

RIGIDITY THEORY IN MULTI-AGENT COORDINATION

A FUNDAMENTAL SYSTEM ARCHITECTURE

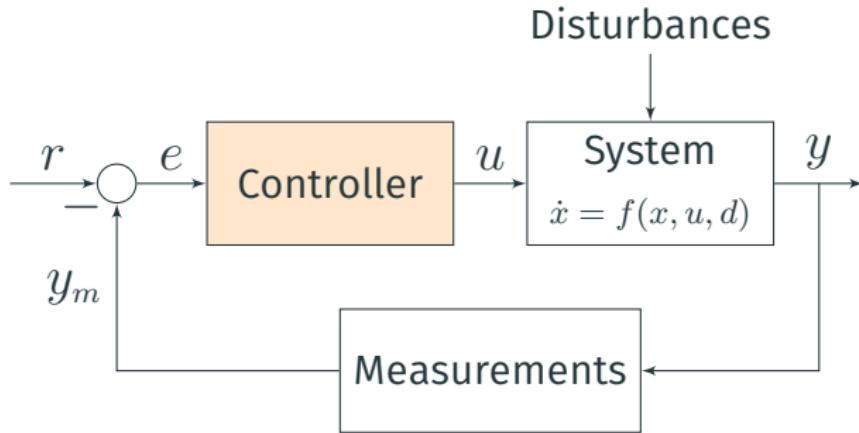
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Geometric Constraint Systems: Rigidity, Flexibility and Applications
Lancaster University



WHAT IS CONTROL THEORY?

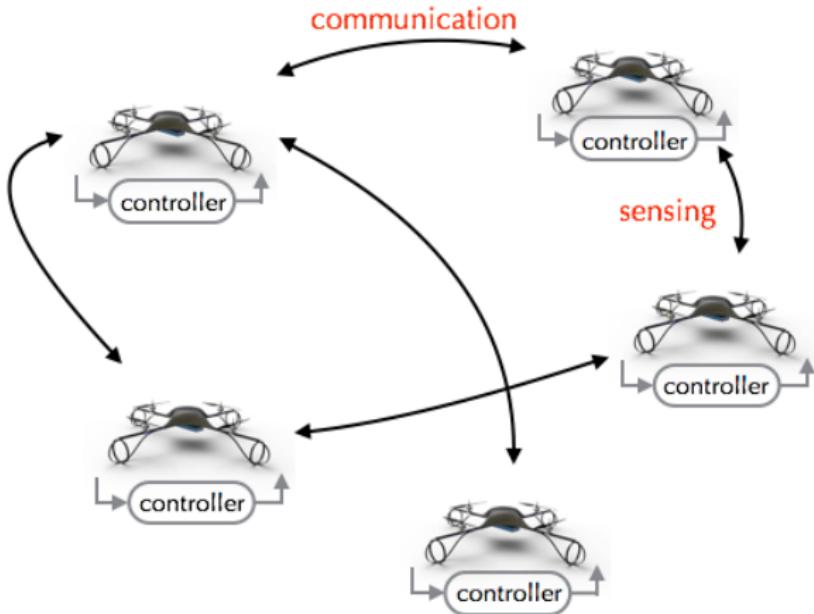
A CLASSIC CONTROL SYSTEM



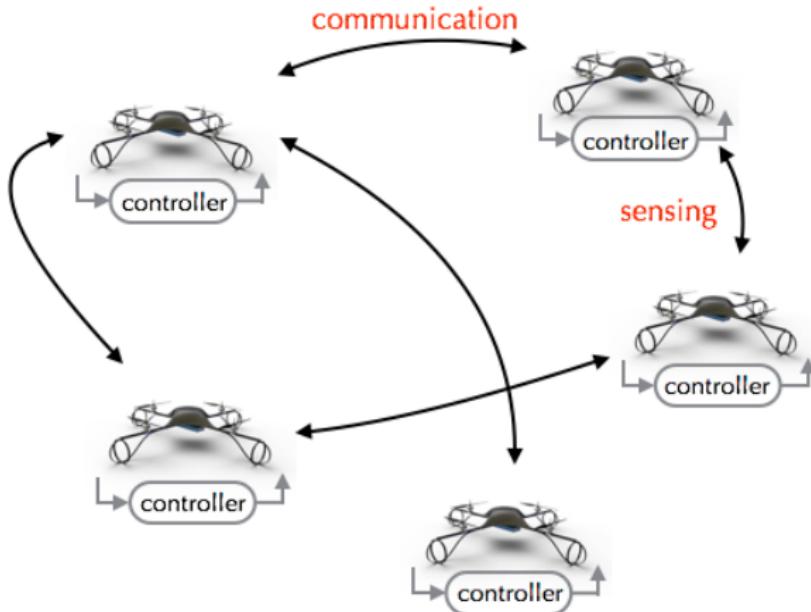
A control systems engineer aims to design a **controller** that ensures the closed-loop system

- ▶ is stable
- ▶ satisfies some performance criteria

WHAT ARE MULTI-AGENT SYSTEMS?



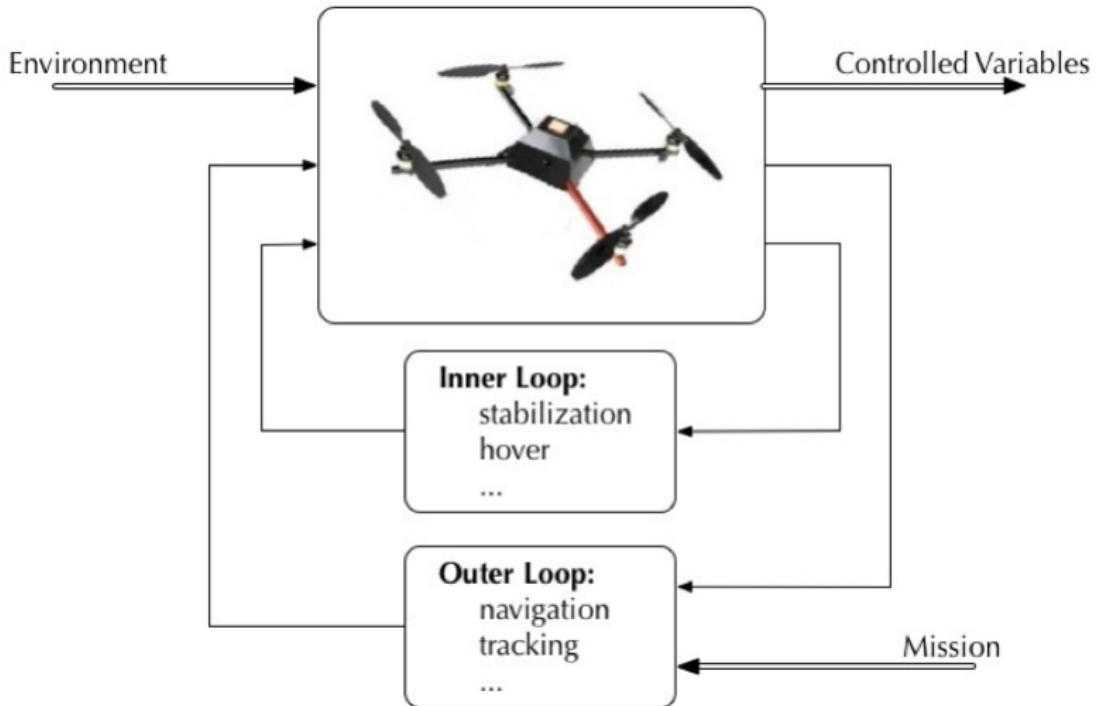
WHAT ARE MULTI-AGENT SYSTEMS?



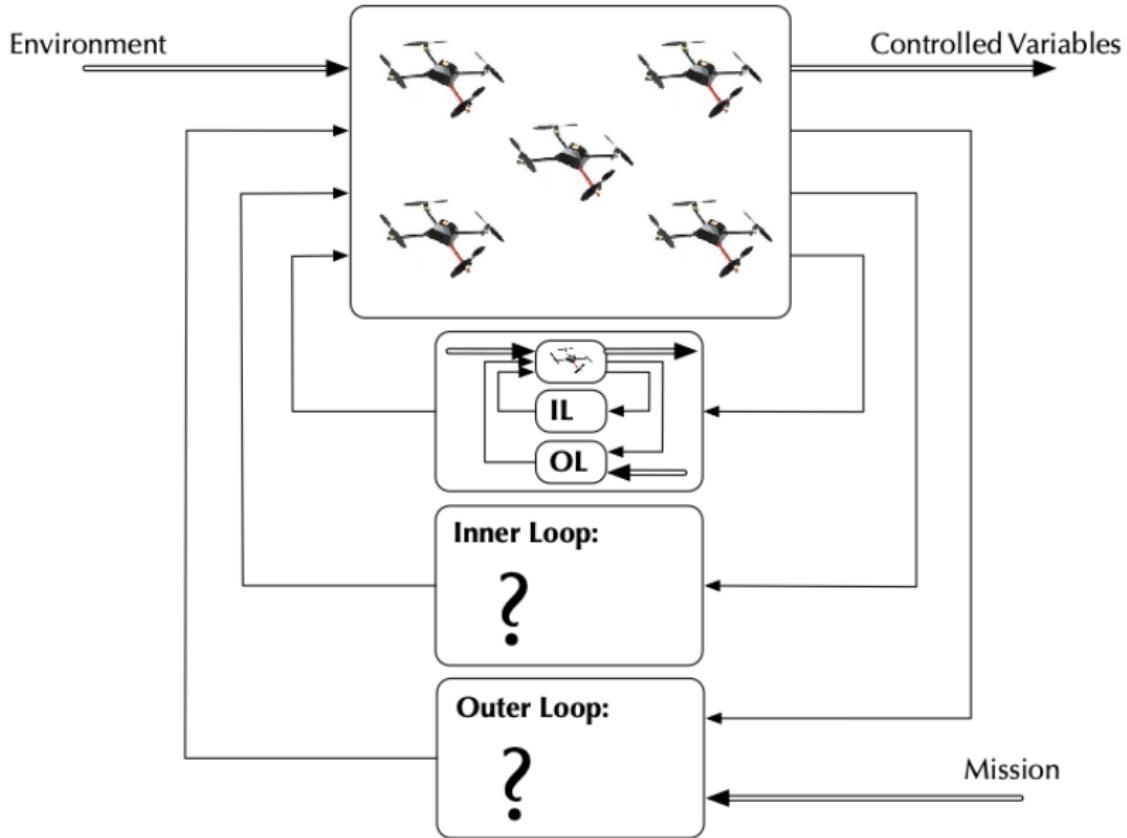
What is the right **control architecture**?

- ▶ of each agent
- ▶ of the information exchange layer

CONTROL ARCHITECTURES



CONTROL ARCHITECTURES



COORDINATION OBJECTIVES

rendezvous

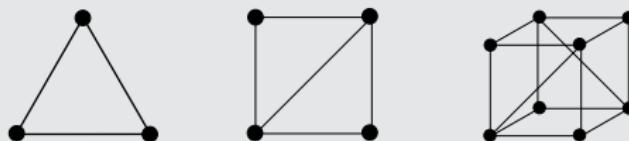
formation control

localization

- ▶ Does the control strategy need to change with different sensing/communication?
- ▶ Are there common architectural requirements that do not depend on the choice of sensing?

Formation Control Objective

Given a team of robots endowed with the ability to sense/communicate with neighboring robots, design a control for each robot using only local information that moves the team into a desired spatial configuration - the **formation**



AGENT DYNAMICS

Control Theory provides us with an analytical justification for using **simple** models!



INTEGRATOR DYNAMICS

$$\dot{x} = u_x$$

$$\dot{y} = u_y$$

$$\dot{z} = u_z$$



UNICYCLE DYNAMICS

$$\dot{x} = v_{\text{lin}} \cos(\psi)$$

$$\dot{y} = v_{\text{lin}} \sin(\psi)$$

$$\dot{\psi} = v_{\text{ang}}$$

AGENT CONFIGURATIONS

- we consider a team of n agents in a d -dimensional Euclidean space

$$p_i(t) \in \mathbb{R}^d$$

- the configuration of the agents at time t is the vector

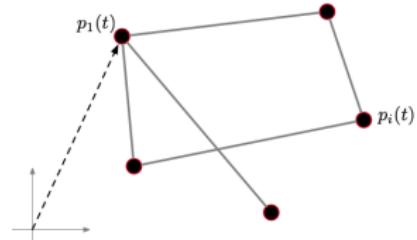
$$p(t) = \begin{bmatrix} p_1(t) \\ \vdots \\ p_n(t) \end{bmatrix} \in \mathbb{R}^{nd}$$

- agents modelled by single integrator dynamics

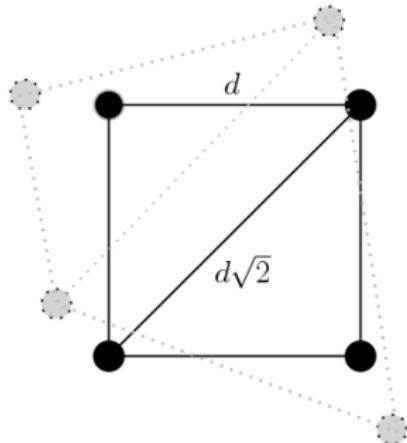
$$\dot{p}_i(t) = u_i(t), i = 1, \dots, n$$

- agents interact according to a sensing graph

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$



A FORMATION POTENTIAL



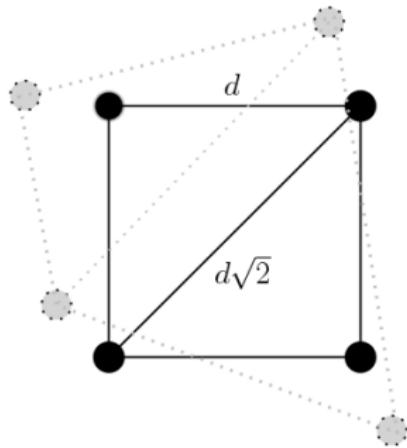
THE "FORMATION" POTENTIAL

$$\Phi(p) = \frac{1}{4} \sum_{i \sim j} (\|p_i - p_j\|^2 - d_{ij}^2)^2$$

A GRADIENT FLOW

$$\dot{p} = -\nabla_p \Phi(p)$$

A FORMATION POTENTIAL



THE "FORMATION" POTENTIAL

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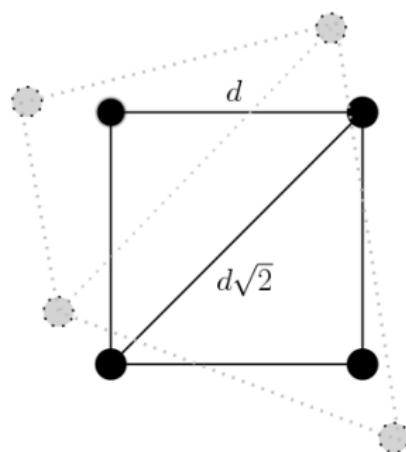
A GRADIENT FLOW

$$\dot{p} = -\nabla_p \Phi(p)$$

Theorem

The gradient dynamical system asymptotically converges to the critical points of the formation potential.

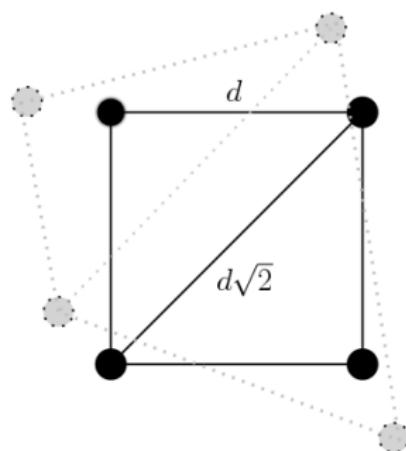
A DISTRIBUTED IMPLEMENTATION



Distributed Control

$$\dot{p}_i = \sum_{i \sim j} (\|p_i - p_j\|^2 - d_{ij}^2) (p_j - p_i)$$

A DISTRIBUTED IMPLEMENTATION



Distributed Control

$$\dot{p}_i = \sum_{i \sim j} (\|p_i - p_j\|^2 - d_{ij}^2) (p_j - p_i)$$

- ▶ Does this strategy solve the formation control problem?
- ▶ Does it reveal a necessary control architecture for the multi-agent system?

RIGIDITY MEETS FORMATION CONTROL

For a framework (\mathcal{G}, p) , we have

Edge Function

$$f_D(p) = \frac{1}{2} \begin{bmatrix} \vdots \\ \|p_i - p_j\|^2 \\ \vdots \end{bmatrix}$$

Rigidity Matrix

$$R_D(p) = \frac{\partial f_D(p)}{\partial p}$$

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For a framework (\mathcal{G}, p) , we have

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Rigidity Matrix

$$R_D(p) = \frac{\partial f_D(p)}{\partial p}$$

$$\begin{aligned}\dot{p} &= \nabla \Phi(p) = \frac{\partial}{\partial p} \left\| f_D(p) - \frac{1}{2} d^2 \right\|^2 \\ &= -R_D(p)^T R_D(p)p - R_D(p)^T d^2\end{aligned}$$

Theorem (Stability and Rigidity)

If the target formation is infinitesimally rigid, then the dynamics are (locally) asymptotically stable and satisfy

$$\lim_{t \rightarrow \infty} p(t) = p^*$$

where $\|p_i^* - p_j^*\|^2 = d_{ij}^2$ for all $\{i, j\} \in \mathcal{E}$.

L. Krick, M. E. Broucke & B. A. Francis, *Stabilisation of infinitesimally rigid formations of multi-robot networks*, International Journal of Control, 82(3):423-439, 2009.

A REAL ROBOT



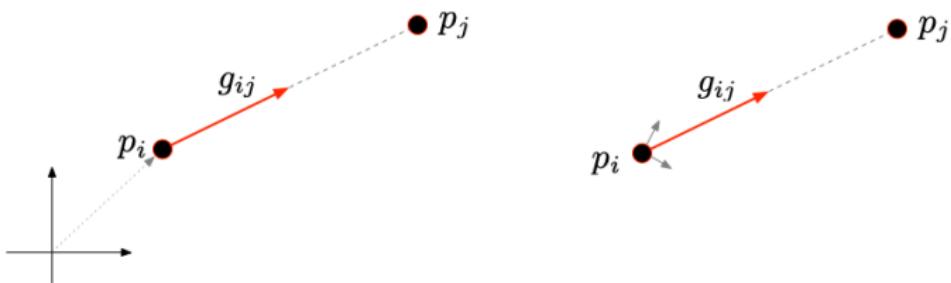
Bearing Sensing

The bearing between two agents is defined as the unit vector

$$g_{ij}(t) = \frac{p_j(t) - p_i(t)}{\|p_j(t) - p_i(t)\|},$$

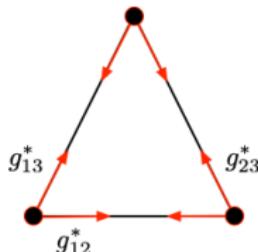
where $p_i(t)$ is the position of agent i .

- ▶ NOTE: g_{ij} can be expressed in a common frame or local frame



BEARING-ONLY FORMATION CONTROL

target formation specified by desired bearings



Formation Control Objective

Design u_i for each agent using only bearing measurements such that

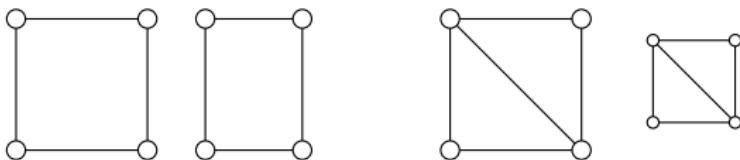
$$\lim_{t \rightarrow \infty} g_{ij}(t) = g_{ij}^*$$

for all pairs (i, j) in the sensing graph.

WHAT IS BEARING RIGIDITY?

Bearing Rigidity

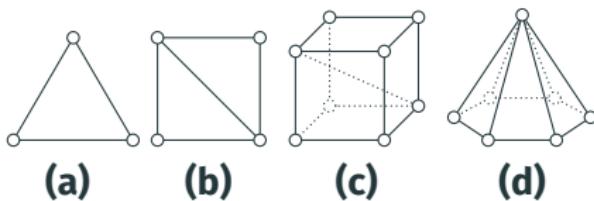
- ▶ If we fix the bearing of each edge in a network, can the geometric pattern of the network be uniquely determined?



- ▶ Intuitive definition: a network is **bearing rigid** if its bearings can uniquely determine its geometric pattern.

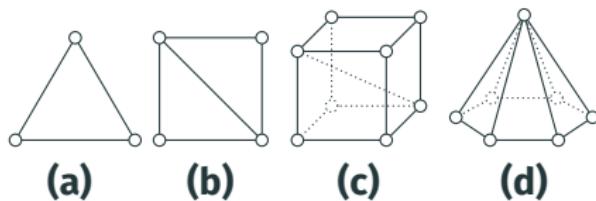
BEARING-EDGE FUNCTION

- ◊ How can one determine if a given network is bearing rigid?



BEARING-EDGE FUNCTION

- ◊ How can one determine if a given network is bearing rigid?



The Bearing-Edge Function

For a network with $|\mathcal{E}| = m$ edges, the **bearing-edge function** is defined as

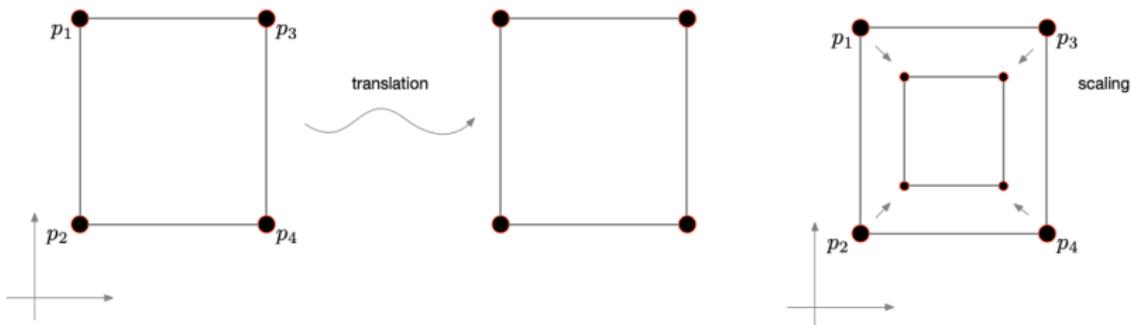
$$f_B(p) \triangleq \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix} \in \mathbb{R}^{dm}.$$

BEARING-PRESERVING MOTIONS

Bearing Trivial Motions

Trivial motions preserve the bearing between all pairs of agents for any framework

- ▶ (rigid body) **translations**
- ▶ **scaling**



INFINITESIMAL MOTION

Consider the Taylor-series expansion of the bearing-edge function:

$$f_B(p + \delta_p) = f_B(p) + \frac{\partial f_B(p)}{\partial p} \delta_p + h.o.t.$$

Infinitesimal Motions

An infinitesimal motion, δ_p , of a network satisfies

$$\frac{\partial f_B(p)}{\partial p} \delta_p = 0.$$

- ▶ first order "bearing-preserving" motions
- ▶ trivial motions are always infinitesimal motions

A RANK TEST

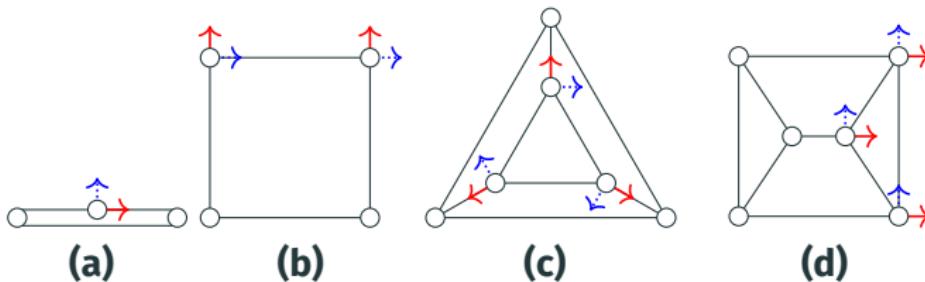
The Rigidity Matrix

$$R_B(p) \triangleq \frac{\partial f_B(p)}{\partial p}$$

Rank-Test for Bearing Rigidity

A network is infinitesimally bearing rigid if and only if
 $\text{rank}(R_B(p)) = dn - d - 1.$

Examples:



A RANK TEST

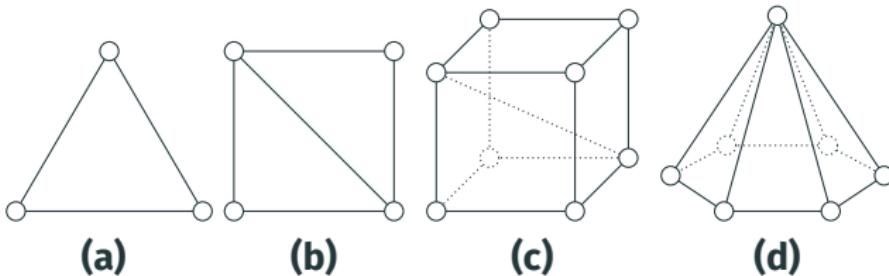
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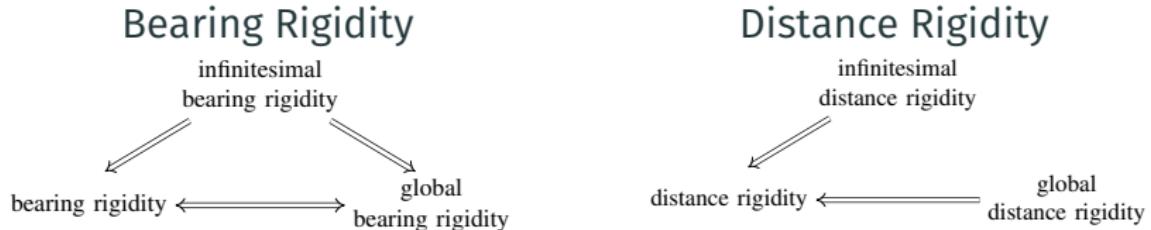
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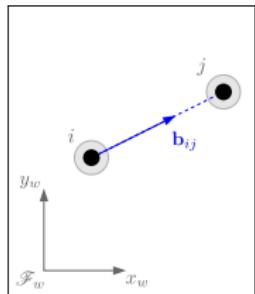
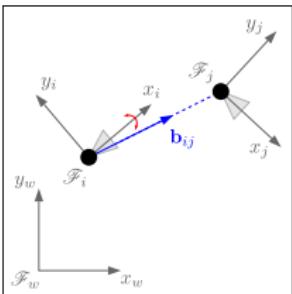
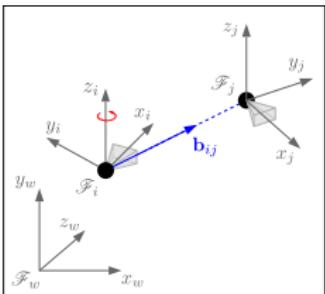
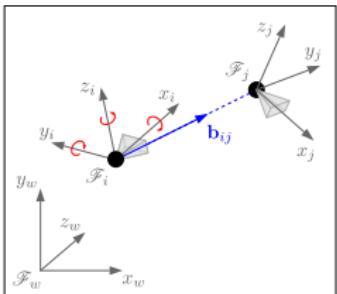


SOME FEATURES OF BEARING RIGIDITY



- ▶ in \mathbb{R}^2 , infinitesimal distance rigidity and infinitesimal bearing rigidity are equivalent
- ▶ infinitesimal bearing rigidity is preserved in lifted spaces
- ▶ Laman graphs are generically bearing rigid in arbitrary dimension
- ▶ at most $2n - 3$ edges are sufficient to ensure bearing rigidity in arbitrary dimension
- ▶ infinitesimal bearing rigid frameworks uniquely define a shape (modulo scale and translation)

DIRECTED BEARING RIGIDITY

(a) \mathbb{R}^2 (b) $\mathbb{R}^2 \times \mathbb{S}^1$ (c) $\mathbb{R}^3 \times \mathbb{S}^1$ (d) $SE(3)$

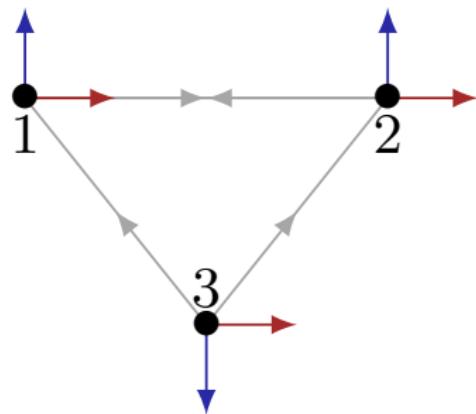
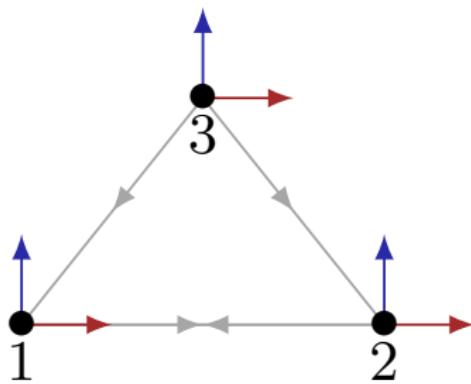
Bearing Rigidity Function

Given a n -agent formation modeled as a framework (\mathcal{G}, χ) in $\bar{\mathcal{D}}$, the bearing rigidity function is the map

$$\mathbf{b}_{\mathcal{G}}: \bar{\mathcal{D}} \rightarrow \bar{\mathcal{M}}, \quad \chi \mapsto \mathbf{b}_{\mathcal{G}}(\chi) = [\mathbf{b}_1^T \quad \cdots \quad \mathbf{b}_m^T]^T$$

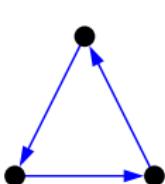
Trivial Motions

Trivial motions in $SE(2)$ are translations, scaling, and coordinated rotations

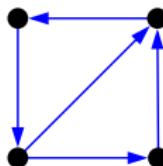


In directed bearing rigidity, local rigidity does not imply global rigidity

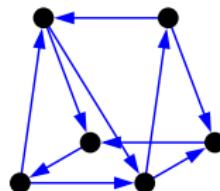
EXAMPLES



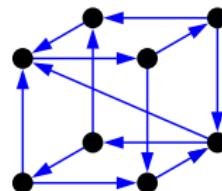
(a) $n = 3$



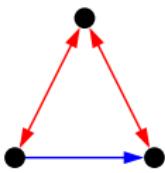
(b) $n = 4$



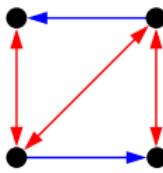
(c) $n = 6$



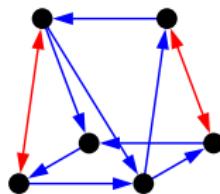
(d) $n = 8$



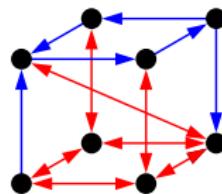
(e) $n = 3$



(f) $n = 4$



(g) $n = 6$



(h) $n = 8$

IBF frameworks in $(\mathbb{R}^2 \times \mathbb{S}^1)^n$ ((a),(b)), in $(\mathbb{R}^3 \times \mathbb{S}^1)^n$ with $\mathbf{n} = \mathbf{e}_3$ ((c),(d)). Examples of IBR frameworks in $(\mathbb{R}^2 \times \mathbb{S}^1)^n$ ((e),(f)) and in $(\mathbb{R}^3 \times \mathbb{S}^1)^n$ with $\mathbf{n} = \mathbf{e}_3$ ((g),(h)).

A GENERAL BEARING RIGIDITY MATRIX

For a framework (\mathcal{G}, χ) , the bearing rigidity matrix takes the form

$$\mathbf{B}_{\mathcal{G}}(\chi) = [\mathbf{B}_p \ \mathbf{B}_o] \in \mathbb{R}^{3m \times 6n},$$

with

$$\mathbf{B}_p = \mathbf{D}_p \bar{\mathbf{E}}^\top \in \mathbb{R}^{3m \times 3n} \quad \text{and} \quad \mathbf{B}_o = \mathbf{D}_o \bar{\mathbf{E}}_o^\top \in \mathbb{R}^{3m \times 3n} \quad (1)$$

\mathcal{D}	\mathbf{p}_i	\mathbf{R}_i	\mathbf{D}_p	\mathbf{D}_o
$SE(3)$	$[p_i^x \ p_i^y \ p_i^z]^\top$	$\mathbf{R}(\alpha_i, \beta_i, \gamma_i, \{\mathbf{e}_h\}_{h=1}^3)$	$\text{diag}(d_{ij} \mathbf{R}_i^\top \mathbf{P}(\bar{\mathbf{p}}_{ij}))$	$\text{diag}(\mathbf{R}_i^\top [\bar{\mathbf{p}}_{ij}]_\times \mathbf{I}_3)$
$\mathbb{R}^3 \times \mathbb{S}^1$	$[p_i^x \ p_i^y \ p_i^z]^\top$	$\mathbf{R}(\alpha_i, \mathbf{n}), \mathbf{n} = \sum_{h=1}^3 n_h \mathbf{e}_h$	$\text{diag}(d_{ij} \mathbf{R}_i^\top \mathbf{P}(\bar{\mathbf{p}}_{ij}))$	$\text{diag}(\mathbf{R}_i^\top [\bar{\mathbf{p}}_{ij}]_\times [\mathbf{0}_{3 \times 2} \ \mathbf{n}])$
$\mathbb{R}^2 \times \mathbb{S}^1$	$[p_i^x \ p_i^y \ 0]^\top$	$\mathbf{R}(\alpha_i, \mathbf{e}_3)$	$\text{diag}(d_{ij} \mathbf{R}_i^\top \mathbf{P}(\bar{\mathbf{p}}_{ij}))$	$\text{diag}(\mathbf{R}_i^\top [\bar{\mathbf{p}}_{ij}]_\times [\mathbf{0}_{3 \times 2} \ \mathbf{e}_3])$
\mathbb{R}^3	$[p_i^x \ p_i^y \ p_i^z]^\top$	$\mathbf{R}(\alpha_i, \mathbf{0}_{3 \times 1}) = \mathbf{I}_3$	$\text{diag}(d_{ij} \mathbf{I}_3^\top \mathbf{P}(\bar{\mathbf{p}}_{ij}))$	$\text{diag}(\mathbf{I}_3^\top [\bar{\mathbf{p}}_{ij}]_\times \mathbf{0}_{3 \times 3})$
\mathbb{R}^2	$[p_i^x \ p_i^y \ 0]^\top$	$\mathbf{R}(\alpha_i, \mathbf{0}_{3 \times 1}) = \mathbf{I}_3$	$\text{diag}(d_{ij} \mathbf{I}_3^\top \mathbf{P}(\bar{\mathbf{p}}_{ij}))$	$\text{diag}(\mathbf{I}_3^\top [\bar{\mathbf{p}}_{ij}]_\times \mathbf{0}_{3 \times 3})$

BEARING RIGIDITY

...back to formation control

THE BEARING POTENTIAL

Consider the potential function of bearing errors:

$$\Phi(t) = \frac{1}{2} \sum \|g_{ij}(t) - g_{ij}^*\|^2$$

A Gradient-descent control

$$\dot{p} = -\nabla_p \Phi(t)$$

$$\dot{p}_i(t) = - \sum_{j \in \mathcal{N}_i} \frac{1}{\|e_{ij}(t)\|} P_{g_{ij}(t)} g_{ij}^*$$

- ▶ $e_{ij}(t) = p_j(t) - p_i(t)$
- ▶ implementation requires distance and bearing measurements!
- ▶ $P_{g_{ij}(t)}$ is an orthogonal projection matrix

BEARING-ONLY STRATEGY

Proposed Control Law

$$\begin{aligned}\dot{p}_i(t) &= - \sum_{j \in \mathcal{N}_i} P_{g_{ij}(t)} g_{ij}^* \\ \dot{p}(t) &= R_B^T(p) \text{diag}\{\|e_{ij}\|\} g^*\end{aligned}$$

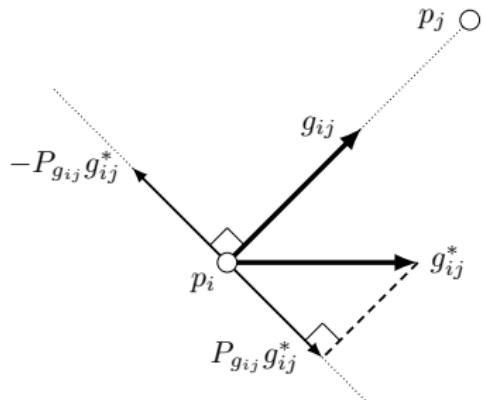


Figure 1: Geometric interpretation

EXAMPLES

EXAMPLES

Centroid and Scale Invariance

- ▶ Centroid of the formation

$$\bar{p} \triangleq \frac{1}{n} \sum_{i=1}^n p_i$$

- ▶ Scale of the formation

$$s \triangleq \sqrt{\frac{1}{n} \sum_{i=1}^n \|p_i - \bar{p}\|^2}.$$

BEARING-ONLY FORMATION CONTROL - STABILITY ANALYSIS

Centroid and Scale Invariance

- ▶ Centroid of the formation

$$\bar{p} \triangleq \frac{1}{n} \sum_{i=1}^n p_i$$

- ▶ Scale of the formation

$$s \triangleq \sqrt{\frac{1}{n} \sum_{i=1}^n \|p_i - \bar{p}\|^2}.$$

Almost global convergence

- ▶ Two isolated equilibriums:
one stable, one unstable

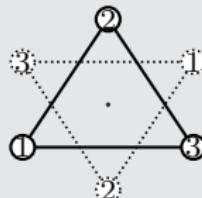


Figure 2: Solid line is target formation.

Reference: S. Zhao and D. Zelazo, "Bearing rigidity and almost global bearing-only formation stabilization," IEEE Transactions on Automatic Control, vol. 61, no. 5, pp. 1255-1268, 2016.

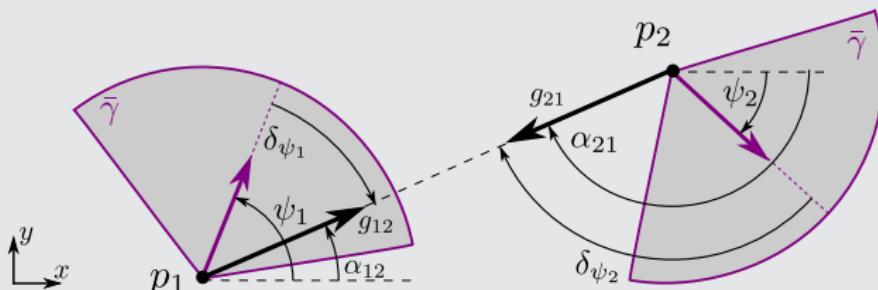
EXTENSIONS - ATTITUDE SYNCHRONIZATION

EXTENSIONS - FORMATION MANEUVERING

LIMITED FIELD-OF-VIEW SENSING

Sensing Model

Visual sensors are bounded by a limited field-of-view



- ▶ Sensing graph can become **directed**
- ▶ Neighbors are not static
- ▶ α_{ij} is the angle of the bearing g_{ij}
- ▶ δ_{ψ_i} is the facing direction error

LIMITED FIELD-OF-VIEW SENSING

$$\dot{p}_i(t) = - \sum_{j \in \mathcal{N}_i} P_{g_{ij}(t)} g_{ij}^*$$

Facing direction is *not* controlled

Problem

Design the control inputs u_i and ω_i such that the desired bearing is reached using only bearing measurements and a given limited field-of-view of the visual sensor.

TWO AGENT CASE - SIMULATIONS

No Sensing: $w_1(0) = w_2(0) = 0$

$$|\delta_{\psi_1}(0)| > \bar{\gamma}/2 \text{ and } |\delta_{\psi_2}(0)| > \bar{\gamma}/2$$

TWO AGENT CASE - SIMULATIONS

Complete Sensing: $w_1(0) = w_2(0) = 1$

$$|\delta_{\psi_1}(0)| < \bar{\gamma}/2 \text{ and } |\delta_{\psi_2}(0)| < \bar{\gamma}/2$$

TWO AGENT CASE - SIMULATIONS

Partial Sensing: $w_1(0) = 1, w_2(t) = 0, t \geq 0$

$|\delta_{\psi_1}(0)| < \bar{\gamma}/2$ and $|\delta_{\psi_2(t)}| > \bar{\gamma}/2$ for all $t \geq 0$

TWO AGENT CASE - SIMULATIONS

Partial Sensing: $w_1(0) = 1, w_2(0) = 0$ and $w_2(t) = 1$ for $t > T$

$$|\delta_{\psi_1}(0)| < \bar{\gamma}/2 \text{ and } |\delta_{\psi_2}(t)| < \bar{\gamma}/2 \text{ for some } t > T$$

Analytical Results for $n = 2$

If the following Assumptions hold:

1. Initially one agent can sense the other
2. The visual sensor satisfies $\bar{\gamma}/2 > 1/d_{12}(0)$

Then, the desired formation g_{12}^* will be reached from almost all initial conditions (except for $g_{12}(0) = -g_{12}^*$).

Analytical Results for $n = 2$

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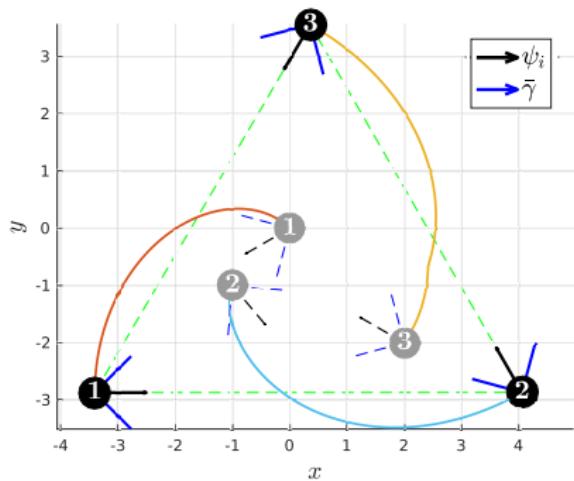
Then, the desired formation g_{12}^* will be reached from almost all initial conditions (except for $g_{12}(0) = -g_{12}^*$).

- ▶ Holds for two agents only
- ▶ Includes directed interactions

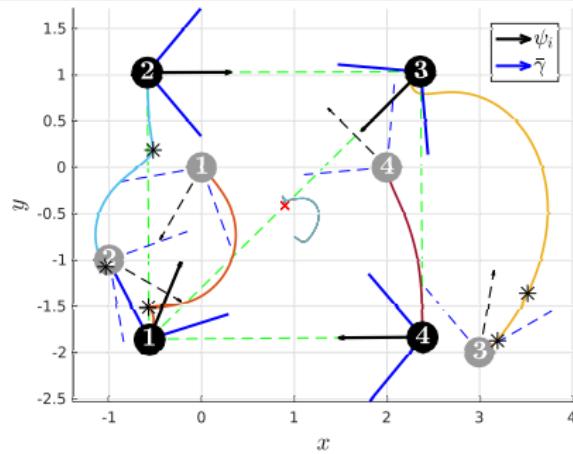
SIMULATION FOR $n > 2$

What changes?

- Desired facing direction is not intuitive
- Rigidity conditions are required



Faces the closest neighbor.



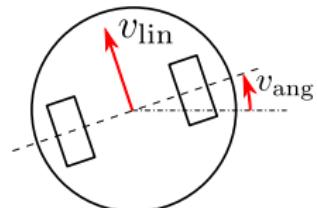
Faces in the middle of the agents that are inside the FOV.

SIMULATION FOR $n > 2$

Facing is controlled by ω_i

EXPERIMENTS

TurtleBotII Robots - Unicycle Model



$$\dot{x}_i = v_{i\text{lin}} \cos(\psi_i)$$

$$\dot{y}_i = v_{i\text{lin}} \sin(\psi_i)$$

$$\dot{\psi}_i = v_{i\text{ang}}$$

Vision sensing with Microsoft Kinect Sensor



Figure 3: Kinect used as a bearing-only sensor.

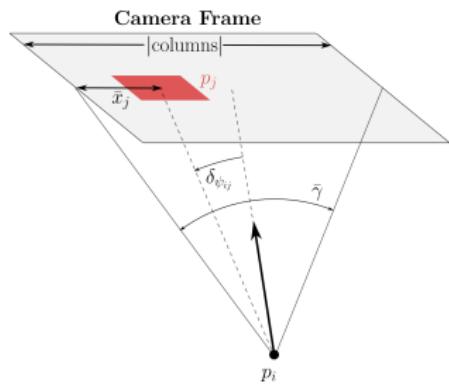


Figure 4: Camera frame that is taken from a visual sensor on agent i , the red square indicates the color of neighbor j within the camera frame.

Bearing-Only Controller for Unicycle Dynamics

$$v_{i_{\text{lin}}} = [\cos(\theta_i) \quad \sin(\theta_i)]^T u_i$$

$$v_{i_{\text{ang}}} = [-\sin(\theta_i) \quad \cos(\theta_i)]^T u_i$$

Inspired by S. Zhao et. al, *A general approach to coordination control of mobile agents with motion constraints*, IEEE Transactions on Automatic Control, 63(5):1509-1516.

Bearing Formation Control with Unicycle

$$\dot{x}_i = - [\cos(\theta_i) \quad \sin(\theta_i)] \sum_{j \in \mathcal{N}_i} P_{g_{ij}} g_{ij}^* \cos(\theta_i)$$

$$\dot{y}_i = - [\cos(\theta_i) \quad \sin(\theta_i)] \sum_{j \in \mathcal{N}_i} P_{g_{ij}} g_{ij}^* \sin(\theta_i)$$

$$\dot{\theta}_i = - [-\sin(\theta_i) \quad \cos(\theta_i)] \sum_{j \in \mathcal{N}_i} P_{g_{ij}} g_{ij}^*.$$

UNICYCLE DYNAMICS BEARING-ONLY CONTROL

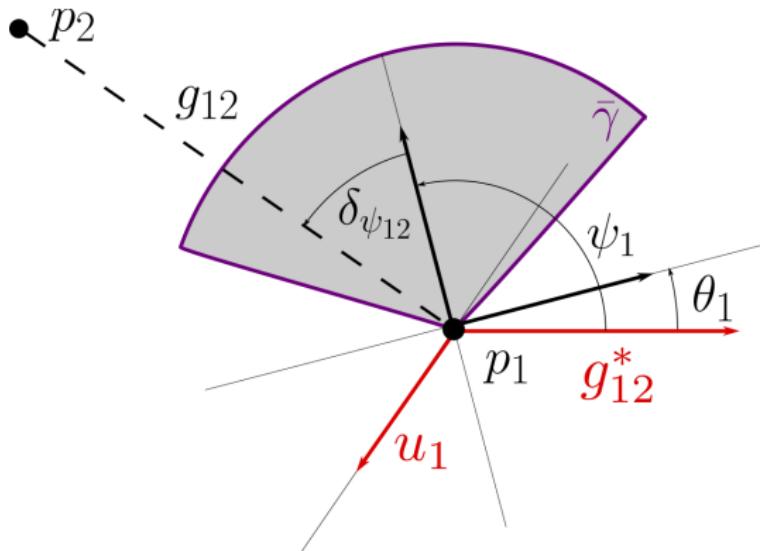


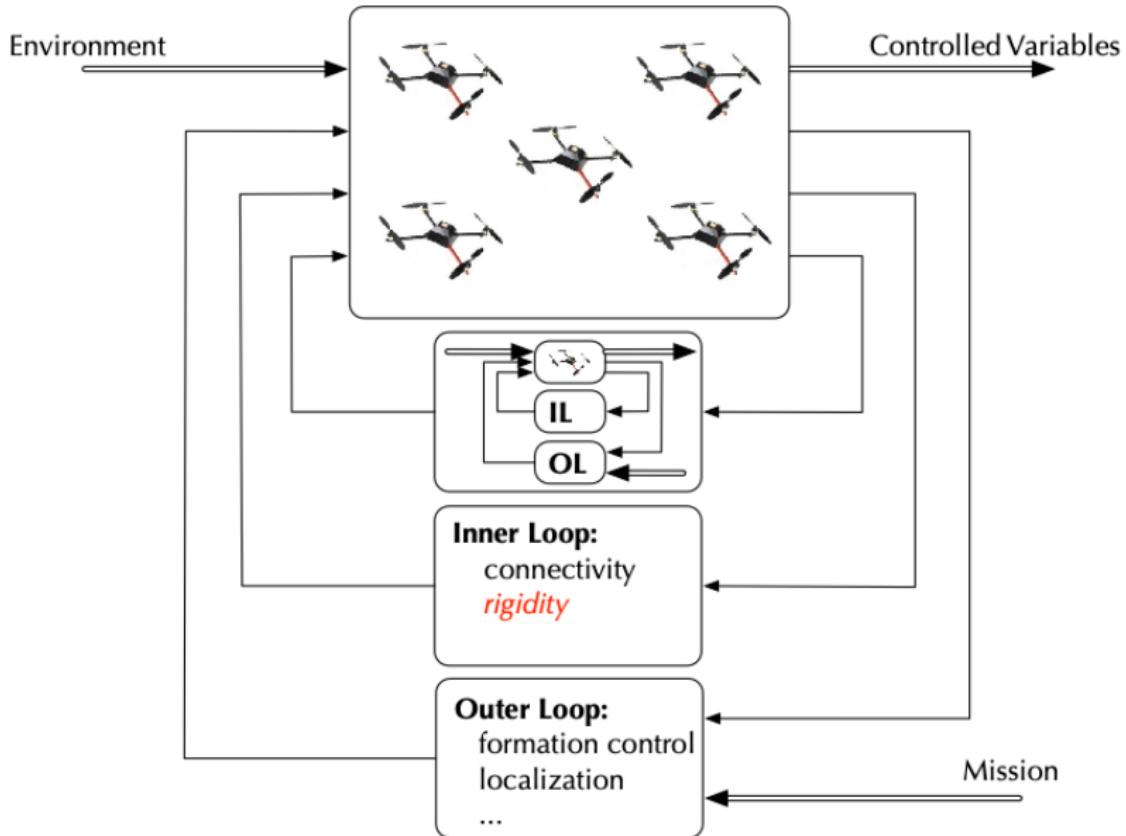
Figure 5: The camera does not align with the moving direction of the unicycle but is turned around $+\pi/2$.

Unique considerations required for unicycle dynamics!

UNICYCLE DYNAMICS BEARING-ONLY CONTROL

UNICYCLE DYNAMICS BEARING-ONLY CONTROL

THE RIGHT ARCHITECTURE



SUMMARY AND OUTLOOK

Topics covered by this talk:

- ▶ Distance rigidity and formation control
- ▶ General Bearing rigidity theory
- ▶ Bearing-only formation control law
- ▶ Field-of-View constrained systems
- ▶ Multi-robot implementation

Where next?

- ▶ directed rigidity theory
- ▶ general non-linear sensors
- ▶ more sophisticated models and robots

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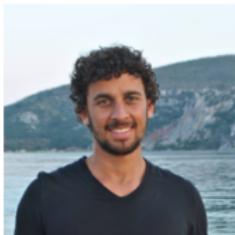
Dr. Antonio Franchi



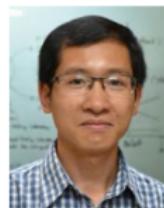
Prof. Hyo-Sung Ahn



Prof. Angelo Cenedese



Dr. Fabrizio Schiano



Dr. Minh Hoang Trinh



Dr. Giulia Michieletto

RELATED PUBLICATION

- ▶ S. Zhao and D. Zelazo, “Bearing rigidity and almost global bearing-only formation stabilization”, *IEEE Transactions on Automatic Control*, vol. 61, no. 5, pp. 1255-1268, 2016.
- ▶ D. Frank, D. Zelazo, and F. Allgower, “Bearing-Only Formation Control with Limited Visual Sensing: Two Agent Case,” *NeCSys 2018*.
- ▶ S. Zhao and D. Zelazo, “Bearing Rigidity Theory and its Applications for Control and Estimation of Network Systems: Life beyond distance rigidity”, *IEEE Control Systems Magazine*, vol. 38, no. 2, pp. 66-83, 2019.