obtain combinatorial results for $\mathcal{C}_s$, $\mathcal{C}_n$, n odd, on $\mathbb{S}^2$ from the corresponding results in $\mathbb{R}^2$.
- In the case of $\mathcal{C}_s$, statements (a),(b) are equivalent to:
  - (c) G can be realised as an infinitesimally rigid Γ-symmetric bar-joint framework in $\mathbb{R}^d$ (with respect to θ and τ) such that the points assigned to X lie on a hyperplane (perpendicular to the mirror hyperplane).
- This transfer also preserves forced Γ-symmetric infinitesimal rigidity.

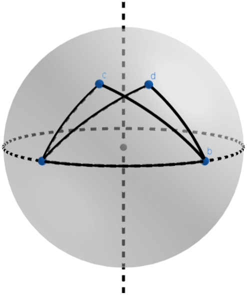


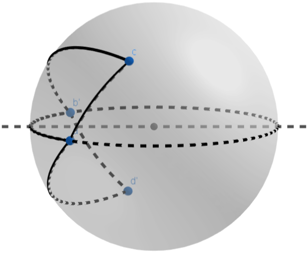


Pairing symmetry groups in $\mathbb{S}^d$ and $\mathbb{R}^d$

$\mathcal{C}_2$ and $\mathcal{C}_s$ on $\mathbb{S}^2$

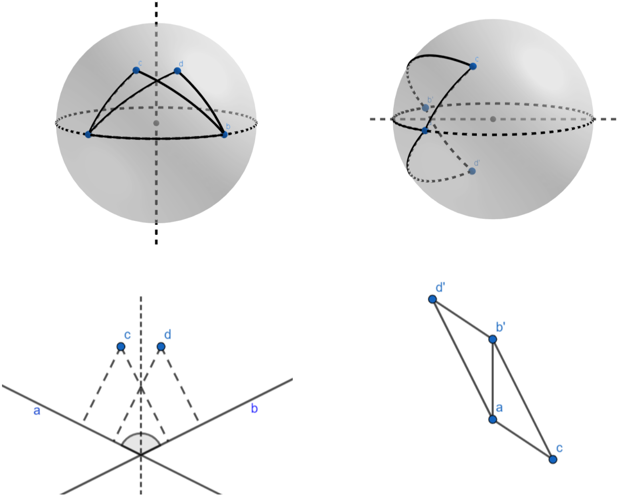
- **Theorem**: Let $G = (V, E)$ be a graph and let $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(G)$ act freely on V. Further, let $X \subseteq V$. TFAE:
  - (a) G can be realised as a $\mathbb{Z}_2$-symmetric (resp. forced $\mathbb{Z}_2$-symmetric) infinitesimally rigid bar-joint framework on $\mathbb{S}^2$ with respect to θ and $\tau : \mathbb{Z}_2 \to \mathcal{C}_s$, where points assigned to X lie on a great circle.
  - (b) G can be realised as a $\mathbb{Z}_2$-symmetric (resp. forced $\mathbb{Z}_2$-symmetric) bar-joint framework on $\mathbb{S}^2$ with respect to θ and $\tau' : \mathbb{Z}_2 \to \mathcal{C}_2$, where points assigned to X lie on a great circle.





$\mathcal{C}_2$ and $\mathcal{C}_s$ in $\mathbb{R}^2$

Now project this pair of frameworks from $\mathbb{S}^2$ to $\mathbb{R}^2$:



Remarks on the $\mathcal{C}_s$ and $\mathcal{C}_2$ pairing

- For $X = \emptyset$, this transfer preserves Γ-regularity of bar-joint frameworks.

Remarks on the $\mathcal{C}_s$ and $\mathcal{C}_2$ pairing

- For $X = \emptyset$, this transfer preserves Γ-regularity of bar-joint frameworks.
- This gives a geometric reason why mirror and half-turn symmetry have the same combinatorial characterisation for $\mathbb{Z}_2$-regular infinitesimal rigidity (resp. forced $\mathbb{Z}_2$-symmetric infinitesimal rigidity) on $\mathbb{S}^2$, as well as in $\mathbb{R}^2$.

Remarks on the $\mathcal{C}_s$ and $\mathcal{C}_2$ pairing

- For $X = \emptyset$, this transfer preserves Γ-regularity of bar-joint frameworks.
- This gives a geometric reason why mirror and half-turn symmetry have the same combinatorial characterisation for $\mathbb{Z}_2$-regular infinitesimal rigidity (resp. forced $\mathbb{Z}_2$-symmetric infinitesimal rigidity) on $\mathbb{S}^2$, as well as in $\mathbb{R}^2$.
- For Γ-regular frameworks, this also allows us to transfer continuous (symmetry-preserving) flexibility between frameworks with mirror and half-turn symmetry.

Remarks on the $\mathcal{C}_s$ and $\mathcal{C}_2$ pairing

- For $X = \emptyset$, this transfer preserves Γ-regularity of bar-joint frameworks.

- This gives a geometric reason why mirror and half-turn symmetry have the same combinatorial characterisation for $\mathbb{Z}_2$-regular infinitesimal rigidity (resp. forced $\mathbb{Z}_2$-symmetric infinitesimal rigidity) on $\mathbb{S}^2$, as well as in $\mathbb{R}^2$.

- For Γ-regular frameworks, this also allows us to transfer continuous (symmetry-preserving) flexibility between frameworks with mirror and half-turn symmetry.

- From the known results for bar-joint frameworks with $\mathcal{C}_2$ symmetry, we obtain a combinatorial characterisation of $\mathbb{Z}_2$-regular (forced) infinitesimally rigid point-line frameworks in $\mathbb{R}^2$ with respect to $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(G)$ and $\tau : \mathbb{Z}_2 \to \mathcal{C}_s$, where $|V_H| = 2$ and θ acts freely on $V = V_P \cup V_H$.

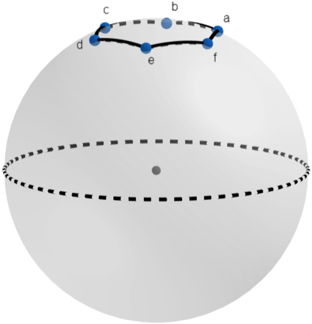
All group pairings in $\mathbb{S}^2$

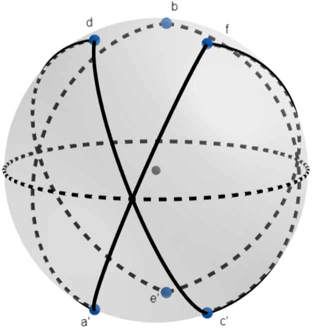
- **Def.:** $\mathcal{G} \leftrightarrow \mathcal{H}$ for symmetry groups $\mathcal{G}$ and $\mathcal{H}$ in dimension 3 (with the same abstract group Γ):
  there exists a Γ-symmetric framework (G, p) on $\mathbb{S}^2$ with respect to θ and $\tau(\Gamma) = \mathcal{G}$, and a Γ-symmetric framework (G, q) on $\mathbb{S}^2$ with respect to θ and $\tau'(\Gamma) = \mathcal{H}$ such that (G, q) is obtained from (G, p) by taking an index 2 subgroup Γ' of Γ and inverting each point of (G, p) assigned to the set $V \setminus \{\gamma v : \gamma \in \Gamma', v \in V_0\}$, where V_0 is a set of representatives for the vertex orbits under the group action θ.

- **Theorem (Clinch, Nixon, S., Whiteley, 2019+):** If $\tau(\Gamma) \leftrightarrow \tau'(\Gamma)$, then it must be one of the following pairings:
  - $\mathcal{C}_2 \leftrightarrow \mathcal{C}_s$;
  - $\mathcal{C}_{2n} \leftrightarrow \mathcal{C}_{nh}$ where n is odd;
  - $\mathcal{C}_{2n} \leftrightarrow \mathcal{S}_{2n}$ where n is even;
  - $\mathcal{C}_{nv} \leftrightarrow \mathcal{D}_n$ for all n;
  - $\mathcal{C}_{2nv} \leftrightarrow \mathcal{D}_{nd}$ where n is even;
  - $\mathcal{C}_{2nv} \leftrightarrow \mathcal{D}_{nh}$ where n is odd;
  - $\mathcal{T}_d \leftrightarrow \mathcal{O}$.

Example: $\mathcal{C}_{2n}$ and $\mathcal{C}_{nh}$ on $\mathbb{S}^2$

Illustration of $\mathcal{C}_6 \leftrightarrow \mathcal{C}_{3h}$:





Remarks on group pairings

- These pairings preserve (forced) infinitesimal rigidity as well as Γ-regularity, and hence give new combinatorial insights and results for bar-joint frameworks on the sphere. For example, from $\mathcal{C}_{nv} \leftrightarrow \mathcal{D}_n$, n odd, we obtain:

Remarks on group pairings

- These pairings preserve (forced) infinitesimal rigidity as well as Γ-regularity, and hence give new combinatorial insights and results for bar-joint frameworks on the sphere. For example, from $\mathcal{C}_{nv} \leftrightarrow \mathcal{D}_n$, n odd, we obtain:

- **Theorem:** Let (G, p) be a Γ-regular framework on $\mathbb{S}^2$ with respect to θ and τ, where $\tau(\Gamma) = \mathcal{D}_n$, n odd. Let (G_0, ψ) be the quotient Γ-gain graph of G. Then (G, p) is forced Γ-symmetric infinitesimally rigid if and only if (G_0, ψ) contains a spanning subgraph that is maximum $\mathcal{D}_n$-tight.

Remarks on group pairings

- These pairings preserve (forced) infinitesimal rigidity as well as Γ-regularity, and hence give new combinatorial insights and results for bar-joint frameworks on the sphere. For example, from $\mathcal{C}_{nv} \leftrightarrow \mathcal{D}_n$, n odd, we obtain:

- **Theorem:** Let (G, p) be a Γ-regular framework on $\mathbb{S}^2$ with respect to θ and τ, where $\tau(\Gamma) = \mathcal{D}_n$, n odd. Let (G_0, ψ) be the quotient Γ-gain graph of G. Then (G, p) is forced Γ-symmetric infinitesimally rigid if and only if (G_0, ψ) contains a spanning subgraph that is maximum $\mathcal{D}_n$-tight.

- Similar pairing results can be established in higher dimensions. We have checked all group pairings in $\mathbb{R}^3$ for groups containing only inversions. The pairs are $\mathcal{C}_s \leftrightarrow \mathcal{C}_i$ and $\mathcal{C}_{2v} \leftrightarrow \mathcal{C}_{2h}$.

Remarks on group pairings

- These pairings preserve (forced) infinitesimal rigidity as well as Γ-regularity, and hence give new combinatorial insights and results for bar-joint frameworks on the sphere. For example, from $\mathcal{C}_{nv} \leftrightarrow \mathcal{D}_n$, n odd, we obtain:

- **Theorem:** Let (G, p) be a Γ-regular framework on $\mathbb{S}^2$ with respect to θ and τ, where $\tau(\Gamma) = \mathcal{D}_n$, n odd. Let (G_0, ψ) be the quotient Γ-gain graph of G. Then (G, p) is forced Γ-symmetric infinitesimally rigid if and only if (G_0, ψ) contains a spanning subgraph that is maximum $\mathcal{D}_n$-tight.

- Similar pairing results can be established in higher dimensions. We have checked all group pairings in $\mathbb{R}^3$ for groups containing only inversions. The pairs are $\mathcal{C}_s \leftrightarrow \mathcal{C}_i$ and $\mathcal{C}_{2v} \leftrightarrow \mathcal{C}_{2h}$.

- This gives a geometric reason why mirror and inversion symmetry have the same combinatorial characterisations for $\mathbb{Z}_2$-regular infinitesimal rigidity (resp. forced $\mathbb{Z}_2$-symmetric infinitesimal rigidity) for body-bar frameworks in $\mathbb{R}^3$ (see S. and Tanigawa, 2014).

Non-free actions on vertices

- **Theorem (S., 2010):** Let $\Gamma = \langle\gamma\rangle$ and let (G, p) be a Γ-regular bar-joint framework (with respect to θ and τ) in $\mathbb{R}^2$, where $\tau(\Gamma) \in \{\mathcal{C}_s, \mathcal{C}_2, \mathcal{C}_3\}$. Then (G, p) is isostatic if and only if G is $(2, 3)$-tight and
  (i) $|E_\gamma| = 1$ for $\tau(\Gamma) = \mathcal{C}_s$.
  (ii) $|V_\gamma| = 0$ and $|E_\gamma| = 1$ for $\tau(\Gamma) = \mathcal{C}_2$.
  (iii) $|V_\gamma| = 0$ for $\tau(\Gamma) = \mathcal{C}_3$.

Non-free actions on vertices

- **Theorem (S., 2010):** Let $\Gamma = \langle\gamma\rangle$ and let (G, p) be a Γ-regular bar-joint framework (with respect to θ and τ) in $\mathbb{R}^2$, where $\tau(\Gamma) \in \{\mathcal{C}_s, \mathcal{C}_2, \mathcal{C}_3\}$. Then (G, p) is isostatic if and only if G is $(2, 3)$-tight and
  (i) $|E_\gamma| = 1$ for $\tau(\Gamma) = \mathcal{C}_s$.
  (ii) $|V_\gamma| = 0$ and $|E_\gamma| = 1$ for $\tau(\Gamma) = \mathcal{C}_2$.
  (iii) $|V_\gamma| = 0$ for $\tau(\Gamma) = \mathcal{C}_3$.

- From this result, together with our transfer results, we can get necessary conditions for symmetric point-line frameworks to be isostatic (see also Owen and Power, 2012). We also obtain a Laman-type result in a special case:

- **Theorem (2019+):** Let $\mathbb{Z}_2 = \langle\gamma\rangle$ and let (G, p, ℓ) be a $\mathbb{Z}_2$-regular point-line framework in $\mathbb{R}^2$ with respect to $\theta : \mathbb{Z}_2 \to \mathrm{Aut}_{PH}(G)$ and $\tau : \mathbb{Z}_2 \to \mathcal{C}_2$. Suppose that γ fixes each $i \in V_H$ and that θ acts freely on V_P. Then (G, p, ℓ) is isostatic if and only if G is $(2, 3)$-tight and $|E_\gamma| = 1$.

Thank you!

Questions?