Uncertainty in Resource Allocation Problems

Multi-Skilled Workforce Planning Models

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Workforce Planning

There are three key stages in most workforce planning problems:

- Strategic Planning
  - Deciding the location of staff centres and the number of staff assigned to each one - up to 2 years in advance.
- Tactical Planning
  - Deciding which workers to train in which skills, and whether to hire new workers or make redundancies - around a year in advance.
- Operational Planning
  - Scheduling, deciding which workers to assign to which specific tasks - at the start of each day or week.

We focus on the operational planning stage.

Basic Premise

We have three things:

- A workforce, with some number of skills and a total number of working hours available.
- The workstack, a list of current jobs for each skill, how long they take, and when they are due.
- A forecast, which estimates the number of new jobs to be received for each skill that day.

We want to assign the total capacity to each skill in such a way that minimises the number of jobs left undone, as this costs the company money and/or reputation.

The Forecast

We forecast the demand using exponential smoothing, where previous data is considered with exponentially less weighting the longer ago it occurred:

$$\begin{aligned} I_{t+1} &= \alpha i_t + (1-\alpha) I_t \\ &= \alpha i_t + \alpha(1-\alpha) i_{t-1} + (1-\alpha)^2 I_{t-1} \\ &= \alpha \left[i_t + (1-\alpha) i_{t-1} + \ldots + (1-\alpha)^{t-1} i_1 \right] + (1-\alpha)^t i_0 \end{aligned}$$

Where I_t is the forecast and i_t the actual intake for day t.

This can be extended to Holt-Winters smoothing, where the data is smoothed three times to account for things such as weekly and yearly trends.

Simplifying the Problem

We reduce the problem to the following formulation:

- We wish to schedule one day in advance.
- We have a total number of hours available each day, a workstack for two days, and a forecast for two days.
- We always complete all jobs due that day from the workstack first.
- We then complete the jobs that arrive that day.
- If spare hours remain, we pull forward some jobs from the next day's workstack.

We wish to decide how many jobs of each skill to pull forward from the next day such as to minimise the number of jobs that roll-over each day.

Notation

- y_j - The number of jobs of skill j to pull forward.
- R_{jt} - The number of jobs of skill j that roll-over from day t.
- h_t - The total hours available for day t.
- D_{jt} - The workstack for skill j on day t.
- I_{jt} - The forecasted intake for skill j on day t.
- a_j - The time in hours taken for a job of skill j.
- c_j - The cost per hour of rolling over a job of skill j.

The Deterministic Model

$$\min_{y,R\geq 0} \sum_{j\in J} \sum_{t=1}^{2} a_j R_{jt} c_j$$

Subject to:

$$
\begin{aligned}
R_{j1} &\geq D_{j1} + I_{j1} + y_j - \left\lfloor \frac{h_1}{a_j} \right\rfloor, && \forall j \in J \\
R_{j2} &\geq R_{j1} + D_{j2} + I_{j2} - y_j - \left\lfloor \frac{h_2}{a_j} \right\rfloor, && \forall j \in J \\
R_{j1} &\leq D_{j1} + I_{j1} + y_j, && \forall j \in J \\
R_{j2} &\leq R_{j1} + D_{j2} + I_{j2} - y_j, && \forall j \in J \\
y_j &\leq \min\left\{ \left\lfloor \frac{h_1}{a_j} \right\rfloor - D_{j1}, D_{j2} \right\}, && \forall j \in J \\
h_1 &\geq \sum_{j\in j} a_j (D_{j1} + I_{j1} + y_j - R_{j1}) \\
h_2 &\geq \sum_{j\in J} a_j (D_{j2} + I_{j2} - y_j + R_{j1} - R_{j2})
\end{aligned}
$$

Introducing Uncertainty

We first introduce uncertainty to the model by allowing an adversary to select the intake from some feasible set $\mathcal{I}$.

$\mathcal{I}$ is the set of all $|J|x2$ matrices where the j^{th} row is a pair of possible intakes for days 1 and 2 respectively for skill j.

This leads to the following robust min-max model:

Adversarial Model

$$\min_{y,R\geq 0} \max_{i\in\mathcal{I}} \sum_{j\in J} \sum_{t=1}^{2} a_j R^i_{jt} c_j$$

Subject to:

$$\begin{aligned}
R^i_{j1} &\geq D_{j1} + i_{j1} + y_j - \left\lfloor \frac{h_1}{a_j} \right\rfloor, && \forall j \in J, i \in \mathcal{I} \\
R^i_{j2} &\geq R^i_{j1} + D_{j2} + i_{j2} - y_j - \left\lfloor \frac{h_2}{a_j} \right\rfloor, && \forall j \in J, i \in \mathcal{I} \\
R^i_{j1} &\leq D_{j1} + i_{j1} + y_j, && \forall j \in J, i \in \mathcal{I} \\
R^i_{j2} &\leq R^i_{j1} + D_{j2} + i_{j2} - y_j, && \forall j \in J, i \in \mathcal{I} \\
y_j &\leq \min\left\{ \left\lfloor \frac{h_1}{a_j} \right\rfloor - D_{j1}, D_{j2} \right\}, && \forall j \in J \\
h_1 &\geq \sum_{j\in j} a_j(D_{j1} + i_{j1} + y_j - R^i_{j1}), && \forall i \in \mathcal{I} \\
h_2 &\geq \sum_{j\in J} a_j(D_{j2} + i_{j2} - y_j + R^i_{j1} - R^i_{j2}), && \forall i \in \mathcal{I}
\end{aligned}$$

Actually Uncertain?

This model has a problem - since higher intakes always result in no less roll-over, the adversary simply always chooses the highest possible intake for every skill every day.

Whilst it is useful to prepare for the worst case scenario, it does not satisfy the original intent of representing the uncertainty in the forecast.

It also does not require this min-max model to do - we could simply choose the maximum intake instead of the forecast and use the original deterministic model instead.

Introducing Uncertainty II

To fix this, we model the intake as a binomial random variable:

$$I_{jt} \sim Bin\left(i_{jt}^{max}, p_{jt}\right)$$

We then allow the adversary to choose $q \in \mathcal{U}$, where $\mathcal{U}$ is the set of $|J|x2$ matrices such that each q_{jt} is some feasible p_{jt}, so:

$$I_{jt} \sim Bin\left(i_{jt}^{max}, q_{jt}\right)$$

Adversarial Probabilistic Model

This gives us a new model, with the new objective function:

$$\min_{y,R\geq 0}\max_{q\in\mathcal{U}}\sum_{j\in J}\sum_{t=1}^{2} a_j c_j \mathbb{E}_q(R_{jt})$$

And the same constraints as the previous model.

Finding Optimal q

For similar reasons as before, if the feasible set for each p_{jt} is $[0, 1]$, or indeed any set containing 1, the adversary would simply choose 1 for every skill every day, as this guarantees the maximum intake.

So, instead, we restrict q_{jt} to sets that do not contain 1 - for example, we may require that the pair of probabilities for each skill sum to no more than 1.

However, since the binomial distribution requires raising to the power i_{jt}^{max}, the objective is non-linear enough for the problem to be computationally intractable.

The model therefore requires some reformulation.

Reformulated Model

We introduce a dummy variable z that we wish to minimise, restricting it to at least as high as each expected cost:

$$\min_{y,R,z\geq 0} z$$

Subject to:

$$z \geq \sum_{j\in J}\sum_{t=1}^{2} a_j c_j \mathbb{E}_q(R_{jt}), \forall q \in \mathcal{U}$$

Along with all the previous constraints.

This causes z to be set to $\max_{q\in\mathcal{U}} \left\{\sum_{j\in J}\sum_{t=1}^{2} a_j c_j \mathbb{E}_q(R_{jt})\right\}$, from which each $\mathbb{E}_q(R_{jt})$ can be recovered, and therefore the optimal q.

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