Fourier Analysis of Stationary and Non-Stationary Time Series

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September 6, 2012
A time series is a stochastic process indexed at discrete points in time i.e \( X_t \) for \( t = 0, 1, 2, 3, \ldots \). The mean is defined as

\[
\mu_t = \mathbb{E}[X_t]
\]

and the autocovariance function is defined as

\[
\gamma(t, s) = \mathbb{E}[(X_t - \mu_t)(X_s - \mu_s)].
\]

A weakly-stationary (or just stationary) process is one in which \( \mu_t \equiv \mu \) where \( \mu \) is constant in time and \( \gamma(t, s) \) only depends on \( h = |t - s| \) called lag.

Stationary time series describe processes whose behavior does not change over time. White noise series are stationary. Often series can be transformed into roughly stationary series.
If autocovariance function is absolutely summable

\[ X_t = \int_{-1/2}^{1/2} A(\omega) e^{2\pi i \omega t} dZ(\omega). \]

Here \( dZ(\omega) \) is an independent mean zero stochastic process with \( \mathbb{E}[|dZ(\omega)|^2] = 1 \) and \( A(\omega) \) is called the transfer function and

\[ |A(\omega)|^2 = f(\omega). \]

\( f(\omega) \) or spectral density is the proportion of the variance concentrated at a component oscillating with frequency \( \omega \). This is called the Cramér model.
Why estimate $f(\omega)$:

- Some series (e.g. the sum of an AR(0.9) and AR(-0.9)) may be difficult to show simply in the time domain but easy in the Fourier domain. The example series will give two broad peaks at low and high frequencies.

- Sometimes the underlying process is periodic in nature (e.g. speech) and $f(\omega)$ might reveal this.
I have been trying to estimate $f(\omega)$. We do this using a *periodogram*. If we have $T$ realizations of a stationary time series. We can fit the model

$$X_t = \sum_{k=0}^{T-1} (a_k \cos(2\pi\omega_k t) + b_k \sin(2\pi\omega_k t)).$$

This model has no error terms as we are using $T$ values to estimate $T$ parameters. The $a_k, b_k$ can be calculated using a fast fourier transform. We then define the periodogram by

$$I(\omega_k) = (a_k^2 + b_k^2)/4\pi T.$$

Where $\omega_k = \frac{k}{T}$ and $k = 0, 1, \ldots, T - 1$. It can be shown that if $\omega_k \rightarrow \omega$ then as $T \rightarrow \infty$ then $I(\omega_k)$ is an asymptotically unbiased estimator of $f(\omega)$.
Time Series Basics
Estimating Spectral Density
Stationary Analysis of FTSE100 data
Evolutionary Spectra
SLEX

Figure: 4 simulated series all of length 1000

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Fourier Domain Analysis
Figure: Periodograms of Series 1-4
However as a point estimator of $f(\omega_k)$ the periodogram is not consistent (i.e. the variance does not tend to zero as $T$ is increased).

To deal with this we have to *smooth* the periodogram.

**Box Kernel Smoother**

$$\hat{P}(\omega_k) = \frac{1}{2m + 1} \sum_{j=k-m}^{k+m} I(\omega_j)$$
I have data showing the activity of the FTSE100 for 7101 daily time steps. This time series is clearly neither mean zero not stationary. The standard transform to improve this is to take logarithmic returns.

**Figure:** FTSE100 time series with logarithmic returns shown below
This periodograms appear to show that most of the variance is concentrated at periodic components with high frequencies with larger peaks at 0.27 and 0.48.
Many time series that people want to study are non-stationary. EEGs will exhibit different behavior before, after and during a seizure. Earthquakes and Explosion detection may show very different behavior.
I generated a series using the R code:

```r
y1 = arima.sim(n=1024,model=list(ar=0.9))
y2 = arima.sim(n=1024,model=list(ar=-0.9))
y = c(y1,y2)
```
Here is the resulting stationary periodogram:
In ... Priestley introduced the idea of a smoothly time varying spectral density function. This changed the Cramér model to one of the form

\[ X(t) = \int_{-1/2}^{1/2} A_t(\omega) e^{2\pi i \omega t} dZ(\omega) \]

where \( A_t(\omega) \) is the time varying transfer function.

\[ \gamma(s, t) = \int_{-1/2}^{1/2} A_s^*(\omega) A_t(\omega) e^{2\pi i \omega (t-s)} d\mu(\omega) \]
I decided to assume that my spectrum was changing slowly with time and that the series would be approximately stationary in small time blocks. With the FTSE series I analyzed the first 4096 time steps by splitting it into 16 blocks and doing stationary fourier analysis on each of these blocks.

**Figure:** A Slowly Changing Sloth
Figure: Periodograms for each of the smaller time series
Here are 8 periodograms produced by splitting the half and half AR non-stationary series:
Issues with the Priestley-Dahlhause method.

- I know no way of selecting or justifying the blocks in which to regard the series as stationary.
- Choosing blocks which are all the same size can miss information when \( f_t(\omega) \) is changing quickly or use a lot of computational power and produce greater error bands when \( f_t(\omega) \) is changing slowly.
The last model I looked at was SLEX. This model also incorporates the idea of a time variable spectrum which is constant in time blocks. SLEX fits the time series in each block to a set of SLEX (Smooth Localized EXponential) vectors which allows the blocks to overlap without producing a bias.

![SLEX vector, freq=2](image)

**Figure:** A SLEX vector
The SLEX algorithm uses a Fourier transform so can be computed using a fast fourier transform making it as efficient as earlier methods.

The SLEX procedure uses the Best Basis Algorithm developed by Coifman and Wickerhauser to automatically select the blocks.

The SLEX procedure minimizes a generalized cross validation deviance function to calculate the smoothing bandwidth.
The SLEX algorithm:

- Split the series dyadically into $2^j$ blocks for $j = 1, \ldots, J$.
- Compute the raw SLEXgrams in each of these blocks.
- Select the optimal bandwidth and smooth the SLEXgrams in each block.
- Select the best blocks to model the series using BBA.
The BBA computes the cost associated with selecting each block. It then compares the cost of each bigger block to the two half blocks beneath it and will reject the bigger block if the cost is greater.

Figure: Segmentation chosen by the BBA for the FTSE100 time series
**Figure:** SLEX blocks shown on the log returns time series
Figure: SLEXgrams
Performing SLEX on the half and half series gives the segmentation:
Here are the resulting SLEX periodograms:
Figure: Any Questions?